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A GEOMETRIC APPROACH TO THE SOLUTION OF CONFORMAL INVARIANT FIELD EQUATIONS

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Геометрический подход к решению конформно-инвариантных уравнений поля

Решения нелинейного конформно-инвариантного волнового уравнения классифицированы по их группам инвариантности с использованием классификации Билялова-Петрова конформно-плоских пространств по их группам изометрий. При этом найден ряд неизвестных ранее решений нелинейного волнового уравнения $\Box\phi + \lambda\phi^3 = 0$. Обсуждается значение этих результатов для решения уравнений Янга-Миллса для группы SU(2) (или SU(2) ⊗ SU(2)). Для полноты изложения приводятся основные сведения о двух понятиях конформной симметрии в произвольных псевдоримановых многообразиях.

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Tagirov E.A., Todorov I.T. **E2 - 11079**

A Geometric Approach to the Solution of Conformal Invariant Field Equations

Solutions of nonlinear conformal invariant scalar wave equation $\Box\phi + \lambda\phi^3 = 0$ are classified according to their invariance groups following the Bilyalov-Petrov classification of conformally flat spaces according to their isometry groups. A number of previously unknown solutions of this equation are found in this way. The relevance of these results to the solution of the SU(2) (or SU(2) ⊗ SU(2)) Yang-Mills equations is pointed out. Background material on the two notions of conformal symmetry for general pseudo-Riemannian manifolds is reviewed in order to make the exposition reasonably self-contained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

Some of the four-dimensional classical field equations with lump-type solutions* , considered recently (see, e.g.,/A1/,/B2-5,9/,/C2,3/,/D1,2,5/,/F1/,/H1/,/J1-3/,/P2/,/R1/,/S1/,/W1,2/,/Y1/)** contain no dimensional parameters and are in fact conformal invariant. The lump-type solutions, having a non-trivial x-dependence, are clearly not translation invariant. They are, therefore, rather unconventional candidates for describing the ground state in a world with spontaneously broken conformal symmetry /F1,D2/. It has been argued (see/D4,E2,G1/) that a more appropriate framework for describing such a type of symmetry breaking is provided by the general theory of relativity.

There is a more fundamental reason for going beyond Minkowski space in the study of conformal invariant equations. The class of conformally flat (pseudo) Riemannian spaces provides a natural

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*We use here the words "lump-type solutions" in a loose sense including singular solutions like the Minkowski space counterparts of "instantons". The term "classical lump" was introduced by S.Coleman /C7/.

** Let us note that the papers by Petiau /P2/ and Castell /C2/ actually precede the present fashion; the latter paper was apparently rediscovered in the midst of the new development by the authors of ref. /D1/.

framework for such a study since these spaces possess isomorphic (local) conformal groups and give room to closely related conformal invariant equations, whose solutions are obtained from one another by a simple transformation law. Moreover, the family of conformally flat spaces contains a distinguished curved space $\tilde{M}$, the universal covering of the conformal compactification of Minkowski space M. Global special conformal transformations (which are always singular on a hypersurface in M) are well defined on $\tilde{M}$ and preserve the natural causal ordering on that space (see /S2-4/,/M2/ and Appendix to ref. /T5/). $\tilde{M}$ is a static (pseudo) Riemannian space of constant scalar curvature; it is diffeomorphic to the 4-dimensional cylinder $R_1 \otimes S_3$ (its constant time surfaces being 3-spheres in R_4).

The (quantum) field theory in a Riemannian space-time has been studied extensively in recent years (see, e.g., /B8/,/C5,6/,/D3/, and references cited therein). In the present context one is confronted with a new problem in that framework; the consideration of (quantum) fluctuations around a non-translation invariant ground state.

The objective of this note is to study classical solutions of the conformal invariant wave equations for a self-interacting scalar field with a prescribed symmetry. It is observed that such solutions are expressed in terms of the factor $\Omega(x)$ relating the intervals ds^2 in different conformally flat spaces of constant scalar curvature.

If the space V_4 with metric tensor $g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu}$ has a κ-parametric isometry group (or group of motions) G_κ ($\kappa \leq 10$), then the corresponding solution is G_κ-symmetric. Its conformal transformations give rise to a $15-\kappa$-parametric family of solutions (with symmetry groups conjugated to G_κ). We also remark that non-static solutions in Minkowski space (like the singular solution described in ref. /F1/) appear as static solutions in terms of the conformal time variable τ of $\tilde{M}$.

We start (in Sect. 1) with a brief review of conformal symmetry in the framework of general relativity. The classical solutions of the massless $\lambda\phi^4$ theory in a conformally flat space are studied in Sect. 2 for both signs of λ. New solutions are found in terms of both elementary and elliptic functions. All new solutions, however, are singular on some surfaces in the Minkowski space. (The only bounded solutions are those related to a compact subgroup of $SO(4,2)$ and they have been previously known). In Sect. 3 we discuss the relevance of the scalar conformal invariant wave equation to the solution of the classical Yang-Mills equations.

1. CONFORMAL SYMMETRY IN THE FRAMEWORK OF GENERAL RELATIVITY

A. The Conformal Group of a Riemannian Space

A point transformation (that is a local diffeomorphism) $x \to x' = x'(x)$ in a (pseudo) Riemannian space V with a metric tensor $g_{\alpha\beta}(x)$ is said to be conformal* if

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* For a lucid discussion of various types of conformal mappings and their interrelations see /E1/ and the first part of /F3/. Note that we are using a metric tensor of opposite sign (our signature is + − − −). Gravitational fields are classified according to their conformal properties in Chapt. VII of ref. /P3/ which also contains a comprehensive bibliography on this and related subjects up to 1964.

$$g_{\mu\nu}({'x})\partial_\alpha {'x}^\mu \partial_\beta {'x}^\nu = \omega^2(x) g_{\alpha\beta}(x) \quad \left(\partial_\alpha = \frac{\partial}{\partial x^\alpha}\right) \tag{1.1a}$$

or

$$'g_{\mu\nu}({'x}) = \frac{\partial x^\alpha}{\partial {'x}^\mu}\,\frac{\partial x^\beta}{\partial {'x}^\nu}\, g_{\alpha\beta}(x) = \omega^{-2}(x)\, g_{\mu\nu}({'x}), \tag{1.1b}$$

where $\omega(x)$ is a (smooth) non-vanishing real valued function. (In other words, a coordinate transformation in V is called conformal if it leaves the form of the metric tensor invariant up to a local factor.)

