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ДУБНА

E2 - 11005

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Submitted to "Reports on Mathematical Physics"

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О конформных преобразованиях в безмассовой модели Тирринга

На основании полученных в предыдущей работе решений для безмассового скалярного поля в двухмерном пространстве-времени построены поля, удовлетворяющие перенормированному уравнению Тирринга. Получены как инфинитезимальные, так и глобальные преобразования этих полей по двухмерной конформной группе. Эти преобразования не совпадают со стандартными. Доказано, что перенормированная модель Тирринга ковариантна по полученным инфинитезимальным преобразованиям конформной группы и по глобальным преобразованиям ее универсальной накрывающей.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1977

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On the Conformal Transformations in the Massless Thirring Model

On the basis of solutions for the massless scalar field in two space-time dimensions, obtained in a previous paper, the field satisfying the renormalized Thirring equation is constructed. Both infinitesimal and global transformations with respect to the two-dimensional conformal group for these fiels are obtained. The latter do not coincide with the standard ones. The renormalized Thirring equation is proved to be covariant under infinitesimal conformal transformations as well as under the global transformations belonging to the representations of the universal covering of the conformal group.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1977

INTRODUCTION

In a previous paper /1/ we have examined the free scalar massless field in two space-time dimensions. The quantum theory of that field was formulated in coordinate as well as in momentum space representation taking into account the existence of two nonzero charges. In the present paper the scalar field is used to construct the solution of the Thirring model and to exhibit the transformation properties of the latter under the two dimensional conformal group.

1. THE SCALAR MASSLESS FIELD

We first briefly remind certain basic definitions and results from paper /1/, which we use in the present paper.

The free quantum scalar massless field in two space-time dimensions satisfies the following equations:

$$\Box\phi(x) = 0, \tag{1.1}$$

$$[\phi(x), \phi(y)] = iD(x-y). \tag{1.2}$$

We note that there are two fields $\phi(x)$ and $\tilde{\phi}(x)$ (sometimes they are called "conjugate" and sometimes "dual") satisfying these equations, related between themselves by the differential equalities

$$\partial_\mu \phi(x) = -\epsilon_\mu^{\ \nu} \partial_\nu \tilde{\phi}(x) \qquad (1.3)$$

and satisfying the nonlocal commutation relation

$$[\tilde{\phi}(x), \phi(y)] = i\tilde{D}(x-y) \qquad (1.4)$$

(The explicit form of the functions $D(x-y)$ and $\tilde{D}(x-y)$ and other commutation functions, which we use in the present paper are listed in the Table at the end of the paper).

For the fields $\phi(x)$ and $\tilde{\phi}(x)$ we can define the creation and annihilation parts $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$, respectively. Starting with eqs. (1.2) and (1.4), we obtain the following commutation functions between these parts

$$[\phi^{\pm}(x), \phi^{\mp}(y)] = [\tilde{\phi}^{\pm}(x), \tilde{\phi}^{\mp}(y)] = \pm D^{\pm}(x-y), \qquad (1.5)$$

$$[\tilde{\phi}^{\pm}(x), \phi^{\mp}(y)] = \pm \tilde{D}^{\pm}(x-y). \qquad (1.6)$$

All other commutators are zero.

The solutions of eqs. (1.1)-(1.6) are found as operator valued generalized functions integrable over the space of test functions $\mathcal{H} \equiv S(R_2)$, i.e., the space of the complex infinitely differentiable rapidly decreasing functions of two variables. Besides, the solutions are supposed to satisfy the subsidiary condition that the quantities

$$a^{\pm}(0) = -i\sqrt{\frac{2}{\pi}} \int \partial_0 \phi^{\pm}(x) dx^1 , \qquad (1.7)$$

$$b^{\pm}(0) = i\sqrt{\frac{2}{2\pi}} \int \partial_0 \tilde{\phi}^{\pm}(x) dx^1 \qquad (1.8)$$

exist and that $a^+(0)$ and $a^-(0)$ (as well as $b^+(0)$ and $b^-(0)$) are not equal to each other.

The decomposition of $\phi(x)$ and $\tilde{\phi}(x)$ in $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$, respectively, is carried out by using the formula

$$\phi^{\pm}(x) = -i \int dz^1 D^{\pm}(x-z) \overleftrightarrow{\partial}{}_0^z \phi(z) \qquad (1.9)$$

(for more details see ref. [1]). With the help of eq. (1.9) one can determine the regularization of the Fourier integral for the fields $\phi(x)$ and $\tilde{\phi}(x)$ provided the regularization of the commutation functions $D^{\pm}(x)$ and $\tilde{D}^{\pm}(x)$ is given.

