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B.S.Getmanov

INTEGRABLE MODEL FOR THE NONLINEAR COMPLEX SCALAR FIELD WITH THE NONTRIVIAL ASYMPTOTICS OF N-SOLITON SOLUTIONS

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Гетманов Б.С. **E2 - 11093**

Интегрируемая модель нелинейного комплексного скалярного поля с нетривиальной асимптотикой солитонных решений

Рассматривается модель поля в двумерном пространстве-времени с лагранжианом

$$\mathcal{L} = \frac{|\partial_\mu \psi|^2}{1 - \lambda^2 |\psi|^2} + m^2 (|\psi|^2 - \lambda^{-2}),$$

которая оказывается вполне интегрируемой гамильтоновой системой. Найден явный вид N-солитонных решений с асимптотикой $|\psi| \to \lambda^{-1}$ при $|x| \to \infty$.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Getmanov B.S. **E2 - 11093**

Integrable Model for the Nonlinear Complex Scalar Field with the Nontrivial Asymptotics of N-Soliton Solutions

Field theory in two space-time dimensions described by the Lagrangian

$$\mathcal{L} = \frac{|\partial_\mu \psi|^2}{1 - \lambda^2 |\psi|^2} + m^2 (|\psi|^2 - \lambda^{-2})$$

is considered. It turns out to be complete integrable Hamiltonian system. The N-soliton solutions subject to the boundary conditions $|\psi| \to \lambda^{-1}$ when $|x| \to \infty$ have been found.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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1. INTRODUCTION

Last years a considerable progress has been achieved in investigation of the nonlinear equations, which possess the localized solutions-solitons [1-3]. In field theory the great interest to such equations is stimulated by the hopes of building of the realistic models of the extended particles on the basis of solitons. In two space-time dimensions in the manifold of nonlinear equations with soliton solutions there is an important class of the integrable systems [2-3] with the unique properties. They are exactly soluble, possess the infinite set of the integrals of motion; the soliton interaction dynamics is beyond the exhaustive description which manifested in the existence of the explicit N-soliton solutions; most of them are the complete integrable systems. The significance of finding and the investigation of integrable systems (in particular, they are of great interest from the pure mathematical point of view) consists also in the possibility of describing a soliton interaction in the "proximate" nonintegrable systems [4]. Lately the only Lorentz-invariant integrable system - the sine-Gordon equation - was known. It seems of great importance the problem of finding some new Lorentz-invariant integrable systems, and due to some considerations [5] it is almost evident that such systems may be the multicomponent field systems only. Recently the essential progress was obtained in this direction. The massive Thirring model has been solved [6] and shown to be complete integrable system using the inverse scattering method (ISM). In ref. [7] the author proposed the model of the complex scalar field described by the Lagrangian

$$\mathcal{L} = \frac{|\partial_\mu \psi|^2}{1 - \lambda^2 |\psi|^2} - m^2 |\psi|^2 \qquad (|\partial_\mu \psi|^2 = |\psi_t|^2 - |\psi_x|^2). \quad (1.1)$$

The equation of motion reads

$$\partial_\mu^2 \psi + \psi^* \frac{(\partial_\mu \psi)^2}{1 - \lambda^2 |\psi|^2} + m^2 (1 - \lambda^2 |\psi|^2) = 0 , \qquad (1.2)$$

the exact multisoliton solutions of (1.2) were found in ref. [7] by Hirota's method and it was conjectured for (1.2) to be the complete integrable system. After publishing ref. [7] we have known about the Pohlmyer work [9]* in which integrability of the class of O_n - invariant relativistic models for the nonlinear field described by Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \vec{q})^2 + \lambda(\vec{q})(\vec{q}^{\,2} - 1) \qquad (1.3)$$

was proved; in the simplest nontrivial case $n = 3$ the system corresponding to (1.3) reduces to the sine-Gordon equation; and for $n = 4$ - ($O_n \cong SU(2) \times SU(2)$ - a nonlinear σ-model) - to the system with the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \operatorname{tg}^2 \frac{\phi}{2} (\partial_\mu \beta)^2 - 2 \sin^2 \frac{\phi}{2} \qquad (1.4)$$

for the fields ϕ, β being the nontrivial generalization of the sine-Gordon equation.

