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METHODS OF MULTILOOP CALCULATIONS AND THE RENORMALIZATION GROUP ANALYSIS OF φ^4 THEORY

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Методы вычисления многопетлевых диаграмм и ренормгрупповой анализ теории ϕ^4

Разработана эффективная техника вычисления параметров ренормализационной группы, позволяющая при счете диаграмм занулять все внешние импульсы. Проведены трех- и четырехпетлевые расчеты функции Гелл-Манна-Лоу теории ϕ^4 в различных ренормировочных схемах. Исследована зависимость этой функции от конкретного выбора фиксированных отношений импульсных аргументов инвариантного заряда.

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Methods of Multiloop Calculations and the Renormalization Group Analysis of ϕ^4 Theory

Effective methods for evaluating renormalization group quantities are worked out, which allow computing the diagrams with all external momenta put equal to zero. The Gell-Mann-Low function of ϕ^4 theory is calculated in the three- and four-loop approximation using different renormalization schemes. The dependence of this function on the ratios of external momenta is studied.

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1. The perturbation theory in coupling constant is the basis of almost all quantum field theory calculations. In the renormalization group (RG) approach a perturbation theory information is also required to write down the basic equations. The typical situation one encounters solving these equations is as follows: RG quantities which in an exact theory are independent of the renormalization scheme, may acquire such a dependence so far as only several lowest orders of perturbation expansions are taken into account. In such cases we have no obvious criteria to prefer one renormalization scheme to another. The reasons of computational convenience seem to be the only possible criterion.

In my previous paper /1/ the general relations between the different renormalization schemes have been presented, and the problem of choosing the most suitable for practical use renormalization scheme have been investigated. In the present paper the comparison of different schemes from this point of view is carried out in the framework of ϕ^4 theory which is the simplest renormalizable field theory model. So, this paper is a sequel to ref. /1/ representing an application of general results to a concrete model. All the notations of ref. /1/ remain unchanged.

2. A comparison of different renormalization schemes in ϕ^4 theory requires at least three-loop calculations of RG parameters, because the Gell-Mann-Low (GML) functions β of various schemes

begin to differ only at three-loop level in this theory. These calculations are nontrivial even in massless scalar theory we investigate. Considerable simplification occurs if it is allowed to put all (or all except one) external momenta of the diagrams equal to zero. The 't Hooft renormalization scheme gives us such an opportunity.

This scheme, based on dimensional regularization, may be formulated in the language of counterterms and renormalization constants [2] or as a recursive subtraction procedure, R-operation [3] (see also ref. [4]). Let KG be the pole part in ϵ of the contribution of diagram G to the corresponding Green function. As usual $\epsilon = \frac{4-n}{2}$, n being the space-time dimension. Then the R-operation of 't Hooft's scheme is of the form

$$R(G) = R'(G) + \Delta(G),$$

$$R'(G) = 1 + \sum_{G = G_1 * \ldots * G_m} \Delta(G_1) \ldots \Delta(G_m), \tag{1}$$

$$\Delta(G_i) = \begin{cases} 0 & \text{if } G_i \text{ is one-particle reducible,} \\ 1 & \text{if } G_i \text{ is an elementary vertex,} \\ -KR'(G_i) & \text{if } G_i \text{ is one-particle irreducible.} \end{cases}$$

The sum is over all partitions of the diagram G into a set of subdiagrams with $1 < m < N$, N being the number of vertices in G. We see that 't Hooft's scheme differs from the standard subtraction procedure [5] only in the definition of the operation K.