If we denote the Fourier transforms of $\phi^{\pm}(x)$ by $A^{\pm}(p^1)$, respectively, then the relation between the functionals of the fields in coordinate and momentum space representations is given by the equation

$$\int \phi^{\pm}(x)\Phi(x)d^2x = \sqrt{\frac{\pi}{2}} \int \frac{dp^1}{|p^1|}[A^{\pm}(p^1)f(p) - a^{\pm}(0)f(0)\theta(\kappa - |p^1|)], \tag{1.10}$$

where $\Phi(x)$ is an arbitrary function from $\mathcal{H}$,

$$f(p^1) = F(|p^1|, p^1) \in S(C_+) \tag{1.11}$$

and $F(p)$ is the Fourier transform of $\Phi(x)$

$$F(p) = \frac{1}{2\pi} \int \Phi(x) e^{-ipx} d^2x. \tag{1.12}$$

An analogous to (1.10) equation for $\tilde{\phi}(x)$ one can obtain by substituting $A^{\pm}(p) \to B^{\pm}(p)$, $a^{\pm}(0) \to b^{\pm}(0)$ into the R.H.S. of eq. (1.10) ($B^{\pm}(p)$ are the Fourier transforms of $\tilde{\phi}^{\pm}(x)$).

It is easy to see that the regularization of the Fourier integrals of the fields $\phi^{\pm}(x)$ is fixed by eq. (1.10) (and analogously for $\tilde{\phi}^{\pm}(x)$), and that this regularization is equal to that obtained in paper [1]. All results of the latter paper can be formulated in terms of functionals using formula (1.10). Since there is no any principal difficulty in such a procedure we do not discuss this problem further. Nevertheless, it seems to us that it is necessary to make a comparison with the regularizations used in certain papers [2,3]. In the latter the functional analogous to (1.10) has the form

$$\int \phi^{\pm}(x)\Phi(x)\,d^2x = \sqrt{\frac{\pi}{2}}\int \frac{dp^1}{|p^1|} A^{\pm}(p)[\,f(p)-f(0)\,]. \qquad (1.13)$$

It is easy to see that the operators of the fields $\phi^{\pm}(x)$ defined by (1.10) and (1.13) differ by the following constant operators:

$$G^{\pm} = \sqrt{\frac{\pi}{2}} \int_{-\kappa}^{\kappa} \frac{dp^1}{|p^1|}[A^{\pm}(p) - a^{\pm}(0)]. \qquad (1.14)$$

The main drawback of the regularization (1.13) is that it is noninvariant with respect to the group of translations in the two-dimensional space-time. Therefore, as is seen in /2/, the commutation relations between such fields are translationally noninvariant. At the same time the operators $G^{\pm}$ compensate the noninvariant terms, and therefore the regularization (1.10) appears to be invariant.

The operators $A^{\pm}(p^1)$ satisfy commutation relations which differ from the canonical ones by a counterterm with support at the point $p^1=q^1=0$ (p^1 and q^1 are the arguments of the commutator). In our case the operators $A^{\pm}(p^1)$ differ from usual ones by two specific features.

a. The condition that the quantities (1.7) are nonzero implies the following representation:

$$A^{\pm}(p^1) = a^{\pm}(p^1) + \epsilon(p^1)b^{\pm}(p^1), \qquad (1.15)$$

where $a^{\pm}(p^1)$ and $b^{\pm}(p^1)$ are certain new operators while $\epsilon(p^1) = \pm 1$ if $p^1>0$ or $p^1<0$, respectively, with a normalization $\epsilon(0)=0$, so that $\epsilon(p^1) = \mathcal{P}\frac{1}{p^1}$ (here $\mathcal{P}$ denotes the principal value).

b. The quantities $a^{\pm}(0)$ from eq. (1.7) should be treated as values of the operators $a^{\pm}(p)$ at $p^1=0$, i.e., $a^{\pm}(p^1)$ must have the following representation:

$$a^{\pm}(p^1) = a^{\pm}(0) + |p^1|^{\alpha}\,\tilde{c}^{\pm}(p^1), \qquad (1.16)$$

where $\mathrm{Re}\,\alpha>0$, while $c^{\pm}(p^1)$ are certain operators that tend to a constant operator when $p^1 \to 0$.

2. THE THIRRING MODEL

The fields $\phi(x)$ and $\tilde{\phi}(x)$ satisfying eqs. (1.1)-(1.6) can be used for the construction of solution for the quantum Thirring model. Such solution is constructed with the help of exponents from the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ only. The solution of this type has first been obtained in papers [3,4]. Since the regularization (1.10) and the condition for finiteness of the operators (1.7) and (1.8) lead to fields with somewhat unusual features, we want to show in this section that the fields $\phi^{\pm}(x), \tilde{\phi}^{\pm}(x)$ are indeed adjustable for construction of solution for the Thirring model. Namely, we shall proove the following statement:

