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FOUR-LOOP APPROXIMATION IN THE φ^4 MODEL

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Четырехпетлевое приближение в модели ϕ^4

Приводятся результаты расчетов в четырехпетлевом приближении функции Гелл-Манна-Лоу в безмассовой модели с четверным взаимодействием. Коэффициенты разложения функции Гелл-Манна-Лоу, вычисленные в рамках теории возмущений, сравниваются с коэффициентами, рассчитанными по асимптотической формуле.

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Dittes F.M., Kubyshin Yu.A., Tarasov O.V. **E2 - 11100**

Four-Loop Approximation in the ϕ^4 Model

The results are presented of the calculation performed within the four-loop approximation for the Gell-Mann-Low function in the massless model with fourfold interaction. The coefficients for decomposition of the Gell-Mann-Low function calculated within perturbation theory are compared with those obtained by asymptotic formula.

The investigation has been parformed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

An attempt of exact calculation of the coefficients in high orders of the perturbation theory (PT) expansion for various quantities in quantum field theory leads to insurmountable difficulties at present time. Various methods for estimation of Feynman diagrams in high orders of PT have been proposed /2,4/. The approach of L.N.Lipatov /2/, which gives the asymptotic formula for the arbitrary coefficient of the series, attracts a particular attention. It is of great interest to compare this formula with the exact PT calculations.

In this paper the result of the 4-loop calculations in the massless φ^4 -model in four-dimensional space is given and the peculiarities of these calculations are discussed. We compare the exact coefficients of the Gell-Mann-Low (GML) function obtained in the framework of PT with that given by the asymptotic formula derived in /2/ and by its generalization to the case of N-component field /7/.

2. The results and peculiarities of calculation.

Consider the model of selfinteracting scalar N-component field. The Lagrangian is of the form:

$$(1)\quad \mathcal{L} = \frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \varphi_i)^2 - \frac{4\pi^2}{3} h \left(\sum_{i=1}^{N} \varphi_i^2 \right)^2 .$$

The invariant charge $\bar{h}(x,h)$ is defined as(see /1/, 51.2)

$$(2)\quad \bar{h}(x,h) = h \cdot \Gamma(x,h) \cdot d^2(x,h) ,$$

where d and Γ are dimensionless and renormalized propagator and one-particle irreducible vertex. All subtractions are made at a symmetrical point:

$$(3)\quad s = t = u = \frac{4}{3} p_i^2 = \lambda^2 .$$

The GML function is defined as :

$$(4)\quad \beta(h) = \left. \frac{\partial \bar{h}(\ln \frac{p^2}{\lambda^2}, h)}{\partial \ln \frac{p^2}{\lambda^2}} \right|_{p^2 = \lambda^2}$$

and can be expanded in a power series in coupling constant

$$(5)\quad \beta(h) = \sum_{k=2}^{\infty} \beta_k h^k .$$

The main result of this paper is the calculation of the coefficient β_5 .

Taking into account this approximation we can represent GML function in the following way:

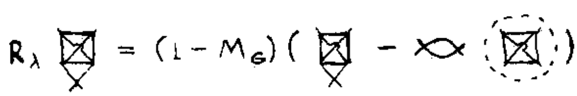
$$(6) \qquad \beta(h) = \frac{N+8}{3} h^2 - \frac{6N+28}{3} h^3 + (1.05 N^2 + 33.51 \cdot N + 119.58) \cdot h^4 - (0.62 \cdot N^3 + 42.1 \cdot N^2 + 579 \cdot N + 1716) \cdot h^5 .$$

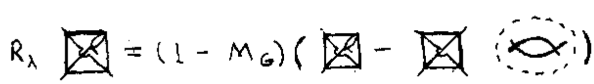
This expression is in agreement with the result obtained in /5/ that was verified for N=1 by the transition to the renormalization scheme used in that article with the help of conversion formulas.

Almost all calculations have been carried out with the help of the Chebyshev polinomial technique proposed in /6/ (see also /3/). However, one fails to calculate the nonplanar 4-loop diagrams 2,3 and 8 (see table 2 in Appendix) within this method since their integrands contain Feynman propagators depending on three momenta to which the technique mentioned above is inapplicable.