Not every Riemannian manifold admits a non-trivial group of conformal transformations. A one-parameter group of coordinate transformations with infinitesimal generator $L = \xi^\alpha \partial_\alpha$ is conformal if there exists a function $f(x)$ such that the conformal Killing equation /E1/, /P3/

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 2f g_{\alpha\beta} \tag{1.2}$$

is satisfied (here ∇_α is the covariant derivative: $\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta - \Gamma^\gamma_{\alpha\beta}\, \xi_\gamma$). If $f = 0$, then the factor $\omega(x)$ in (1.1) is one and the corresponding transformation is an isometry. The group $C(V_n)$ of conformal symmetries of an n-dimensional Riemannian manifold V_n has at most $\frac{1}{2}(n+1)(n+2)$ parameters (15 parameters for V_4). A criterion for two spaces V and $\bar{V}$ to have the same (local) group of conformal transformations is given by the following statement. Let there exist a mapping $x \to \bar{x}(x)$ of some neighbourhood $O = O_x$ of each point $x \in V$ onto a neighbourhood $\bar{O}$ of a point $\bar{x} \in \bar{V}$ such that, if we use the x's as (local) coordinates in both $O \subset V$ and $\bar{O} \subset \bar{V}$, the corresponding metric tensors are proportional:

$$\bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \quad (\Omega^2(x) > 0 \ \text{ for } x \in O). \tag{1.3}$$

Under these conditions the (Weyl) conformal curvature tensor*

$$C^\lambda{}_{\mu\nu\rho} \equiv R^\lambda{}_{\mu\nu\rho} - \frac{1}{n-2}\left(R_{\mu[\nu}\delta^\lambda_{\rho]} + g_{\mu[\nu} R^\lambda_{\rho]}\right) + \frac{1}{(n-1)(n-2)} R g_{\mu[\nu}\delta^\lambda_{\rho]}, \tag{1.4}$$

$$\left(g_{\mu[\nu} R^\lambda_{\rho]} \equiv g_{\mu\nu} R^\lambda_\rho - g_{\mu\rho} R^\lambda_\nu\right).$$

is the same in the two neighbourhoods and the conformal groups in O and $\bar{O}$ are isomorphic. In particular, the solution $\bar{\xi}_\alpha$, $\bar{f}$ of the Killing equation in $\bar{O}$ is related to the solution ξ_α, f of the (1.2) in O by

$$\bar{\xi}_\alpha(x) = \Omega^2(x)\xi_\alpha(x),$$

$$\bar{f}(x) = f(x) + g^{\alpha\beta}(x)\xi_\alpha(x)\partial_\beta \ln\Omega(x). \tag{1.5}$$

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*We are using the standard notation (see, e.g., ref. /E1/,/P3/) for the Riemann curvature tensor $R^\lambda{}_{\mu\nu\rho} = \partial_\nu \Gamma^\lambda_{\mu\rho} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - \Gamma^\lambda_{\sigma\rho}\Gamma^\sigma_{\mu\nu}$, the Ricci tensor $R_{\mu\nu} = R^\sigma_{\mu\nu\sigma}$, and the scalar curvature $R = R^\alpha{}_\alpha$, $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$ being the Christoffel symbols.

A (local) mapping of Riemannian manifolds, in which the metric tensor at identified points is changed according to (1.3) is called conformal (Domokos [D4] uses the term "Weyl transformation" for it). It should be clearly distinguished from the conformal transformations defined by (1.1)* . Two spaces V and $\bar{V}$ for which the metric tensors are (locally) related by (1.3) are called conformal (to each other). It can be shown, that light-like geodesics go into light-like geodesics under a conformal mapping, and classical equations of motion for massless particles and dimensionless coupling remain invariant.

The conformal group of a Riemannian space is maximal (and locally isomorphic to $SO_0(n,2)$) if and only if the space V_n is conformally flat, i.e., if

$$g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu} \quad (\eta = \mathrm{diag}(1,-1,\dots,-1)\,, \qquad (1.5')$$

or, equivalently, if $C^{\lambda}{}_{\mu\nu\rho} = 0$.

*The conformal transformations are point or coordinate transformations within the same Riemannian manifold V which only exist in a restricted class of Riemann spaces and from at most a $(n+1)(n+2)/2$ parameter group. The conformal mapping, by contrast, relates one Riemann space, V, to another, $\bar{V}$, (with a different metric structure). The group of such mapping is isomorphic to the group of all positive local factors $\Omega(x)$ and is, therefore, infinite dimensional.

B. Conformal Invariant Wave Equation for a Scalar Field

The analogue of a massless scalar field equation with a dimensionless selfcoupling in a curved (n -dimensional) space-time is

$$(\Box + \frac{n-2}{4(n-1)}R)\phi + \lambda\phi^{\frac{n+2}{n-2}} = 0\,, \qquad (1.6)$$

where

$$\Box = g^{\alpha\beta}\nabla_\alpha\partial_\beta = \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}\,g^{\alpha\beta}\partial_\beta)\,,\quad g \equiv \det g_{\alpha\beta}\,,$$

is the invariant d'Alembert operator. It has been noted already by Penrose [P1] (for the case $\lambda = 0$) and further justified in [C4] (see also [M1],[T1,2]) that the term proportional to the scalar curvature ($R/6$ for $n = 4$) is necessary both for the invariance of Eq. (1.6) with respect to the conformal mapping (1.3) and for the correct physical interpretation of the theory.

Indeed, if $\phi(x)$ is a solution of (1.6) in V_n, then

$$\bar{\phi}(x) = \Omega^{1-\frac{n}{2}}\phi(x) \qquad (1.7)$$

is a solution of the same equation in $\bar{V}_n$ (with $\Box$ and R replaced by $\bar{\Box}$ and $\bar{R}$). That follows from the identity

$$(\Box + \frac{n-2}{4(n-1)}R)\phi = \Omega^{\frac{n+2}{2}}(\bar{\Box} + \frac{n-2}{4(n-1)}\bar{R})\Omega^{\frac{2-n}{2}}\phi$$

which can be obtained from the known transformation law for the scalar curvature

$$\bar{R} = \Omega^{-2}\{R + 2(n-1)\Box \ln\Omega + (n-1)(n-2)\, g^{\alpha\beta}\, \partial_\alpha \ln\Omega\, \partial_\beta \ln\Omega\} \qquad (1.8)$$

under the mapping (1.3) (see, e.g.,[P3] Eq. (35.8)).