Theorem: Let $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ be the quantum fields satisfying eqs. (1.1)-(1.6), defined as operator valued generalized functions with the help of eq. (1.10), then the two component quantity

$$\Psi_k(x) = e^{i(-1)^k\beta\tilde{\phi}^-(x)}\, e^{-i\alpha\phi^-(x)}\, e^{-i\alpha\phi^+(x)}\, e^{i(-1)^k\beta\tilde{\phi}^+(x)}\, u_k, \tag{2.1}$$

where

$$|u_1|^2 = |u_2|^2 = U = \frac{1}{2\pi}(\mu^2)^{\frac{\alpha^2+\beta^2}{4\pi}}$$

(μ is a constant with dimension of mass squared, related to the regularization constant κ), while

$$\alpha = \sqrt{\pi\frac{1-h}{1+h}}, \quad \beta = \sqrt{\pi\frac{1+h}{1-h}}, \quad h = \frac{g}{2\pi},$$

satisfies the renormalized quantum equation of the Thirring model

$$i\gamma^{\nu}\partial_{\nu}\Psi(x) = -g\gamma^{\nu}:J_{\nu}(x)\Psi(x): \quad , \tag{2.2}$$

where γ^{ν} are the Dirac matrices in two-dimensional space-time (f.i. $\gamma^{o}=\sigma_1$, $\gamma^{1}=i\sigma_2$, and σ are the Pauli matrices). The renormalized current is defined as a bilinear form of the fields $\Psi(x)$ in the following way:

$$\begin{aligned} J_{\nu}(x) &= \tfrac{1}{2}[j_{\nu}(x) + \tilde{j}_{\nu}(x)], \\ j_{\nu}(x) &= \lim_{\substack{\epsilon^{o}=0 \\ \epsilon^{1}\to 0}} j_{\nu}(x,\epsilon); \quad \tilde{j}_{\nu}(x) = \lim_{\substack{\epsilon^{o}=0 \\ \epsilon^{1}\to 0}} \tilde{j}_{\nu}(x,\epsilon), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} j_{\nu}(x,\epsilon) &= \tfrac{1}{2}(-\epsilon^{2})^{\frac{\alpha^2+\beta^2}{4\pi}-\frac{1}{2}} [\bar{\Psi}(x+\epsilon)\gamma_{\nu}\Psi(x) - \bar{\Psi}(x)\gamma^{o}\gamma_{\nu}\Psi^{+}(x-\epsilon)], \\ \tilde{j}_{\nu}(x,\epsilon) &= j_{\nu}(x,\tilde{\epsilon}), \quad \tilde{\epsilon} = -\epsilon^{2}, \quad \tilde{\epsilon},\epsilon = 0 . \end{aligned} \tag{2.4}$$

The proof of the statement should be carried in two stages. First one must substitute the quantity (2.1) in (2.2) and (2.3) and by direct calculation show that it is a formal solution of (2.2), and second one must prove that exponentials of the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ are well defined as generalized functions.

The first stage of the proof has two parts. We first calculate the current (2.3). For this purpose we substitute (2.1) into (2.4). If we reorder the operator exponentials for sufficiently small ϵ^{μ}, we have

$$\begin{aligned} \bar{\Psi}(x+\epsilon)\gamma_{\mu}\Psi(x) = \sum_{k=1}^{2} (-1)^{\mu(k+1)} |u_k|^{2} \exp[(\alpha^{2}+\beta^{2})D^{+}(\epsilon) - \\ - 2\pi(-1)^{k}\tilde{D}^{+}(\epsilon)][1 - i\alpha\epsilon^{\mu}\partial_{\mu}\phi(x) - i\beta(-1)^{k}\epsilon^{\mu}\partial_{\mu}\tilde{\phi}(x)]. \end{aligned} \tag{2.5}$$

If we denote

$$V_k(\epsilon) = \left(\frac{\epsilon^o - \epsilon^1 - i0}{\epsilon^o + \epsilon^1 - i0}\right)^{-(-1)^k/2}, \tag{2.6}$$

where

$$\lim_{\epsilon^o=0,\,\epsilon^1\to 0} V_k(\epsilon) = e^{i\pi(-1)^k/2} = (-1)^{(-1)^k/2}.$$

Then we have

$$j_\mu(x,\epsilon) = \frac{i\epsilon^\nu}{2\pi\sqrt{-\epsilon^2}} \sum_{k=1}^{2} (-1)^{\mu(k+1)+1} V_k(\epsilon)\,\partial_\nu [a\phi(x) + \beta(-1)^k \tilde{\phi}(x)]. \tag{2.7}$$