But using the α -representation for Feynman diagrams one can show that either the sum of two nonplanar diagrams or the difference between nonplanar and planar diagrams doesn't depend on the relation between external momenta. Therefore, putting some momenta equal to zero and using the Chebyshev polynomial technique one can easily calculate the obtained integrals.

Let us consider the diagrams 2 and 3. Using the notation of /1/ § 30, R-operation for them can symbolically be represented in the form:





The coefficients before the linear logarithms are necessary for calculating the GML function. Using the α -representation one can write them in the following way (see /3/):

$$
(7) \qquad f_2 = -\int d\alpha \; \delta(1-\sum\alpha)\left(\frac{1}{D_2^2(\alpha)} - \frac{A_a(\alpha)}{D_a^2(\alpha)\, D_b^2(\alpha)[A_a(\alpha)+A_b(\alpha)]}\right),
$$

$$
(8) \qquad f_3 = -\int d\alpha \cdot \delta(1-\sum\alpha)\left(\frac{1}{D_3^2(\alpha)} - \frac{A_b(\alpha)}{D_a^2(\alpha)\, D_b^2(\alpha)[A_a(\alpha)+A_b(\alpha)]}\right).
$$

Here $A(\alpha) = \dfrac{Q(\alpha;p)}{p^2 \, D(\alpha)}$, where $D(\alpha)$ and $Q(\alpha;p)$ are the structural functions of the given diagrams expressed through the sums over trees and 2-trees, respectively (/1/, § 29). The diagrams a and b are given in figure 1.

From (7) and (8) the sum of these integrals is equal to:

$$
(9) \qquad f_2 + f_3 = -\int d\alpha \; \delta(1-\sum\alpha)\left(\frac{1}{D_2^2(\alpha)} + \frac{1}{D_3^2(\alpha)} - \frac{1}{D_a^2(\alpha)\, D_b^2(\alpha)}\right)
$$

and doesn't depend on the relation between the external momenta, entering into $A(\alpha)$ only.

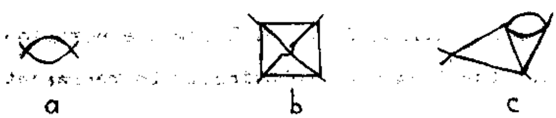


Fig.1.

To determine their contribution to the GML function, it is necessary to calculate at computer the coefficient before the linear logarithm of one of them since they enter into the vertex with different combinatorial factors. This is the main inaccuracy of the result.

Analogously one can show that the difference between the diagrams 8 and 14 is independent of the relation between the external momenta. Owing to this fact the diagram 8 can be calculated exactly. The results of calculation for each diagram, combinatorial factors and the coefficient β_5 of the GML function are given in Appendix.

3. The discussion of the results

When deriving the asymptotic formula, the contribution of propagator has not been taken into account since it enters into the correction $O(\frac{1}{K})$. It is interesting to note in this connection that for $N=1$ the propagators give 5,9%, 4,6% and 4,8% in the 2-, 3- and 4-loop approximations, respectively. Diagram 1 without divergent subgraphs gives the greatest contribution (43%) to the GML function for N=1 . The comparative contribution is given by diagrams 2 (24%) and 3 (12%) containing one divergent subgraph and also by diagram 6 (21%) with three divergent subgraphs. Note the fact that for the 3-loop approximation the contribution of diagram C (fig.1) (62%) is twice larger as the contribution of the sceleton diagram b (37%).

Thus, the contribution of diagrams without subgraphs or with one divergent subgraph to the GML function increases for the 4-loop approximation in comparison with the previous order, and reaches 78%.

Let us compare the first (see table 1) exact coefficients of the GML function β_K with the asymptotic ones $\breve{\beta}_K$ which can be determined from formula (see /7/):

$$\tilde{\beta}_K = \alpha(N) \left(\frac{2K}{e}\right)^K K^{b+\frac{1}{2}} .$$

where

$$b = 3 + \frac{N}{2}$$

$$\alpha(N) = \begin{cases} 0.0087 & N=1 \\ 0.0043 & N=2 \\ 0.0020 & N=3 \end{cases} .$$

For N=1 this expression coincides with that obtained by L.N. Lipatov /2/.