The system of equations

$$\begin{aligned} &\partial_\mu^2 \phi + \operatorname{tg}^2 \frac{\phi}{2} (\partial_\mu \beta)^2 / \sin\phi + \sin\phi = 0 , \\ &\partial_\mu^2 \beta + (\partial_\mu \beta)^2 / \sin\phi = 0 , \end{aligned} \qquad (1.5)$$

following from (1.4) was obtained also by Lund and Regge [10] by analyzing the motion of vortices in

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* The author thanks P.P.Kulish who referred him to papers [9,10].

superfluid. In subsequent papers [11-12] Lund proposed the elegant geometric interpretation of Polmyer's results for σ-model *. He showed that the Gauss-Weingarten equations of surface theory for constructing the normal and tangent vectors to a surface embedded in the three-dimensional Euclidean space can be written in the form of the system of linear equations

$$\begin{aligned} \partial_x w &= i\vec{\sigma}\vec{\omega}_1 w \\ \partial_t w &= i\vec{\sigma}\vec{\omega}_1 w, \qquad w = \binom{w_1}{w_2}, \end{aligned} \tag{1.6}$$

(here σ_i - the Pauli matrices, $\vec{\omega}_1$, $\vec{\omega}_2$ depend on parameter, functions $\phi(x)$, $\beta(x)$ and their derivatives, see eq. (9-10) in ref. [11]); the (Gauss-Codazzi) integrability conditions for (1.6) are precisely the equations (1.5).

Thus (1.5) may be solved by the ISM. Its using consists of considering of the linear problem (1.6). Recently Lund solved [12] the initial value problem for (1.5) and proved its complete integrability.

It is an easy task to ckeck up, that the substitution (in dimensionless variables) $\psi = \sin\phi/2 \; e^{i\beta/2}$ transfers (1.1) into (1.4) and, therefore, model (1.1) turns out to be the integrable system. The corresponding linear problem is (1.6), where $\phi = 2 \arcsin|\psi|$, $\beta = -2i \ln(\psi/|\psi|)$.

The aim of this paper consists of finding N-soliton solutions of (1.1) with the inverse sign of m^2. This model is integrable too, because for $m^2<0$ (1.1) transforms to (1.3) by substitution $\psi = \cos\phi/2 \; e^{i\beta/2}$, but the structure of solutions appears a quite different. It turns out that this difference relates to arbitrariness of choice of the boundary conditions for function β in (1.5). In particular, using of the boundary conditions $\phi \to 0$, $\beta_x \to A''\phi^2$, $\beta_t \to A'\phi^2$, when $|x| \to \infty$ for solving the linear problem (1.6) in

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* Similar results have been obtained by P.P. Kulish [13].

ref.[12] corresponds to $m^2<0$ in (1.1). The "physical meaning" of the field β is not quite clear, and to our mind, the attractiveness of the formulation (1.1) consists of the larger clearness and closeness to the traditional field theory, and of the simplicity in the structure of N-soliton solutions (the last ones are not written in ref.[12]).

Sec. 2 is devoted to finding and discussing 1-soliton solution of (1.2) for $m^2<0$. In Sec. 3 we found by means of Hirota's method 2-soliton solution that is generalized for the N-soliton case. We look for the solution in form $\psi=g/f$; the knowledge of the explicit form of the 1-soliton solution allows one to find and solve the system of Hirota's equations for g and f.

Some comments about Hirota's method will be useful here. The ISM provides, of course, the more complete information, in particular, we can solve the initial value problem and find explicitly the infinite set of integrals of motion. But the regular methods of finding the corresponding linear problem for the given nonlinear equation are unknown now, in this sense Hirota's method is more "direct". In particular, in some cases (for equation $\partial_\mu^2\psi+m^2\psi-\lambda\,\psi|\psi|^2=0$ for example) we are able immediately to show the absence of the soliton solutions for $N\geq 2$ and, therefore, to "eliminate suspicion" of the integrability of the model. It seems that the region of the applicability of the Hirota's method is more wider than that of ISM: there are known equations[14,15]* in two close but not transformed from one to another forms, which are solvable by Hirota's method but only one of them is solvable by the ISM.

In conclusion, to our mind, the Hirota's method may turn out to be perspective one for finding "integrable" models with localized solutions in more than one space dimensions and to shed the light

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* The author thanks V.E.Zakharov, who referred him to paper[14].

on the problem of soliton interaction in the real space-time.

2. THE 1-SOLITON SOLUTION

As was pointed out, the knowledge of the 1-soliton solution is desirable to use Hirota's method. Here we find it by the Bogomolny's method which is applicable in the cases of soliton solutions with the nontrivial boundary conditions.