To deduce the basic RG equations it proves to be more convenient to use the other equivalent formulation of the 't Hooft scheme in terms of renormalization constants [2]. While in the R-operation approach the renormalized Green function is

$$\Gamma_R\left(\frac{k_i^2}{\mu^2}, H\right) = \lim_{\epsilon\to 0} R\Gamma((\mu^2)^\epsilon H, k_i^2, \epsilon),$$

where H is a coupling constant, k_i are external momenta, and μ is a renormalization parameter, in another approach it is

$$\Gamma_R\left(\frac{k_i^2}{\mu^2}, H\right) = \lim_{\epsilon\to 0} Z_\Gamma\left(H, \frac{1}{\epsilon}\right)\Gamma(H_B, k_i^2, \epsilon), \qquad (2)$$

where in the case of ϕ^4 theory

$$H_B = (\mu^2)^\epsilon H Z_1\left(H, \frac{1}{\epsilon}\right) Z_D^{-2}\left(H, \frac{1}{\epsilon}\right).$$

Here Z_1 and Z_D^{-1} are the renormalization constants of four-point vertex and propagator, respectively, given (as well as H_B) by power series in H and $\frac{1}{\epsilon}$:

$$H_B = (\mu^2)^\epsilon \left[H + \sum_{\nu=1}^{\infty} \frac{1}{\epsilon^\nu} \sum_{\lambda=\nu+1}^{\infty} a_{\nu\lambda} H^\lambda\right], \qquad (3)$$

$$Z_\Gamma = 1 + \sum_{\nu=1}^{\infty} \frac{1}{\epsilon^\nu} \sum_{\lambda=\nu}^{\infty} c_{\nu\lambda} H^\lambda. \qquad (4)$$

The coefficients $a_{\nu\lambda}$ and $c_{\nu\lambda}$ enter the expressions for GML function

$$\beta(H) = \sum_{\lambda=2}^{\infty} (\lambda - 1) a_{1\lambda} H^\lambda$$

and anomalous dimension of Γ_R

$$\gamma_\Gamma(H) = -\sum_{\lambda=1}^{\infty} \lambda c_{1\lambda} H^\lambda,$$

which are the parameters of RG equation

$$\left(\mu^2 \frac{\partial}{\partial\mu^2} + \beta(H)\frac{\partial}{\partial H} - \gamma(H)\right)\Gamma_R\left(\frac{k_i^2}{\mu^2}, H\right) = 0.$$

Let us now rewrite Z_Γ in terms of R′-operation defined by (1). Consider N-th order in H of eq. (2). From (4) it follows that $Z_\Gamma = 1 + \sum_{\lambda=1}^{\infty} H^\lambda Z_\lambda\left(\frac{1}{\epsilon}\right)$. All subtractions prescribed by R-operation may be regarded as an effect of adding the counterterms to the

Lagrangian. The singular in ϵ coefficients of these counterterms are constructed from $Z_\lambda(\frac{1}{\epsilon})$. The terms in the r.h.s. of eq. (2) which does not contain Z_λ correspond to unsubtracted diagrams; those which contain $Z_\lambda, \lambda < N$, correspond to the subtraction of subdiagrams, and the $Z_N H^N$ term corresponds to the subtraction of diagrams as a whole, i.e., to the operation $\Delta\Gamma$. We arrive at the relation

$$Z_\Gamma(H, \frac{1}{\epsilon}) = 1 - KR'\Gamma((\mu^2)^\epsilon\ H, k_i^2, \epsilon), \tag{5}$$

which holds for any renormalizable theory. Eq. (5) proves to be very useful for RG calculations in the 't Hooft scheme. It is shown [3,6] that $KR'G$ for any diagram G is a polynomial in external momenta (and in masses as well if they are present in the theory). This allows one to put momenta and masses equal to zero when computing the contributions to Z_Γ from the logarithmically divergent diagrams, where the above-mentioned polynomial is a constant.

3. We proceed now to the calculations at the three-loop level of ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \frac{h}{4!}\phi^4, \quad H \equiv \frac{h}{16\pi^2}.$$

To prevent the appearance of $\ln\pi$ and the Euler constant it is convenient to multiply the standard dimensionally regularized integrals [7] by $\pi^\epsilon\Gamma(1-\epsilon)$ that is equivalent to finite renormalization [8].

To simplify the computation of three-loop diagrams two different methods have been used. One of them consists in putting all external momenta equal to zero, while several internal lines are supplied by a mass to avoid the infrared trouble. The other method deals with all propagators massless, one of the external momenta being nonzero. As a rule three-loop diagrams can easily be calculated by both methods.