The relative errors $\eta = 1 - \frac{\tilde{\beta}_K}{\beta_K}$ are also given in table 1.

Table I

	K	2	3	4	5
N=1	β_K	3	-34/3	154.3	-2338
	$\tilde{\beta}_K$	0.30	-7,6	167.1	-3664
	η	90%	33%	-8%	-57%
N=2	β_K	3,3	-13,3	190,8	-3048
	$\tilde{\beta}_K$	0,21	- 6,5	165,2	-4049
	η	94%	51%	13%	-32%
N=3	β_K	3,7	-15,3	229,6	-3850
	$\tilde{\beta}_K$	0,14	- 5,2	153,6	-4211
	η	96%	66%	33%	-9%

For examined N in first orders in h the coefficients of perturbation expansion exceed the asymptotic ones. But the latter grow more rapidly with K and become greater than the coefficients of PT in fifth order. This explains the fact that for N=1,2 the relative error diminishes rapidly for $K \leq 4$; but it increases in absolute value in the next order. For N=3 η diminshes up to fifth order inclusively.

Thus the asymptotic formula describes the first five terms of the expansion of $\beta(h)$ badly, and no numerical estimates for low orders can be made on its basis. It may serve as a source of additional information about the analytic properties of the GML function which can be taken into account in the construction of approximants. An optimistic hypothesis (see /2/,/8/) that in fifth order the asymptotic coefficients will draw near to the exact ones isn't confirmed. Let us note in this connection that for such a comparatively simple model as anharmonic oscillator the asymptotics sets on slowly and the relative error diminishes to 10% only in 17 th order (see /9/).

For the 4-loop approximation the GML function possesses zero for all N (for N=1 when h =0.118). By the general classification this would correspond to the finite renormalization of coupling constant. But as it is mentioned in /1/ no stable conclusions can be made on the basis of the finite number of terms of the perturbation expansion.

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Appendix

The results of calculation for each diagram and their contributions to the GML function for N=1 are given in table 2. Diagrams are groupped with the increasing number of divergent subgraphs. The second column gives the momenta integrals

$$\frac{1}{\pi^8}\int \frac{dk_1\, dk_2\, dk_3\, dk_4}{\ldots (p_i - k_j)^2 \ldots}$$

for the corresponding diagrams whereas the third one the combinatorical factors. The following notation is used in table 2:

$$L = \ln\left(-\frac{p^2}{\lambda^2}\right)$$

$$J = \frac{1}{i\pi^2}\int dq \frac{\ln\frac{(q-k)^2}{q^2}}{q^2\,(p-q)^2}\Bigg|_{k^2=(p-k)^2=\frac{3}{4}p^2} = 0.74939$$

$$I = \frac{1}{i\pi^2}\int dq \frac{\ln\frac{q^2}{k^2}\,\ln\frac{(q-k)^2}{q^2}}{(q-p)^2\,(q-k_1)^2}\Bigg|_{p^2=k_1^2=\frac{3}{4}k^2} = -0.805$$

J, a_c and S can be expressed through the sums

$$\sum_{n=1}^{\infty}\frac{C_n(\hat{pk})}{b_n}\left(\frac{p}{k}\right)^n\Bigg|_{p^2=(p-k)^2=\frac{3}{4}k^2}$$

and

$$\sum_{n=1}^{\infty}\frac{C_n(-\hat{p_1 p_2})}{b_n}\Bigg|_{p_1^2=p_2^2=\frac{3}{4}(p_1+p_2)^2}$$

where $C_n(\beta\kappa)$ are the Chebyshev polynomials, and b_n are the products of factors n, $n+1$ and $n+2$ in various powers. This enables us to calculate the values J, a_c and S with great accuracy at computer,