The invariance property thus established is related to the conformal invariance of Eq. (1.6) in the sense of coordinate transformations. If the space V admits a non-trivial conformal group (of transformations $x \to {}'x$ satisfying (1.1))

$$({}'\Box_{'x} + \frac{n-2}{4(n-1)} R({}'x))\, \omega^{\frac{n-2}{2}}(x)\, \phi({}'x) =$$

$$= \omega^{\frac{n+2}{2}}(x)\,(\Box_{'x} + \frac{n-2}{4(n-1)} R({}'x))\phi({}'x). \qquad (1.9)$$

Consequently, if $\phi(x)$ satisfies Eq. (1.6) so does

$${}'\phi(x) = \omega^{\frac{n-2}{2}}(x)\, \phi({}'x(x)). \qquad (1.10)$$

It is remarkable that the factor Ω in the mapping (1.3) not only allows one to relate the solutions of Eq. (1.6) in V and $\bar{V}$, but for $n=4$ and $\bar{R}$ = constant it is proportional to a solution of (1.6) invariant under the isometry group of $\bar{V}$. Indeed, the transformation law (1.8) for the scalar curvature can be rewritten in the form

$$(\Box + \frac{R}{2(n-1)})\Omega - \frac{\bar{R}}{2(n-1)}\Omega^3 + \frac{n-4}{2\Omega} g^{\alpha\beta}\, \partial_\alpha \Omega\, \partial_\beta \Omega = 0. \quad (1.11)$$

For $n=4$ the last term drops out and we obtain

$$(\Box + \frac{1}{6}R)\Omega - \frac{1}{6}\bar{R}\Omega^3 = 0$$

that is nothing but Eq. (1.6) for $n=4$ and $\lambda = -\frac{1}{6}\bar{R}$ (=const). A real solution of

$$(\Box + \frac{1}{6}R)\phi(x) + \lambda\phi^3(x) = 0 \qquad (1.12)$$

for arbitrary λ (of the same sign as $-\bar{R}$) is given by

$$\phi(x) = \sqrt{-\frac{\bar{R}}{6\lambda}}\,\Omega(x). \qquad (1.13)$$

If the space V admits a nontrivial conformal group C(V), then Eq. (1.10) gives rise, in general, to more solutions of Eq. (1.6). The particular solution (1.13) is invariant under the isometry subgroup of $C(\bar{V})$ [= C(V)], or, in other words, under the group of motions of the space $\bar{V}$.

We shall exploit the above observation for studying symmetric solutions of Eq. (1.12) in a (4-dimensional) conformally flat space - particularly in Minkowski space - in Sect. 2 below. First of all, however, we shall review in the two following subsections some general properties of the symmetric solutions of Eq. (1.6) in an n-dimensional (pseudo) Riemannian manifold.

C. <u>Solutions of the Nonlinear Wave Equation with a Given Symmetry</u>

The observation made at the end of the preceding subsection allows one to apply known results on the classification of Riemann spaces according to their isometry groups in order to find solutions of Eq. (1.12) invariant under a given subgroup G of the conformal group $C(V_4)$. The following auxiliary statement will be useful in the sequel.

Proposition 1. Let V_n admit a nontrivial conformal group $C(V_n)$ and let $\phi(x)$ be any smooth function on V_n conformal invariant under one-parameter subgroup of $C(V_n)$ with generator $L = \xi^\alpha(x)\partial_\alpha$; then

$$L\left(\frac{\Box\phi + \frac{n-2}{4(n-1)} R\phi}{\phi^{\frac{n+2}{n-2}}} \right) = 0 . \tag{1.14}$$

Proof. Let $x \to {}'x(x,\epsilon)$ be indeed a one-parameter (conformal) transformation group (with parameter ϵ) satisfying (1.1) with $\omega = \omega(x,\epsilon)$ and such that

$$'x(x, 0) = x, \qquad \frac{\partial}{\partial\epsilon}\, {}'x(x,\epsilon)\Big|_{\epsilon = 0} = \xi(x),$$

$$\frac{\partial}{\partial\epsilon}\,\omega(x, \epsilon)\Big|_{\epsilon = 0} = f(x).$$

Then differentiating Eq. (1.9) with respect to ϵ we obtain for $\epsilon = 0$

$$(L + \frac{n+2}{2} f)(\Box + \frac{n-2}{4(n-1)}R)\phi =$$

$$= (\Box + \frac{n-2}{4(n-1)}R)(L + \frac{n-2}{2} f)\phi .$$

Now we assume that the transformation (1.10) leaves ϕ conformal invariant so that

$$(L + \frac{n-2}{2} f)\phi = 0 . \tag{1.15}$$

Since L is a differentiation we obtain

$$\left[\frac{\Box\phi + \frac{n-2}{4(n-1)}R\phi}{\phi^{\frac{n+2}{n-2}}} \right] = \phi^{-\frac{n+2}{n-2}} L(\Box\phi + \frac{n-2}{4(n-1)}R\phi) -$$

$$- \frac{n+2}{n-2}\phi^{-\frac{n+2}{n-2}-1}(L\phi)(\Box + \frac{n-2}{4(n-1)})\phi = 0 .$$

That completes the proof of Proposition 1.

Corollary. If $\phi(x)$ is conformal invariant under a κ-parameter group $G_\kappa \subseteq C(V_n)$, and G_κ is transitive, then $\phi(x)$ satisfies Eq. (1.6) with some (constant) λ .

Indeed the transitivity of G_κ means that the matrix ξ $(i = 1, \ldots, \kappa)$ has rank n. Therefore, the set of all equations of the form (1.14) implies

$$\partial_\mu \left[\frac{(\Box + \frac{n-2}{4(n-1)}R)\phi}{\phi^{\frac{n+2}{n-2}}} \right] = 0 ,$$

or

$$\frac{(\Box + \frac{n-2}{4(n-1)}R)\phi}{\phi^{\frac{n+2}{n-2}}} = \lambda \ (= \mathrm{const}), \tag{1.16}$$

which coincides with (1.6). (Note, however, that the constant in the last equation may happen to be zero; that is the case, for instance, when G_κ is the (n-parameter) translation group in the flat space-time).