The following expression for GML function $\beta(H)$ has been obtained at the three-loop level:

$$\beta(H) = \frac{3}{2}H^2 - \frac{17}{6}H^3 + [\frac{145}{16} + 6\zeta(3)]H^4. \qquad (6)$$

The three-loop term of eq. (6) does not coincide with the result of previous calculations /9/. This is an example of the general property of GML function $\beta(H)$ namely of its dependence on the renormalization scheme used. However, this does not yet mean the absence of the physical sense of the third coefficient in the expansion (6). Indeed, the momentum dependence of the invariant charge $\bar{H}(\frac{k^2}{\mu^2}, H)$ is explicitly given not by the equation with GML function β

$$(k^2 \frac{\partial}{\partial k^2} - \beta(H)\frac{\partial}{\partial H})\bar{H}(\frac{k^2}{\mu^2}, H) = 0,$$

but by the other equation

$$k^2 \frac{\partial}{\partial k^2}\bar{H}(\frac{k^2}{\mu^2}, H) = f(\bar{H}(\frac{k^2}{\mu^2}, H))$$

with function f related to β as follows:

$$f(q(H)) = \beta(H)\frac{dq(H)}{dH}, \quad q(H) \equiv \bar{H}(1, H). \qquad (7)$$

The function f (as well as ultraviolet behaviour of invariant charge) does not depend on the choice of renormalization scheme.

An invariant charge $\bar{H}$ is treated here as a function of k^2, with other momentum arguments assumed to be proportional to it. Let k^2 denote the invariant momentum variables, and let $t = \alpha s$, $k_i^2 = \alpha_i s$ $(i=1,2,3,4)$. The coefficients α and α_i are hidden in the definition of $\bar{H}$. It follows from (5) that GML function $\beta(H)$ in the 't Hooft scheme does not depend on α and α_i while the function f(H) does. To find this dependence explicitly it is required to calculate q(H) at the two-loop level and use (6) and (7). The result is

$$f_\alpha(H) = \frac{3}{2}H^2 - \frac{17}{6}H^3 + H^4\{\frac{31}{2} + 6\zeta(3) + \frac{1}{16}\sum_{i=1}^{4} \ln\alpha_i -$$
$$- \frac{1}{12}\ln(\alpha\alpha_5) - \frac{3}{8}\ln^2(\alpha\,\alpha_5) + \frac{9}{8}(\ln^2\alpha + \ln^2\alpha_5) - \qquad (8)$$
$$-\frac{3}{2}[I(\alpha_1,\alpha_2) + I(\frac{\alpha_1}{\alpha}, \frac{\alpha_3}{\alpha}) + I(\frac{\alpha_1}{\alpha_5}, \frac{\alpha_4}{\alpha_5})]\},$$

where $\alpha_5 = \frac{u}{s} = \sum_{i=1}^{4} \alpha_i - \alpha - 1$,

$$I(a,b) = \int_0^1 dx \int_0^1 \frac{dy}{1-y} \ln[y + a(1-y) + b\frac{xy(1-y)}{1-x}].$$

For symmetric asymptotics $\alpha = 1, \alpha_i = \frac{3}{4}$ the known result /9/ is reproduced:

$$f_{symm}(H) = \frac{3}{2}H^2 - \frac{17}{6}H^3 + H^4(\frac{31}{2} + 6\zeta(3) + \frac{1}{4}\ln\frac{3}{4} - \frac{9}{2}I(\frac{3}{4}, \frac{3}{4})).$$

We see that f depends essentially on α and α_i, i.e., on the choice of asymptotical regime. Due to this dependence $f_\alpha(H)$ may be equal to zero at $H \neq 0$. For instance it takes place when $\alpha_i = \frac{3}{4}$, $10^{-16} < \alpha < 10^{-12}$. Whether this connection between the asymptotical regime and the existence of nontrivial zero remains in an exact theory, or it is an effect of the truncation of perturbation series at a finite order, this is an open question.