The starting equation reads

$$\partial_\mu^2\psi + \psi^* \frac{(\partial_\mu\psi)^2}{1-\lambda^2|\psi|^2} - m^2\psi(1-\lambda^2|\psi|^2) = 0, \quad m^2>0. \qquad (2.1)$$

Look for the solutions to (2.1) subject to boundary conditions $|\psi|\to\lambda^{-1}$, $\psi_x\to 0$ when $|x|\to\infty$.

In the following we shall use the dimensionless variables excluding constants m, λ by the transformation $\lambda\psi\to\psi$, $mx_\mu\to x_\mu$. In such variables the energy functional for a static * solution is

$$H = \frac{m^2}{\lambda^2}\int_{-\infty}^{\infty} dx\left[\frac{|\psi_x|^2}{1-|\psi|^2} + 1 - |\psi|^2\right]. \qquad (2.2)$$

Let us rewrite this expression identically in the following form

$$H = \frac{m^2}{\lambda^2}\left\{\int_{-\infty}^{\infty}\left|\frac{\psi_x}{\sqrt{1-|\psi|^2}} - \sqrt{\overline{1-|\psi|^2}}\right|^2 + \int_{-\infty}^{\infty} dx(\psi_x + \psi_x^*)\right\}. \qquad (2.3)$$

The second term is $I_2 = 2\frac{m^2}{\lambda^2}(D_2 - D_1)$, where $D_1 = \psi'|_{x=-\infty}$, $D_2 = \psi'|_{x=\infty}$, ψ' and ψ'' are the real and imaginary parts of ψ, respectively. Without loss of generality let $D_2 > D_1$. The minimal value of H is evidently to be achieved when ψ obeys the equation

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* It is possible to show that the substitution in (2.1) $\psi = e^{i\omega t}\phi(x)$. , where $\phi(x)$ is a localized function, leads to $\omega = 0$.

$$\psi_x - 1 - |\psi|^2 = 0. \tag{2.4}$$

From (2.4) we get $\psi'' = c$ (const), then from the equation for ψ': $\psi'_x = 1 - c^2 - \psi'^2$ and the boundary condition $\psi'_x|_{x=\pm\infty} = 0$ we get

$$\psi'^2|_{x=\pm\infty} = 1 - c^2, \quad D_2 = -D_1 = \sqrt{1 - c^2} = A.$$

By integrating the equation for ψ' one obtains

$$\psi = \psi' + i\psi'' = A\ \mathrm{th}Ax - i\sqrt{1 - A^2}, \quad |A| \le 1. \tag{2.5a}$$

The general solution of (2.1) contains the arbitrary constant phase multiplyer $e^{i\nu}$; boosting (2.5a) we get, finally, the solution that depends on four arbitrary parameters A, v, x_0, ν:

$$\begin{aligned}\psi &= A\cos\nu\,\mathrm{th}[A\gamma(x - x_0 - vt)] + \sqrt{1 - A^2}\sin\nu + \\ &+ i[A\sin\nu\,\mathrm{th}[A\gamma(x - x_0 - vt)] - \sqrt{1 - A^2}\cos\nu]; \quad \gamma = (1 - v^2)^{-1/2}.\end{aligned} \tag{2.5b}$$

By using the formalism of the complex Lorentz-vectors proposed in ref. /7/ the solution (2.5b) may be written in the more compact and convenient for the following purposes form

$$\psi = e^{i\bar{\nu}}\,\frac{1 + e^{z + z^* + i\alpha}}{1 + e^{z + z^*}}. \tag{2.6}$$

Here

$$\begin{aligned}\bar{\nu} &= \nu - \frac{\alpha}{2} + \frac{\pi}{2}, \\ z &= z' + iz'' = k_\mu (x - x^{(0)})^\mu -\end{aligned} \tag{2.7}$$

- Lorentz-invariant complex variable, k_μ - complex space-like vector in the two-dimensional pseudoeuclidean space-time:

$$k_\mu^2 = k_0^2 - k_1^2 = -1; \tag{2.8}$$

$x_\mu^{(0)}$ is an arbitrary constant vector. $x_\mu^{(0)} = 0$ fixes the soliton location in the centre of frame of reference of the pseudoeuclinean space-time.