Thus in the case of a transitive group G_κ of conformal transformations, the G_κ-invariant solution of the non-linear equation (1.6) is found to satisfy the set of linear equations (1.15). If the conformal group in V_n has N parameters $(N \geq \kappa)$, then transforming our solution according to (1.10) we obtain a $N - \kappa$-parameter family of solutions.

Next, we shall consider the case when the rank of the matrix (ξ_i^μ) is $n-1$ and will show how to reduce Eq.(1.6) in that case to a (non-linear) ordinary differential equation. Under the action of a group G_κ with the above property the space V_n splits into a one-dimensional family of G_κ-transitive hypersurfaces $\sigma(x) = \text{const}$. G_κ leaves each such surface invariant:

$$L_i \sigma \equiv \xi_i^\mu \partial_\mu \sigma = 0, \quad i = 1, \ldots, \kappa \ (\geq n-1). \qquad (1.17)$$

If σ satisfies this set of equations and F is an arbitrary (differentiable) function, then $F(\sigma(x))$ also satisfies eq. (1.17) (That exhausts the freedom left by the G_κ-invariance). Any function ϕ which satisfies the invariance condition (1.15) can be written in the form

$$\phi = \psi(x) F(\sigma), \qquad (1.18)$$

where $\psi(x)$ is some fixed solution of Eq. (1.15). In particular, any G_κ-invariant solution of (1.6) should be of this form.

Proposition 2. The substitution of (1.18) where ψ and σ are nontrivial solutions of (1.15) and (1.17), respectively into Eq. (1.6) leads to an ordinary differential equation for F:

$$A_2 F'' + A_1 F' + A_0 F + \lambda F^{\frac{n+2}{n-2}} = 0. \qquad (1.19)$$

The coefficients

$$A_0 = \psi^{-\frac{n+2}{n-2}} \left(\Box + \frac{n-2}{4(n-1)} R\right)\psi, \qquad (1.20a)$$

$$A_1 = \psi^{-\frac{n+2}{n-2}} \left(\Box(\psi\sigma) - \sigma\Box\psi\right), \qquad (1.20b)$$

$$A_2 = \psi^{-\frac{4}{n-2}} \partial_\alpha \sigma \partial^\alpha \sigma \qquad (1.20c)$$

of this equation depend on x through σ only.

Proof. The derivation of (1.19) and (1.20) is straightforward. In order to prove the last assertion, it is necessary and sufficient to verify the equations

$$L_i A_a = 0 \quad \text{for} \quad i = 1, \ldots, \kappa; \ a = 0, 1, 2. \qquad (1.21)$$

For A_0 this is true because of (1.14), since $\psi(x)$ is assumed to satisfy the conformal invariance condition (1.15). Then we express $\Box\psi$ in A_1 in terms of A_0 and ψ; Eq. (1.21) is again verified, since $\sigma\psi$ also satisfies (1.15) (if ψ does).

Finally,

$$L_i A_2 = \psi^{-\frac{n+2}{n-2}} \Big\{ -\frac{4}{n-2} \partial_\alpha \sigma \partial^\alpha \sigma L_i \psi + {}$$
$$+ 2\psi \partial^\alpha \sigma (\partial_\alpha L_i \sigma - \nabla_\alpha \xi^\beta \partial_\beta \sigma) \Big\} = 0,$$

because according to (1.2) $2\partial^\alpha \sigma \nabla_\alpha \xi_{i\beta} \partial^\beta \sigma = 2 f_i \partial_\alpha \sigma \partial^\alpha \sigma$ and ψ satisfies (1.15). That completes the proof of Proposition 2.

2. SYMMETRIC SOLUTIONS OF THE NONLINEAR WAVE EQUATION IN MINKOWSKI SPACE

In this section we shall exploit the observation of Sect. 1.B that the symmetric solutions of Eq. (1.12) in 4-dimensional space-time are expressed in terms of the conformal factors $\Omega(x)$ mapping, say Minkowski space, onto a (conformally flat) space V_4 of constant scalar curvature. That allows one to relate the solutions of a sufficiently high symmetry to the known classification of conformally flat spaces according to the isometry groups (see /P3/).

A. Conformally Flat Spaces of Maximal Isometry

We are primarily interested in finding and classifying symmetric solutions of the equation

$$\Box\phi + \lambda\phi^3 = 0 \qquad (2.1)$$

in Minkowski space $M = M_4$.

Three conformally flat spaces of non-vanishing constant scalar curvature and transitive isometry groups are well known. These are the de Sitter spaces of constant positive and negative curvature (also known as de Sitter and anti-de Sitter space-times) and the cylindric (Einstein) static universe $\tilde{M}$. Although the corresponding conformal factors only provide previously known solutions, these simple cases can serve as a good illustration to the method and we proceed to their description.

The de Sitter spaces $S_{\rho\epsilon}$ can be defined as hyperboloids in 5-space

$$S_{\rho\epsilon} = \{ y \in \boldsymbol{R}_5 ;\ (y^0)^2 - y^2 - \epsilon(y^4)^2 + \epsilon\rho^2 = 0, \epsilon = \pm, \rho > 0 \}. \qquad (2.2)$$

Here ϵ is the sign of the curvature and ρ is its radius; $S_{\rho+}$ is the "closed isotropic universe" with isometry group $O(4,1)$, $S_{\rho-}$ corresponds to the open-universe model with isometry group $O(3,2)$. The space $S_{\rho\epsilon}$ is conformal to the domain $\rho^2 > \epsilon x^2$ of Minkowski space. Indeed, setting $y^\mu = 2\rho^2(\rho^2 - \epsilon x^2)^{-1} x^\mu$ $y^4 = \rho(\rho^2 - \epsilon x^2)^{-1}(\rho^2 + \epsilon x^2)$ we obtain the following expression for the (anti) de Sitter space-time interval

$$ds_\epsilon^2 = (dy^0)^2 - (dy)^2 - \epsilon(dy^4)^2 = \Omega_\epsilon^2(x) dx^2, \qquad (2.3)$$

where

$$\Omega_\epsilon(x) = \frac{2\rho^2}{\rho^2 - \epsilon x^2}, \quad \epsilon = \pm . \qquad (2.4)$$