The α-dependence of the invariant charge forces us to put also another question: can it serve as a tool for investigating the ultraviolet properties of a given theory? Let us consider the RG equation for the propagator $D_R(\frac{k^2}{\mu^2}, H)$. It can be written in two various forms /1/

$$k^2 \frac{\partial}{\partial k^2} \ln D_R\left(\frac{k^2}{\mu^2}, H\right) = -\psi_D\left(\bar{H}\left(\frac{k^2}{\mu^2}, H\right)\right), \qquad (9)$$

$$k^2 \frac{\partial}{\partial k^2} \ln D_R\left(\frac{k^2}{\mu^2}, H\right) = -\bar{\psi}_D\left(\xi\left(\frac{k^2}{\mu^2}, H\right)\right), \qquad (10)$$

where

$$\psi_\Gamma(q(H)) = \bar{\psi}_\Gamma(H), \; q(\xi) = \bar{H},$$

$$k^2 \frac{\partial}{\partial k^2} \xi\left(\frac{k^2}{\mu^2}, H\right) = \beta\left(\xi\left(\frac{k^2}{\mu^2}, H\right)\right).$$

It is known /1/ that ψ_Γ does not depend on the renormalization scheme and depends on the asymptotical regime. Conversely, the function $\bar{\psi}_\Gamma$ does not depend on asymptotical regime and depends on the choice of scheme. Thus, in exact theory the l.h.s. of (9) and (10) given by $\psi_D(\bar{H})$ or $\bar{\psi}_D(\xi)$ in ultraviolet limit loses both these dependences. However, it is not so in any finite order of perturbation theory, where ψ_Γ and $\bar{\psi}_\Gamma$ are polynomials. For example, using an implicit α-dependence of $\bar{H}$ in eq. (9) one can change the ultraviolet behaviour of D_R drastically from "zero-charge" to the ultraviolet stable fixed point. Analogously in the case of eq. (10) in different renormalization schemes the propagator D_R may display absolutely different asymptotical behaviour. Consequently, as concerns the investigation of the ultraviolet properties of $D_R(\frac{k^2}{\mu^2}, H)$, the invariant charge $\bar{H}$ in spite of its independence of the renormalization scheme, has no advantages over an "effective charge" ξ. It is easy to observe that an analogous conclusion may be drawn in any other field of application of RG equations, when an invariant charge itself is not studied but is used as a tool for studying ultraviolet behaviour of other objects. The effective charge ξ may serve for these purposes equally well.

However, in the computational aspect the charge ξ possesses obvious advantages over the change $\bar{H}$. It is because we need only GML function $\beta(H)$ to find $\xi(\frac{k^2}{\mu^2}, H)$, and there are effective methods for computing $\beta(H)$ in the 't Hooft scheme. It will be shown that similar methods can be applied also in some other renormalization schemes.

4. We consider now one of these schemes that appears to be extremely suitable for RG calculations in the massless ϕ^4 theory. This scheme is based on the ultraviolet cutoff of momentum integrals at some Λ after the Wick rotation is done. This scheme (Λ^2 -scheme, for brevity) is a correct regularization procedure /10/, its $\Lambda^2 \to \infty$ limit being invariant under RG transformations, where Λ^2 plays the role of the renormalization parameter. In the case of massless scalar theory the following valuable features of Λ^2-scheme may be used: 1) The Chebyshev polynomial techniques /11/ is applicable, because the momentum integrations are performed in four space-time dimensions. 2) The renormalization parameter Λ^2 of this scheme is introduced quite independently of the asymptotical regime. Therefore, the RG functions $\beta(H)$ and $\gamma(H)$, as in 't Hooft's scheme, don't depend on this regime /1/. 3) In Λ^2 -scheme, one can calculate the logarithmically divergent contributions to Z_Γ with all external momenta and all internal masses equal to zero.