$a_c = -0.31554$

$S = 0.46905$

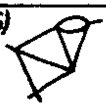
$D = -11.5 \pm 0.2$ coefficient before the linear logarith of diagram 2. Using these parameters we can write the coefficient before h^5 of the GML function in the following way:

$$
\begin{aligned}
\beta_5 = \frac{1}{81}\Big\{ & 47N^3 - 505N^2 - 11964N - 32668 - \\
& - (64N^3 + 1184N^2 + 7232N + 14848)\cdot S + \\
& + (32N^3 + 896N^2 + 7040N + 15360)\cdot a_c - \\
& - (12N^3 + 216N^2 + 1152N + 1536)\cdot \ln\tfrac{3}{4} + \\
& + (4N^3 + 72N^2 + 384N + 512)\cdot \ln^2\tfrac{3}{4} - \\
& - (8N^3 + 224N^2 + 1760N + 3340)\cdot \dot{I} - \\
& - (56N^3 + 2144N^2 + 20224N + 57280)\cdot \zeta(3) - \\
& - (640N^2 + 17600N + 59520)\,\zeta(5) - \\
& - (80N^2 + 992N + 2816)\cdot D\Big\}
\end{aligned}
$$

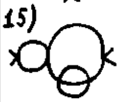
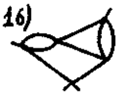
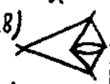
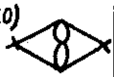
$\zeta(3) = 1.2020569$ $\qquad$ $\zeta(5) = 1.0369278$

$$
\beta_5 = \begin{cases} -2338 \pm 10 & N=1 \\ -3048 \pm 13 & N=2 \\ -3850 \pm 16 & N=3 \end{cases}
$$

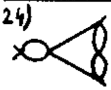
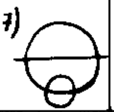
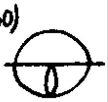
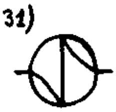
Table 2

Diagram	The integral value	Combinatorical factor	(N=1) Contribution to β_5
1	2	3	4
1) 	$-20\zeta(5)L$	$\frac{16}{81}(2N^2+55N+186)$	-995.5
2) 	$3\zeta(3)L^2+DL$	$\frac{16}{81}(5N^2+62N+176)$	-553.2
3) 	$3\zeta(3)L^2-(12\zeta(3)+D)L$	$\frac{32}{81}(5N^2+62N+176)$	-278.4
4) 	$\frac{L^3}{6}-(\frac{1}{2}+\frac{1}{4}\ln\frac{3}{4})L^2+(1-\frac{J}{2}+\frac{1}{2}\ln\frac{3}{4})L$	$\frac{16}{81}(5N^2+32N+44)$	7.7
5) 	$\frac{L^3}{12}-(\frac{1}{2}+\frac{1}{4}\ln\frac{3}{4})L^2+(1-\frac{J}{2}+\frac{1}{2}\ln\frac{3}{4})L$	$\frac{16}{81}(5N^2+32N+44)$	7.7
6) 	$\frac{L^4}{24}-\frac{L^3}{2}+\frac{5}{2}L^2-5L$	$\frac{32}{81}(7N^2+72N+164)$	-480
7) 	$\frac{L^4}{8}-\frac{5}{6}L^3+(\frac{5}{2}-J)L^2+(1+a_0-2S-I)L$	$\frac{32}{81}(7N^2+72N+164)$	52.9
8) 	$\frac{L^4}{12}-\frac{2}{3}L^3+(3-J)L^2+(I+2S+2a_0+2\zeta(3))L$	$\frac{8}{81}(7N^2+72N+164)$	45.7
9) 	$\frac{L^4}{24}-\frac{L^3}{2}+\frac{5}{2}L^2+(4\zeta(3)-5)L$	$\frac{32}{81}(7N^2+72N+164)$	-18.4
10) 	$\frac{L^4}{4}-L^3+3(1-J)L^2-6SL$	$\frac{4}{81}(3N^3+24N^2+80N+136)$	-33.8

(cont.)