The Riemann tensor and the scalar curvature are given by

$$R^\alpha_{\ \mu\nu\beta} = \epsilon \frac{1}{\rho^2} g_{\mu[\nu} \delta^\alpha_{\beta]}, \quad R = \epsilon \frac{12}{\rho^2}. \qquad (2.5)$$

Hence, according to (1.13) the corresponding ($O(4,1)$ or $O(3,2)$ -symmetric) solution of Eq. (2.1) is given by

$$\phi_{\rho\epsilon}(x) = \sqrt{-\epsilon\frac{2}{\lambda}}\ \frac{2\rho}{\rho^2 - \epsilon x^2}. \qquad (2.6)$$

The positive λ solutions ($\epsilon = -1$), which correspond to a positive classical Hamiltonian, have been used by Fubini et al. /F1/, /D2/ in an attempt to describe spontaneously broken conformal symmetry, while the negative λ solutions are applied by Lipatov and others /L1/, /B6,7/ in their study of the large order behaviour of the perturbation series in the Euclidean quantum field theory. (Note that $\phi_{\rho+}$ is finite for pure imaginary times.) Following the general prescription of Sect. 1.C we obtain (by translating (2.6)) a five parameter family of solutions

$$\phi_{\rho\epsilon}(x-a) = \sqrt{-\epsilon\frac{2}{\lambda}}\ \frac{2\rho}{\rho^2 - \epsilon(x-a)^2}. \qquad (2.7)$$

(The fifth parameter is the radius ρ of the universe which plays the role of scale or dilatation parameter).

The Einstein static universe $\tilde{M} = R_1 \otimes S_3$ can be identified with the universal covering of the conformal compactification $\bar{M}$ of Minkowski space /U1/, considered by Penrose and others /P1/,/S2-4/,/T5/,/M2/. Representing the points in Minkowski space M by 2x2 hermitian matrices

$$\mathbf{X} = x^{\mu}\sigma_{\mu} = \begin{pmatrix} x^{o}+x^{3} & x^{1}-ix^{2} \\ x^{1}+ix^{2} & x^{o}-x^{3} \end{pmatrix},$$

and using the Cayley transformation $u=(1-i\mathbf{X})^{-1}(1+i\mathbf{X})$ we can define $\bar{M}$ as the set U(2) of all unitary 2x2 matrices. Parametrizing the points in $U(2) = S_1 \otimes S_2 / \mathbf{Z}_2$ by four angles

$$u = e^{ir}(\cos\chi + i\sin\chi\,\underline{n}\underline{\sigma}),\quad \underline{n} = (\sin\theta\ \cos\phi,\ \sin\theta\sin\phi,\ \cos\theta),$$

$$-\pi < \tau \le \pi,\quad 0 \le \chi \le \pi,\quad 0 \le \chi \le \pi,\ 0 \le \theta < \pi,\ 0 < \phi < 2\pi,$$

we will have

$$\sin\chi\underline{n} = \Omega(x)x,\quad \cos\chi = \Omega(x)\frac{1+x^2}{2},$$

$$\cos\tau = \Omega(x)\frac{1-x^2}{2},\ \sin\tau = \Omega(x)x^{o}, \qquad (2.8)$$

where

$$\Omega(x) = [\frac{1}{4}(1-x^2)^2 + (x^{o})^2]^{-1/2} \quad (=\cos\chi + \cos\tau) \qquad (2.9)$$

(Eqs. (2.8) define a map of M on the submanifold $\cos\chi + \cos\tau > 0$ of $S_1 \otimes S_3$). The isometry group of M is the maximal compact subgroup $K = SO(2) \otimes SO(4)$ of $SO_0(4,2)$. The K-invariant quadric of $\bar{M}$ is given by

$$ds^2 = d\tau^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2) = \Omega^2(x)dx^2. \quad (2.10)$$

The scalar curvature of $\bar{M}$ is constant, $R = 6$, (although the Riemann curvature is not). The compactified Minkowski space $\bar{M}$ and its covering spaces are distinguished by the fact that global conformal transformations are everywhere defined and nonsingular on each of these spaces. The universal covering space $\tilde{M}$ (obtained from $\bar{M}$ by replacing the circle S by the real line $-\infty < \tau < \infty$) is distinguished for having a conformal invariant causal ordering on it (see /S2-4/,/T5/, /M1/). It is the carrier space of the infinite sheeted universal covering $\tilde{C}(\bar{M}) = C(\tilde{M})$ of the Minkowski space conformal group* $SO_0(4,2)/Z_2$.

The K-symmetric solution of (2.1) is $\phi(x) = \frac{1}{\sqrt{-\lambda}}\Omega(x)$, where Ω is the factor (2.9). This solution was described by Castell /C2/ and further explored in /D1/. It gives rise (through conformal transformations) to an 8-parameter family of solutions:

$$\phi_{\ell}(x-a,u) = \frac{2}{\ell\sqrt{-\lambda}}\{[1-(\frac{x-a}{\ell})^2]^2 + [2\frac{u(x-a)}{\ell}]^2\}^{-1/2}, \qquad (2.11)$$

where u is a unit time-like vector ($u^2 = 1$). These solutions are distinguished for being bounded on the entire Minkowski space and for carrying a finite energy.

We conclude this subsection with a remark on the solution of Eq. (1.12) in other conformally flat spaces.

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*$\tilde{C}$ is also called the quantum mechanical conformal group since all projective representations of C(M) can be lifted (according to Bargmann /B1/) to suitable unitary representations of $\tilde{C}$.

Given a solution of Eq. (2.1) in M we can write according to (1.7) the corresponding solution $\bar{\phi}(x) = \Omega^{-1}(x)\phi(x)$ of (1.12) in any other conformally flat space. If V is a conformally flat space of constant scalar curvature R_V then the solution corresponding to (1.13) for the conformal factor $\Omega_V(x)$ is constant in V:

$$\bar{\phi}_V(x) = \sqrt{\frac{-R_V}{6\lambda}} . \qquad (2.12)$$

(That is also a direct consequence of Eq. (1.12)). Thus, in particular, non-static solutions in Minkowski space go into static solutions in the corresponding conformally flat spaces. It is interesting to note that the Fubini solution $\phi_{\rho-}(x)$ (2,6) also goes into a τ-independent solution on the cylindric space $\tilde{M}$ of radius ρ:

$$\bar{\phi}_{\rho-}(x) = \Omega^{-1}\left(\frac{x}{\rho}\right)\phi_{\rho-}(x) = \sqrt{\frac{2}{\lambda}}\ \frac{1}{\rho\cos\chi} . \qquad (2.13)$$