To prove the last statement let us consider the RG formalism of Λ^2 -scheme in more detail. The renormalized Green functions of this scheme which satisfy RG equations are constructed as follows: the momentum integrals are cutted off at Λ, the quadratic divergences proportional to Λ^2 are subtracted (to retain $m^2=0$ in the renormalized theory), and the asymptotical limit $\Lambda^2 \to \infty$ is taken, i.e., all terms of the type $(\ln \Lambda^2)^M (\Lambda^2)^{-N}$, $N>0$, are dropped out. After that the Green functions appear to be logarithmically dependent on Λ^2. In brief notation

$$\Gamma(\ell n\frac{\Lambda^2}{k_i^2},H)=\Gamma(\ell n\frac{\Lambda^2}{M^2}-\ell n\frac{k_i^2}{M^2},H)\equiv\Gamma(L-k,H),$$

where M^2 fixes the momentum scale. To arrive at the RG equation for $\Gamma(L-k,H)$ let us consider the R-operation removing the L-dependence from $\Gamma(L-k,H)$. The recursive subtraction procedure acts analogously to the 't Hooft scheme (1). The operator K now picks up all "singular" in Λ^2 terms (i.e., all the terms which contain L).

$$R\Gamma(L-k,H)=(1-K)R'\Gamma(L-k,H)=\Gamma(-k,H).$$

It may be rewritten in the language of renormalization constants,

$$Z_\Gamma(L,H_0)\Gamma(L-k,H(L,H_0))=\Gamma(-k,H_0), \tag{11}$$

where $Z_\Gamma(L,H_0)=1-KR'\Gamma(L-k,H_0)$,

$$H(L,H_0)=H_0Z_1(L,H_0)Z_D^{-2}(L,H_0).$$

The RG equation for $\Gamma(L-k,H)$ can be obtained by differentiating eq. (11) with respect to L. We have

$$(\frac{\partial}{\partial L}+\beta(H)\frac{\partial}{\partial H}-\gamma_\Gamma(H))\Gamma(L-k,H)=0$$

with

$$\beta(H)=\frac{\partial H(L,H_0)}{\partial L}\Big|_{H_0(L,H)=\text{const}}, \tag{12}$$

$$\gamma_\Gamma(H)=-\frac{\partial\,\ell n Z_\Gamma(L,H_0)}{\partial L}\Big|_{H_0(L,H)=\text{const}}.$$

In r.h.s. of eqs. (12) one can put L=0. Given any diagram G from the definition of K it follows that $KG(L-k,H)|_{L=0}=0$. Hence $Z_\Gamma(0,H)=1$, $H_0(0,H)=H$ and

$$\gamma_\Gamma(H) = -\frac{\partial}{\partial L}\ln Z_\Gamma(L,H))|_{L=0} = -\frac{\partial Z_\Gamma(L,H)}{\partial L}|_{L=0} ,$$

$$\beta(H) = \frac{\partial}{\partial L}H(L,H)|_{L=0} =$$

$$= H[\frac{\partial}{\partial L}Z_1(L,H) + 2\frac{\partial}{\partial L}Z_D^{-1}(L,H)]_{L=0} = -H[\gamma_1(H) + 2\gamma_D(H)] .$$

Using (5) we get

$$\gamma_\Gamma(H) = \frac{\partial}{\partial L}KR'\Gamma(L-k,H)|_{L=0} . \tag{13}$$

The quantity $KR'G$ contributes immediately to the counterterms and must be a polynomial in external momenta k_i of a diagram G. Thus, formula (13) gives the recipe of computing $\gamma_\Gamma(H)$ in the momentum independent way. However, some further simplifications can be achieved.

The condition $KG|_{L=0} = 0$ shows that there are no contributions to $\gamma_\Gamma(H)$ from the factorized diagrams as well as from the products $\Delta(G_i)\Delta(G_j)$ in eq. (1) when both G_i and G_j are one-particle irreducible. Furthermore, the symbol K may be omitted under the sign of $\frac{\partial}{\partial L}$. It results in

$$\frac{\partial}{\partial L}R'G|_{L=0} = [\frac{\partial G}{\partial L} - \sum_m (G/G_m)(\frac{\partial}{\partial L}R'G_m)_{L=0}]_{L=0} , \tag{14}$$

where the sum is over all one-particle irreducible subdiagrams of G, and G/G_m is a result of reducing G_m into a point.