The complex vector k_μ depends on four parameters from which only two are independent in spite of (2.8). The following parametrization of k_μ is convenient

$$k_0 = \mathrm{sh}\beta; \quad k = \mathrm{ch}\beta; \quad \beta = \beta' + i\beta''. \tag{2.9}$$

Then in (2.6), (2.5) $a = 2\beta'' + \pi$, $\cos a = (k_\mu k^{\mu^*})$, $A = \cos\beta''$, $\gamma = \mathrm{ch}\beta'$. Evidently in the real limit $\beta'' = 0$ the solution (2.6) becomes (up to the precision of the constant phase multiplyer) $\psi = \mathrm{th}\, z' = \cos\phi/2$, where $\phi = 4\,\mathrm{arctg}\, e^{z'}$ is the solution of the sine-Gordon equation.

The first integrals of motion, - the energy, the momentum, the charge

$$H = \frac{m^2}{\lambda^2}\int_{-\infty}^{\infty} dx \left[\frac{|\psi_t|^2 + |\psi_x|^2}{1 - |\psi|^2} + 1 - |\psi|^2 \right];$$

$$P = \frac{m^2}{\lambda^2}\int_{-\infty}^{\infty} dx \frac{\psi_t \psi_x^* + \psi_t^* \psi_x}{1 - |\psi|^2}; \quad Q = i\frac{m}{\lambda^2}\int_{-\infty}^{\infty} dx \frac{\psi^* \psi_t - \psi \psi_t^*}{1 - |\psi|^2}$$

- for the solution (2.9) are equal, respectively,

$$E = \gamma M, \quad P = \gamma v M, \quad M = 4Am\lambda^{-2}, \quad Q = 0. \tag{2.10}$$

The charge is equal to zero because in contrast to the solution of equation (1.2)/7/

$$\psi = A\, \mathrm{sech}\, z' e^{iz''} \tag{2.11}$$

the solution (2.5) in the rest frame is time-independent. Instead of this we have nonzero topological charge I_0; it is convenient to define I_0 by the expression

$$I_0 = \left| \int_{-\infty}^{\infty} J_0 \, dx \right| = 2A$$
$$J_\mu = \epsilon_{\mu\lambda} \partial^\lambda \psi . \qquad (2.12)$$

It is useful to draw an analogy of equations (1.2), (2.1) with the nonlinear Schrödinger equation /8,17,18/

$$i\psi_t + \psi_{xx} + \kappa^2 \psi |\psi|^2 = 0 \qquad (2.13)$$

for various signs of κ^2. For $\kappa^2 > 0$ the functional form of solutions of (2.13) and (1.2) coincides (the difference is only in the form of the dispersion equations). But for $\kappa^2 < 0$ the analogy (2.13) and (1.2) is broken: the solution of (2.13) has a time-dependent phase multiplyer (in paper /18/ in formulae (3), (4) this multiplyer is absent. Such a form of the solution corresponds to the presence of a term $\mu^2 \psi$ in (2.13)).

Just the same situation is for the analogy to the nonlinear Klein-Gordon equation: for the equation

$$\partial_\mu^2 \psi + \psi - 2\psi |\psi|^2 = 0 \qquad (2.14)$$

the solution is of the form (2.11), and the interaction of the solitons of (2.14) is determined by the "proximity" to (1.2); the localized solution of the complex Higgs equation

$$\partial_\mu^2 \psi - 2\psi (1 - |\psi|^2) = 0$$

reads

$$\psi = A \,\mathrm{th}\, Ax \, e^{i\sqrt{2(A^2 - 1)}t} .$$

In spite of this the divergent term depending on the arbitrary constant A appears in the Hamiltonian and, therefore, in contrast to the above-discussing equations we can't eliminate the divergence in the Hamiltonian by the subtraction procedure for all the solutions simultaneously.

3. N-SOLITON SOLUTIONS

Let's look for solutions of (2.1) in the form $\psi = g/f$ where f is a real function. In the following it is convenient to use the Lorentz-invariant D-operators generalizing ones proposed by Hirota /8/:

$$D_\mu^2 g(x)\cdot f(x) = (\partial_\mu - \partial_\mu')^2 g(x) f(x')|_{x=x'} = \tag{3.1}$$
$$= f\partial_\mu^2 g - 2\partial_\mu g \partial_\mu f + g\partial_\mu^2 f.$$

The basic important property of D-operators for calculations consists of the easily checked identity

$$D_\mu^2 e^{z_i} \cdot e^{z_j} = (k_i - k_j)_\mu^2 e^{z_i + z_j}, \tag{3.2}$$

where $z_i = k_i^\mu x_\mu$ is given by (2.7)