B. Solutions of Eq. (2.1) Corresponding to Homogeneous Spaces *

The general study of the symmetric solutions of Eq. (2.1) is greatly facilitated by the classification of conformally flat space-time according to their isometry groups due to Bilyalov and Petrov *. We shall only present here an extract of the relevant results. In order to facilitate comparison with the Petrov listing we shall denote the group under consideration by $G_\kappa^{(A)}$, κ being the number of parameters of G and A for the number in the Petrov list of groups with given κ (see /P3/ p. 314-318). Solutions of Eq. (2.1) will be written in a Lorentz covariant form; for this purpose we introduce an orthonormal basis in M_4 constituted by four constant vectors u, q, r, s:

$$u^2 = 1, \quad q^2 = r^2 = s^2 = -1, \quad uq = ur = us = qr = qs = rs = 0$$

and two isotropic vectors

$$n = u - s, \quad p = u + s \qquad (p^2 = n^2 = 0).$$

Generators of the subgroup of $SO_0(2,4)$ (of the isometry group of the corresponding conformally flat space-time) under which a given solution is symmetric, may be obtained by projection of the corresponding generators from the Bilyalov-Petrov classification onto the basis $\{u, q, r, s\}$, the generators of translations, Lorentz transformations dilatations and special conformal transformations being considered, respectively, as a four-vector, an antisymmetric tensor, a scalar and a four-vector.

The only 4-dimensional conformally flat homogeneous spaces of isometry groups G_κ with $\kappa \geq 7$ are the Minkowski space M and the spaces $S_{\rho\epsilon}$ and $\tilde{M}$ considered in the preceding subsection. There are two homogeneous spaces with 6-parameter

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*We recall that a space V with isometry group G_κ is called homogeneous if G_κ is transitive on V (that is if any two points of V can be transformed into one another by an element of G_κ).

* This classification is presented in a systematic fashion in the book by A.Z.Petrov /P3/ (see, in particular, p. 310-318).

isometry groups $G_6^{(4)}$ and $G_6^{(6)}$ which both have zero scalar curvature and thus lead to singular solutions of the linear d'Alembert equation.

There remain two more homogeneous spaces with isometry groups with five and four parameters which lead to the solutions of the nonlinear equations (2.1):

$$G_5^{(1)}: \phi = \sqrt{\frac{2}{-\lambda}}[4qx - (px)^2]^{-1},$$

$$G_4^{(10)}: \phi = \sqrt{\frac{2}{-\lambda}}[1 + (px)^2]^{1/2}[qx + px \cdot rx]^{-1}. \tag{2.14}$$

C. Isometry Groups with a Transitive Hypersurface

We proceed to the case when the matrix (ξ_i^{μ}) ($i=1,2,\ldots,\kappa$, $\mu=0,1,2,3$) of the Killing vectors' components of a subgroup $G_\kappa \subset C(M_4)$ (the isometry group of the corresponding conformally flat space-time) has rank 3. In this case the Bilyalov-Petrov classification provides us with the invariant $\sigma(x)$ and the factor $\psi(x)$ in the substitution (1.18) which reduces the solution of Eq. (2.1) to the solution of the ordinary differential equation (1.19) (with $n = 4$).

For some invariance subgroups G_κ the choice of ψ and σ is rather trivial. For instance, for $G_6 = O(3,1)$ we just set $\sigma(x) = x^2$, $\psi(x) = 1$; the corresponding (elliptic functions') solution of Eq.(1.19) has been found long ago by Petiau /P2/ . In most cases, however, this substitution is far from obvious. Since it is not clear that the choice of /P3/

is the most appropriate one, we consider the additional change of variables

$$F(\sigma) = f(\sigma)H(z), \tag{2.15a}$$

$$z = z(\sigma), \tag{2.15b}$$

where $z(\sigma)$ and $f(\sigma)$ are given and $H(z)$ is a new unknown function. We shall write down conditions on the coefficients A_a (1.20) under which Eq.(1.19) can be transformed by a substitution of the type (2.15) to the form

$$H''(z) + (1 - 2k^2)H'(z) + 2k^2H^3(z) = 0, \tag{2.16}$$

where k is a constant. This latter equation is satisfied by the Jacobi elliptic cosine.

Inserting (2.15) into (1.19), we obtain

$$A_2 f z'^2 H'' + [A_2(f z'' + 2f'z') + A_1 f z']H' +$$

$$+ (A_2 f'' + A_1 f' + A_0 f)H + \lambda f^3 H^3 = 0.$$

In order to reduce this equation to the form (2.16) the functions f and z should satisfy the system

$$\frac{d}{d\sigma}\ln(z'f^2) + \frac{A_1}{A_2} = 0,$$

$$f^2 = \frac{2k^2}{\lambda} A_2 z'^2,$$

$$A_2 f'' + A_1 f' + A_0 f = (1 - 2k^2)A_2 z'^2 f.$$

Solving the first two equations with respect to z′ and f

$$z' = A_2^{-1/3} \exp\{-\frac{1}{3}\int_{\sigma_0}^{\sigma} \frac{A_1}{A_2} d\sigma'\}, \quad \sigma = \text{const},$$

$$f = \sqrt{\frac{2k^2}{\lambda} A_2^{1/3}} \exp\{-\frac{1}{3}\int_{\sigma_0}^{\sigma} \frac{A_1}{A_2} d\sigma'\},$$

and inserting them into the third one, we obtain the following relation for the coefficients A_a of Eq. (1.19):

$$6A_2(A_2'' - 2A_1' + 6A_0) - (5A_2' - 4A_1)(A_2' - 2A_1) =$$

$$= 36(1 - 2k^2) A_2^{4/3} e^{-\frac{2}{3}\int_{\sigma_0}^{\sigma} \frac{A_1}{A_2} d\sigma}. \qquad (2.17)$$

If this condition is satisfied then the general G_κ-invariant solution of Eq. (2.1) can be written in the form*

$$\phi(x) = \sqrt{\frac{2k^2}{\lambda}} A_2^{1/6} \psi(x) \exp\{-\frac{1}{3}\int_{\sigma_0}^{\sigma} \frac{A_1}{A_2} d\sigma'\} \operatorname{cn}(z + z_0, k),$$

$$z = \int \frac{d\sigma}{A_2^{1/3}} \exp\{-\frac{1}{3}\int_{\sigma_0}^{\sigma} \frac{A_1}{A_2} d\sigma'\}; \qquad (2.18)$$

$$\sigma_0 = \text{const}, \quad z_0 = \text{const},$$

*See also the Note added in proof at the end of the paper, p. 31.

where k^2 and A_a are determined respectively from (2.17) and (1.20) for given ψ and σ; the sign of λ must be the same as the sign of $A_2 = \psi^{-2}\partial_a\sigma\partial^a\sigma$, that is the sign of the Lorentz square of the normal vector to the surface σ = const. (In other words, $\lambda > 0$ for space-like G_κ-homogeneous hypersurface, and $\lambda < 0$ if $\sigma(x)$ = const has a tangent time-like vector).