It will be proved inductively that the condition $L=0$ may be dropped out as well. Let us denote $\frac{\partial G}{\partial L}$ by ∂G and $[\ldots]$ in (14) by DG. At the one-loop level $DG=\partial G$ is a function of $L-k$, but $DG|_{L=0}$ does not depend on k (because $KR'G$ does not. It is valid for the propagator diagrams too if we always treat them as multiplied by the free propagator $\frac{1}{k^2}$). Therefore, at this level DG does not depend on L too. Let this be valid for N loops and let G be a (N+1)-loop diagram. In eq. (14) DG_m does not depend on L and k, while ∂G and G/G_m and, consequently, DG are the functions of L−k. But $DG|_{L=0}$ is again independent of k and consequently, DG does not depend on L and k. Thus, the contribution of the diagram G to the anomalous dimension $\gamma_\Gamma(H)$ depends on H only and is given by the recursive relation

$$DG = \partial G - \sum_m (G/G_m) DG_m ; \quad \gamma_{\Gamma'}(H) = D\Gamma . \qquad (15)$$

It should be noted that the proof of eq. (15) remains valid for any scheme with regularization parameter being simultaneously the renormalization parameter (for example the Feynman cutoff or Pauli-Villars methods) and for the 't Hooft scheme.

Formula (15) allows one to put all external momenta equal to zero. To prevent infrared divergences one has to cut off the momentum integrals from below at some κ. Using the method proposed in ref. /6/ it is easy to show that KR′G and DG are independent of κ^2. Really, the differentiation with respect to the lower limit of integration κ^2 removes the "singularities" in Λ^2 from R′G because all subdiagrams are subtracted, and so an integrand does not include L.

With the help of eq. (15) it proved to be possible to calculate the four-loop RG functions of ϕ^4 theory in an analytical form. The logarithmically divergent diagrams have been treated with all external momenta being zero. The lower cutoff κ^2 has been introduced in case of need, into one or more momentum integrations. The propagator diagrams have been calculated with its external momentum $k^2 \neq 0$. The following expression for $\beta(H)$ has been obtained:

$$\beta(H) = \frac{3}{2} H^2 - \frac{17}{6} H^3 + [\frac{109}{8} + 6\zeta(3)] H^4 -$$

$$-[60\zeta(5) + 18\zeta(4) + \frac{69}{2}\zeta(3) + \frac{1115}{12}] H^5 .$$

With the use of conversion formulas of ref. [1] this expression has been compared with the result of calculations of ref. [12], where the renormalization scheme based on the subtractions at the symmetrical point was applied. This comparison has confirmed the correctness of both results.

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REFERENCES

1. Vladimirov A.A. TMF, 1975, 25, p.335.
2. 't Hooft G. Nucl. Phys., 1973, B61, p.455.
3. Speer E.R. J.Math.Phys., 1974, 15, p.1.
4. Collins J.C. Nucl.Phys., 1975, B92, p.477.
5. Bogoliubov N.N., Shirkov D.V. Introduction to the Theory of Quantized Fields, 1976.
6. Collins J.C. Nucl.Phys., 1974, B80, p.341.
7. Collins J.C., Macfarlane A.J. Phys.Rev., 1974, D10, p.1201.
8. Kang J.S. Phys.Rev., 1976, D13, p.851.
9. Belokurov V.V. et al. TMF, 1974, 19, p.149.
10. Akhiezer A.I., Berestetsky V.B. Quantum Electrodynamics, 1969.
11. Rosner J.L. Ann.Phys., 1967, 44, p.11.
12. Dittes F., Kubyshin Yu.A., Tarasov O.V. JINR, E2-11100, Dubna, 1977.

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