1	2	3	4
11) 	$\frac{L^4}{4} - L^3 + 3(1-J)L^2 - (2I + 6S)L$	$\frac{8}{81}(N^3+14N^2+76N+152)$	-28.9
12) 	$\frac{L^4}{12} - \frac{2}{3}L^3 + L^2(3-J) + (2a_0+2S)L$	$\frac{16}{81}(N^3+14N^2+76N+152)$	14.7
13) 	$\frac{L^3}{6} - \frac{5}{4}L^2 + \left(\frac{5}{2} + \frac{3}{2}\ln\frac{3}{4} - \frac{1}{2}\ln^2\frac{3}{4}\right)L$	$\frac{4}{81}(N^3+18N^2+96N+128)$	24.3
14) 	$\frac{L^4}{12} - \frac{2}{3}L^3 + (3-J)L^2 + (I+2S+2a_0)L$	$\frac{8}{81}(N^3+10N^2+72N+160)$	-12.0
15) 	$\frac{L^3}{4} - \frac{L^2}{2} - \frac{1}{2}\ln\frac{3}{4}L^2$	$\frac{8}{81}(N^3+8N^2+32N+40)$	0
16) 	$\frac{L^4}{8} - \frac{5}{6}L^3 + \left(\frac{5}{2}-J\right)L^2 + (1+a_0-2S)L$	$\frac{16}{81}(11N^2+76N+156)$	-12.2
17) 	$\frac{5}{24}L^4 - \frac{5}{6}L^3 + \frac{1}{2}L^2 + (2S+I+3-4\zeta(3))L$	$\frac{8}{81}(11N^2+76N+156)$	$-40,2$
18) 	$\frac{L^4}{12} - \frac{2}{3}L^3 + L^2 - 2SL$	$\frac{8}{81}(11N^2+76N+156)$	-22.5
19) 	$\frac{L^4}{12} - \frac{2}{3}L^3 + (1+J)L^2 - (4\zeta(3)+2a_0)L$	$\frac{8}{81}(N^3+14N^2+76N+152)$	-100.3
20) 	$\frac{L^4}{6} - \frac{2}{3}L^3 + (4S+8-12\zeta(3))L$	$\frac{2}{81}(N^3+18N^2+80N+144)$	-27.3
21) 	L^4	$\frac{N^4+10N^3+40N^2+80N+112}{81}$	0
22) 	$\frac{1}{2}L^4 - L^3$	$\frac{4}{81}(3N^3+24N^2+80N+136)$	0
23) 	$\frac{1}{4}L^4 - L^3 + L^2$	$\frac{4}{81}(11N^2+76N+156)$	0

(cont.)

1	2	3	4
24) 	$\frac{1}{3}L^4 - L^3 + 2(1-J)L^2$	$\frac{4}{81}(N^3+18N^2+80N+144)$	0
25) 	$\frac{1}{6}L^4 - L^3 + 2L^2$	$\frac{16}{81}(N^3+14N^2+76N+152)$	0
26) 	$\frac{1}{3}L^4 - L^3 - 2L^2 + 2JL^2$	$\frac{4}{81}(3N^3+24N^2+80N+136)$	0
27) 	$\frac{L^2}{8} - \frac{5}{8}L$	$\frac{4}{81}(3N^2+12N+12)$	-1.7
28) 	$\frac{1}{4}L^2$	$\frac{4}{81}(N^2+4N+4)$	0
29) 	$\frac{L^3}{2} + \left(\frac{3}{2}\ln\frac{3}{4} - \frac{9}{4}\right)L^2 + \left(\frac{21}{4} + \frac{3}{2}\ln^2\frac{3}{4} - \frac{9}{2}\ln\frac{3}{4}\right)L$	$\frac{2}{81}(N^3+8N^2+32N+40)$	26.7
30) 	$\frac{L^3}{6} + \left(\frac{1}{2}\ln\frac{3}{4} - \frac{5}{4}\right)L^2 + \left(\frac{9}{4} + J - \frac{5}{2}\ln\frac{3}{4} + \frac{1}{2}\ln^2\frac{3}{4}\right)L$	$\frac{4}{81}(5N^2+32N+44)$	30.1
31) 	$\frac{L^3}{3} + \left(\ln\frac{3}{4} - 2\right)L^2 + \left(5 + J + \ln^2\frac{3}{4} - 4\ln\frac{3}{4}\right)L$	$\frac{4}{81}(5N^2+32N+44)$	55.9

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