It turns out that Eq. (2.17) is satisfied for all five 6-parameter groups with homogeneous hypersurfaces, considered in /P3/. Hence, for all these cases the general $G_6^{(A)}$-invariant solution of (2.1) has the form (2.18). For three of them (with $G_6^{(1)} = E(3) = R^3 \otimes O(3)$, $G_6^{(3)} = O(4)$ and $G_6^{(7)} = O(3,1)$) these solutions have been known (see /P2/ and /C2/).

We list the values of ψ, σ, A_a, z and k^2 to be substituted in Eq. (2.18) for the G_6-invariant solutions in Table 1. All 5-parameter groups are exhausted by the transitive $G_5^{(1)}$ considered in the previous subsection. We have also considered all 4-parameter groups and list the cases solvable in terms of Jacobi elliptic functions in Table 2.

According to the Bilyalov-Petrov classification there are 19 three-parameter subgroups of $SO_0(4,2)$ which are transitive on a hypersurface of M_4 and thus lead to Eq. (1.19). We are not exploring these here.

3. RELATION TO THE YAHG-MILLS EQUATIONS. DISCUSSION

A. The Corrigan-Fairlie-'t Hooft-Wilczek Ansatz

It turns out that the solutions of the scalar wave equation (2.1) (or (1.12)) can be used for solving the physically more interesting (or, at least, more fashionable) Yang-Mills equations.

Table 1

G_6-symmetric solutions of Eq. (2.1) expressed in terms of Jacobi elliptic functions: expressions for the functions and constants which appear in Eq. (2.18) and Eq. (2.18a) (see Note added in proof, p. 31)

The symmetry group	$\psi(x)$	$\sigma(x)$	A_0	A_1	A_2	z	$1-2k^2$
$G_6^{(1)} = R_1 \otimes O(3)$	1	ux	0	0	1	σ	0
$G_6^{(2)}$	$(qx)^{-1}$	$px(qx)^{-1}$	-2	-4σ	$-\sigma^2$	$-\sigma_0^{4/3}\sigma^{-1}$	0
$G_6^{(3)} = O(4)$	$(ux)^{-1}$	$(1-x^2)(ux)^{-1}$	2	4σ	σ^2+4	$\frac{1}{2}(\sigma_0^2+4)^{2/3}\cdot\operatorname{arctg}\frac{\sigma}{2}$	$4(\sigma_0^2+4)^{-4/3}$
$G_6^{(5)}$	$(qx)^{-1}$	$rx(qx)^{-1}$	-2	-4σ	$-(\sigma^2+1)$	$(\sigma_0^2+1)^{2/3}\cdot\operatorname{arctg}\sigma$	$(\sigma_0^2+1)^{-4/3}$
$G_6^{(7)} = O(3,1)$	1	x^2	0	8	4σ	$4^{-1/3}\sigma_0^{2/3}\ln\sigma$	$-4^{-1/3}\sigma_0^{-4/3}$

Table 2

G_4-symmetric solutions of Eq. (2.1) expressed in terms of Jacobi elliptic functions: expressions for the functions and constants which appear in Eq. (2.18)

The symmetry group	$\psi(x)$	$\sigma(x)$	A_0	A_1	A_2	z	$1-2k^2$
$G_4^{(2)}$* $(w=\pm\frac{v}{3})$	$(x^2)^{-1/2}$	$(px)^{2v}(x^2)^{-v\mp\frac{v}{3}}$	-1	$-4(v+\frac{8}{9}v^2)\sigma$	$-\frac{32}{9}v^2\sigma^2$	$(\frac{16}{3}v\sigma_0)^{(1+\frac{9}{8v})1/3}\sigma^{-\frac{3}{8v}}$	0
$G_4^{(2)}(v=0)$	$(x^2)^{-1/2}$	$(x^2)^{-w}$	-1	$4w^2\sigma$	$4w^2\sigma^2$	$(2w)^{-2/3}\sigma_0^{1/2}\ln\sigma$	$-4^{1/3}w^{-2/3}\sigma_0^{-2/3}$
$G_4^{(6)}$	$(px)^{-1}$	$\frac{qx}{px}-\ln px$	0	0	-1	$-\sigma$	0
$G_4^{(7)}$	1	$px\,e^{-qx}$	0	$-\sigma$	$-\sigma^2$	$\sigma_0^{1/3}\ln\sigma$	0
$G_4^{(8)}$	1	$4(rx)-(px)^2$	0	0	-4	$-4^{-1/3}\sigma$	0
$G_4^{(9)}$	$e^{px/2}$	$(rx)^2\,e^{px}$	0	-2	-4σ	$-2^{1/3}\sigma_0^{1/6}\sigma^{1/2}$	0

* In this case the group algebra depends on two constants, v and w, in general.