Now (2.1) reads

$$[fD_\mu^2 g\cdot f - gD_\mu^2 f\cdot f] + \frac{g^*}{f^2 - |g|^2}[gfD_\mu^2 g\cdot f - \tag{3.3}$$
$$- \frac{1}{2} g^2 D_\mu^2 f\cdot f - \frac{1}{2} f^2 D_\mu^2 g\cdot g] - g(f^2 - |g|^2) = 0 .$$

By using the known 1-soliton (2.6) we can decompose (3.3) on the system of two polylinear on g and f equations

$$(D_\mu^2 + 2) f\cdot f - 2|g|^2 = 0$$
$$f(D_\mu^2 + 1) g\cdot f - \frac{1}{2} g^* (D_\mu^2 + 2) g\cdot g = 0 , \tag{3.4}$$

so that the function (2.6) $(g = 1 + e^{z + z^* + i\alpha}, f = 1 + e^{z + z^*})$ satisfies (3.4).

We should stress, that in contrast to all the known up to now cases of Hirota's method using, similarly to ref. /7/, one of the equations (3.4) appears to be <u>trilinear</u> one on g and f. This fact appreciably complicates the calculations; in particular, we have not been able to get the rigorous proof of the general N-soliton solution. We point out also, that

(3.4a) may be written as

$$|\psi|^2 = 1 + \partial_\mu^2 \ln f. \tag{3.5}$$

Let us look for g and f as a power series on a parameter ϵ

$$\begin{aligned} g &= 1 + \epsilon^2 g_2 + \epsilon^4 g_4 + \dots , \\ f &= 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \dots . \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.4) and collecting terms with the same power of ϵ, we obtain the overdetermined system

$$\epsilon^2: \ (D_\mu^2 + 2)1 \cdot f_2 - g_2 - g_2^* = 0, \tag{3.6.1}$$

$$\epsilon^4: \ (D_\mu^2 + 2)(2 \cdot f_4 + f_2 \cdot f_2) - 2(g_4 + g_4^* + |g_2|^2) = 0, \tag{3.6.2}$$

$$\epsilon^6: \ (D_\mu^2 + 2)(f_2 \cdot f_4) - g_2^* g_4 - g_2 g_4^* = 0, \tag{3.6.3}$$

$$\epsilon^8: \ (D_\mu^2 + 2) f_4 \cdot f_4 - 2|g_4|^2 = 0, \tag{3.6.4}$$

$$\epsilon^2: \ (D_\mu^2 + 2) f \cdot 1 - g_2 - g_2^* = 0, \tag{3.7.1}$$

$$\begin{aligned} \epsilon^4: \ & (D_\mu^2 + 1)(1 \cdot f_4 + g_2 \cdot f_2 + 1 \cdot g_4) + \\ & + f_2 (D_\mu^2 + 1)(1 \cdot f_2 + 1 \cdot g_2) + \\ & + f_4 - \frac{1}{2}(D_\mu^2 + 2)(2 \cdot g_4 + g_2 \cdot g_2) - g_2^* (D_\mu^2 + 2) 1 \cdot g_2 - g_4^* = 0, \end{aligned} \tag{3.7.2}$$

. . . . ,

$$\epsilon^{12}: \ f_4 (D_\mu^2 + 1) g_4 \cdot f_4 - \frac{1}{2} g_4^* D_\mu^2 g_4 \cdot g_4 = 0. \tag{3.7.6}$$

Choosing $f_2 = e^{z + z^*}$ we get from (3.6.1) $g_2 = e^{z + z^* + i\alpha}$, $\cos\alpha = k^\mu k_\mu^*$, , and the rest of the equations turn out to be the identities for $g_4 = f_4 = 0$.

To find 2-soliton solution choose f_2 in the form

$$f_2 = \sum_{i,j=1}^{2} a(i,j^*)\, e^{z_i + z_j^*} . \tag{3.8}$$

The reality condition for f which dictates the most general form of f_2 (3.8) leads to the conditions on coefficients $a(i,j^*)$: $a^*(i,j^*) = a(j,i^*)$, following $a^*(i,i^*) = a(i,i^*)$. The substitution (3.8) into (3.6.1) gives

$$\begin{aligned} g_2 = {} & b(1,1^*)e^{z_1 + z_1^* + i\alpha_1} + b(1,2^*)e^{z_1 + z_2^* + i(\alpha_1 + \alpha_2)/2} + \\ & + b(2,1^*)e^{z_2 + z_1^* + i(\alpha_1 + \alpha_2)/2} + b(2,2^*)e^{z_2 + z_2^* + i\alpha_2} . \end{aligned} \tag{3.9}$$

Here

$$\cos\alpha_i = k_i^{\mu} k_{i\mu}^*, \; (\alpha_i = 2\beta_i'' + \pi), \; b(i,j^*) = a(i,j^*)\exp(\beta_i' - \beta_j'). \tag{3.10}$$

Now from (3.6.2–3.6.7) we can get f_4, g_4:

$$\begin{aligned} & f_4 = a(1,1^*,2,2^*)\, e^{z_1 + z_2 + z_1^* + z_2^*} ; \\ & g = a(1,1^*,2,2^*) e^{z_1 + z_2 + z_1^* + z_2^* + i(\alpha_1 + \alpha_2)} ; \\ & a(i,j,k,n) = a(i,j)\, a(i,k)\, a(i,n)\, a(j,k)\ \ a(j,n)\, a(k,n) ; \\ & a(i,j^*) = -[(k_i + k_j^*)^2_{\mu}]^{-1} ; \; a(j^*,i) = a^*(j,i^*) ; \\ & a(i,j) = -(k_i - k_j)^2_{\mu} ; \; a(i^*,j^*) = a^*(i,j) ; \end{aligned} \tag{3.11}$$

the rest of the equations turn out to be the identities.

Finally, we have the following compact form of 2-soliton solution

$$\psi = e^{i\nu} \frac{1+\sum_{i,j=1}^{2} a(i,j^*)e^{\bar{z}_i+\bar{z}_j^*} + a(1,1^*,2,2^*)e^{\bar{z}_1+\bar{z}_2+\bar{z}_1^*+\bar{z}_2^*}}{1+\sum_{i,j=1}^{2} a(i,j^*)e^{z_i+z_j^*} + a(1,1^*,2,2^*)e^{z_1+z_2+z_1^*+z_2^*}}, \quad (3.12)$$

where

$$\bar{z}_i = z_i + \beta_i + i\frac{\pi}{2}; \quad \bar{z}_i^* = z_i^* - \beta_i^* + i\frac{\pi}{2} \quad (\bar{z}_i^* \neq (\bar{z}_i)^*!); \qquad (3.13)$$
$$z_i \neq z_j \quad \text{for} \quad i \neq j \; ;$$

z_i, $a(i,j,\ldots,k)$ are defined by (2.7-2.9, 3.11), the arbitrary constant ϵ is given to be equal to unity. The form (3.12) is manifestly Lorentz-invariant because of the Lorentz-invariance of the rapidity difference

$$\operatorname{ch}(\beta_i - \beta_j^*) = -k_i^{\mu} k_{j\mu}^* . \qquad (3.14)$$

In the limit $t \to \pm\infty$ the solution (3.12) transfers to the direct sum of two 1-soliton solutions with easily calculable phase shifts.

The generalization (3.12) for $N \geq 3$ is evident enough; we can write the N-soliton solution in the following compact form

$$\psi = e^{i\nu} \frac{\det\|I + \bar{M}\bar{M}^T\|}{\det\|I + MM^T\|}, \quad (M^T)_{ij} = M_{ji}, \qquad (3.15)$$

$$M_{ij} = \frac{1}{2}\operatorname{sech}\frac{\beta_i - \beta_j^*}{2} e^{\frac{z_i + z_j^*}{2}}; \quad \bar{M}_{ij} = \frac{1}{2}\operatorname{sech}\frac{\beta_i - \beta_j^*}{2} e^{\frac{\bar{z}_i + \bar{z}_j^*}{2}};$$

$$i,j = 1,2,\ldots N$$

z_i, $\bar{z}_i$ are defined by (3.13). But we have not been able to prove rigorously (3.15) by means of the standard Hirota techniques because of the trilinear form of the equation (3.4.2).

It is clear from the foregoing the possibility of checking "suspicious on integrability" systems: almost in all the cases we are able to write the system of (3.4) type using 1-soliton solution (to find the last one is not usually too difficult problem). Then for nonintegrable equations the system (3.6-3.7) turns out to be inconsistent (of course, this is not the rigorous proof of nonintegrability). The emergence of a lot of nontrivial identities in integrable systems seems to be the intrigue fact far from comprehension for the present.

It seems to be the interesting problem of generalizing systems (1.2), (2.1) for the case of multicomponent (≥ 3) fields. This problem is now under investigation.

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