In order to fix the ideas, we consider an SU(2) Yang-Mills field

$$A_\mu(x) = \frac{e}{2} q_i A^i_\mu(x), \qquad (3.1)$$

where q_i are the quaternion units satisfying

$$q_i q_j = -\delta_{ij} + \epsilon_{ijk} q_k, \quad i,j,k = 1,2,3 \qquad (3.2)$$

and e is the Yang-Mills coupling constant. In the 2-dimensional representation q_j can be expressed in terms of the Pauli matrices

$$q_j = -i\sigma_j \qquad \text{so that } \operatorname{tr} q_i q_j = -2\delta_{ij}. \qquad (3.3)$$

In a Reimannian space-time manifold V we introduce the extended covariant derivative

$$D_\mu = \nabla_\mu + A_\mu \qquad (3.4)$$

(where ∇_μ is the usual covariant derivative, defined after Eq. (1.2)). The Maxwell Yang-Mills field will be defibed by

$$F_{\mu\nu} = [D_\mu, D_\nu]. \qquad (3.5)$$

Note, that with this definition $F_{\mu\nu}$ does not in general vanish even if $A_\mu = 0$, since it contains the term $[\nabla_\mu, \nabla_\nu]$ proportional to the curvature tensor. The advantage of such a definition comes from the fact that the second set of Maxwell equations,

$$\epsilon^{\kappa\lambda\mu\nu} [D_\lambda, F_{\mu\nu}] = \epsilon^{\kappa\lambda\mu\nu} [D_\lambda, [D_\mu, D_\nu]] = 0, \quad (3.6)$$

in this case is nothing but the Jacobi identity for the double commutator (just like the Bianchi identities in general relativity).

It should be noted, however, that with the alternative definition

$$F'_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu] \qquad (3.5')$$

we also have

$$\epsilon^{\kappa\lambda\mu\nu} [D_\lambda, F'_{\mu\nu}] = 0, \qquad (3.6')$$

since $[A_\lambda, [\nabla_\mu, \nabla_\nu]] = 0$. The choice (3.5) leads to a generalization of Weyl's conformal formulation of Einstein's theory (for a recent discussions see /Y2/,/D4/). Both choices coincide in a flat space-time and the subsequent discussion is applicable to either of them.

In terms of $F_{\mu\nu}$ the general covariant Yang-Mills action is written in the form

$$S(A) = -\frac{1}{2e^2} \int \operatorname{tr} F_{\kappa\lambda} F_{\mu\nu} g^{\kappa\mu} g^{\lambda\nu} \sqrt{|g|}\, d^4x \qquad (3.7)$$

(see, e.g., /B9/); it is manifestly invariant under the conformal mapping (1.3) (we should just remember, that for $g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$, $g^{\mu\nu}$ is transformed according to the inverse formula: $g^{\mu\nu} \to \bar{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}$ while $|g| = |\det g_{\alpha\beta}| \to |\bar{g}| = \Omega^8 |g|$). Consequently, the Yang-Mills equation

$$g^{\lambda\mu} [D_\lambda, F_{\mu\nu}] = 0 \qquad (3.8)$$

is conformal invariant (SO(4,2) - invariant in any conformally flat space).

Classical finite action solutions of the Yang-Mills equations in 4-dimensional Euclidean space have attracted lately much attention (see, e.g., /B2,3,5,9/,/A1/,/D1/,/J1–3/,/N1,2/,/O1/,/P4,5/,/S1/,/T3,4/,/W1,2/,/Y1/). It turns out that the basic 1-instanton solutions are obtained from the Euclidean space counterpart of the solution $\phi_{\rho+}(x-a)$ (2.7) through the simple ansatz *

$$A_\mu(x) = g_{\sigma\mu}\,\partial_\sigma \ln\phi(x) \tag{3.9}$$

provided that the matrices $g_{\mu\nu}$ satisfy the O(4) commutation relations

$$[q_{\kappa\lambda}, q_{\mu\nu}] = \delta_{\kappa\mu} q_{\lambda\nu} + \delta_{\lambda\nu} q_{\kappa\mu} - \delta_{\kappa\nu} q_{\lambda\mu} - \delta_{\lambda\mu} q_{\kappa\nu}. \tag{3.10}$$

In the 2-dimensional realization (3.1)-(3.2) the matrices have the form

$$q_{i4} = \tfrac{1}{2}\epsilon_{ijk} q_{jk} = \tfrac{1}{2} q_i\,. \tag{3.11}$$

We remark that the whole construction is readily extended to the gauge group SU(2) ⊗ SU(2) (~ SO(4)) (see, e.g., /B2/,/J2/). Extensions to other compact semi-simple gauge groups have also been considered (see /B5/).

The ansatz (3.9) has been also applied to find Minkowski space solutions of Yang-Mills equations /C3/,/B4/,/R1/ which is being studied by other methods as well (see /D1/,/L2/,/C8/).

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* This ansatz is attributed to unpublished work by Corrigan, Fairlie, Wilczek and 't Hooft. We note that originally the instanton solutions were found by Belavin, Polyakov, Schwartz and Tyupkin /B2/ without using such an ansatz.

B. Possible Applications

We intentionally did not commit ourselves to any particular physical interpretation of the classical solutions of Eq. (2.1), but have listed all solutions satisfying certain symmetry properties independent of whether they are singular or not. In this way, we had to deal with the well defined mathematical problem in its natural generality.

In the recent boom of interest in classical solutions of non-linear wave equations most physicists have in mind either finite energy particle-like solutions in Minkowski space or finite action "pseudoparticle" solutions in Euclidean 4-space. We have no reason to challenge such views. We feel, however, that the possibility for using classical solutions of conformal invariant field equations in the framework of general relativity have not been exhausted. We would like to call attention, for exapmle (in addition to the papers mentioned in the introduction) to a recent work of Hawking and others (see /H1/,/D5/,/C1,4/) where instanton-type solutions of Einstein's equation have been discussed in connection with black holes and related cosmological problems.

Note added in proof.

Apparently Eq. (2.18) gives real solutions of Eq. (1.9) only for λ whose sign coincides with that of A_2 (i.e., mostly for $\lambda < 0$ as Tables 1 and 2 show). Actually an explicit real solution with the opposite sign of λ can be written for any case indicated in Tables 1 and 2. To this end an imaginary $z(\sigma)$, , i.e, $z = iy$, , should be introduced on the way that leads to Eq. (2.18). This gives instead of Eq. (2.18)

$$\phi(x) = \sqrt{-\frac{2k^2}{\lambda} A_2^{1/3}}\,\psi(x)\, e^{-\frac{1}{3}\int_{\sigma_0}^{\sigma}\frac{A_1}{A_2}d\sigma'}\,[\mathrm{cn}(y+y_0, k')]^{-1}, \tag{2.18a}$$

$$y = -\int \frac{d\sigma}{A_2^{1/3}} e^{-\frac{1}{3}\int_{\sigma_0}^{\sigma} \frac{A_1}{A_2} d\sigma'}, \quad k' = \sqrt{1-k^2},$$

and $1-2k^2$ should be taken with the opposite sign from Tables 1 and 2.

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