From the latter expression we can obtain according to eq. (2.3) the quantities $j_\mu(x)$ and $\tilde{j}_\mu(x)$

$$j_\mu(x) = \frac{i}{2\pi} \sum_{k=1}^{2} (-1)^{\mu(k+1)+1+\frac{1}{2}(-1)^k} \partial_1 F_k(x), \tag{2.8}$$

$$\tilde{j}_\mu(x) = \frac{i}{2\pi} \sum_{k=1}^{2} (-1)^{\mu(k+1)+\frac{1}{2}} \partial_0 F_k(x), \tag{2.9}$$

where

$$F_k(x) = a\phi(x) + \beta(-1)^k \tilde{\phi}(x).$$

Inserting (2.8) and (2.9) into (2.3), we obtain

$$J_\mu(x) = \frac{1}{2\pi}(a+\beta)\,\partial_\mu \phi(x). \tag{2.10}$$

Now to complete the proof we must insert the quantities $\Psi(x)$ and $J_\mu(x)$ determined by eqs. (2.1) and (2.10), respectively, into eq. (2.2). As a result we have the following relation for a and β

$$h(a+\beta) = \beta - a. \tag{2.11}$$

It is easily seen that a and β satisfy this identity, and this accomplishes the formal proof.

If we remind the difference between Klaiber's and our regularization (see section 2) the second stage of the proof is implicit. As we have seen the scalar fields of Klaiber differ from ours by the constant operator (1.14). Therefore we come to the conclusion that the quantities $:\phi^n(x):$ and $:e^{\phi(x)}:$ (the symbol :...: denotes normal ordering as usual) can be regarded as operator-valued generalized functions in the space $S(R_2)$ in our case too.

3. TRANSFORMATION PROPERTIES OF THE FIELDS $\phi(x)$ AND $\Psi(x)$

In paper /1/ the transformations of the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ under the two-dimensional Lorentz group have been found. It appeared there that the latter were nonhomogeneous with constant operator nonhomogeneous terms. Thus, one can show that the $\Psi(x)$ field is not a true spinor. Indeed if we substitute in eq. (2.1) the values of $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ obtained after a Lorentz transformation Λ_χ:

$$\phi^{\pm}(x) \xrightarrow{\Lambda_\chi} \phi^{\pm}(\Lambda_\chi x) - \frac{\chi}{2\sqrt{2\pi}} b^{\pm}(0), \qquad (3.1)$$

$$\tilde{\phi}^{\pm}(x) \xrightarrow{\Lambda_\chi} \tilde{\phi}^{\pm}(\Lambda_\chi x) + \frac{\chi}{2\sqrt{2\pi}} a^{\pm}(0) \quad \Lambda(\chi) = \begin{pmatrix} ch\chi & sh\chi \\ sh\chi & ch\chi \end{pmatrix} \qquad (3.2)$$

we obtain the transformation law for the Thirring field in the form

$$\Psi(x) \xrightarrow{\Lambda_\chi} :e^{i\chi(\alpha S+\beta\gamma^5 L)}\Psi(\Lambda_\chi x): \; , \qquad (3.3)$$

where L and S are the following operators

$$L = \frac{1}{2\sqrt{2\pi}}[a^+(0) + a^-(0)], \quad S = \frac{1}{2\sqrt{2\pi}}[b^+(0) + b^-(0)]. \qquad (3.4)$$

It is evident from this transformation law that the quantity $(\bar{\Psi}\Psi)$ is not Lorentz invariant, while the quantity $(\bar{\Psi}\gamma_\mu\Psi)$ is a true Lorentz vector.

Now we pass to the transformation properties of the fields with respect to the group of dilatations.

However, the method which we have exploited in order to obtain the representation of the Lorentz group is not useful here. The transformation laws of the dilatation group, under which the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ are transformed, we shall establish using the condition of invariance of the commutators (1.5) and (1.6). Unfortunately, the latter procedure does not determine this representation uniquely.

Going ahead for a while, we point out that when the special conformal transformations are considered this arbitrariness is ruled out. Meanwhile we fix the representation by making use of the condition for dilatation invariance of the procedure of separation the creation and annihilation parts of the field operators, i.e., we impose the condition that the equalities

$$\phi^{\pm}(x)|0> = \tilde{\phi}^{\pm}(x)|0> = 0 \qquad (3.5)$$

are invariant. This constraint implies that the non-homogeneous terms in the transformation laws for $\phi^{+}(x)$ and $\tilde{\phi}^{+}(x)$ must also annihilate the vacuum.

After these remarks it is easy to obtain the explicit form of the dilatation transformation D for the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$. Namely, if $\lambda > 0$ is the parameter of this transformation, we have

$$\phi^{\pm}(x) \xrightarrow{D\lambda} \phi^{\pm}(\lambda x) + \frac{a^{\pm}(0)}{2\sqrt{2\pi}} \ln\lambda , \qquad (3.6)$$

$$\tilde{\phi}^{\pm}(x) \xrightarrow{D\lambda} \tilde{\phi}^{\pm}(\lambda) - \frac{b\ (0)}{2\sqrt{2\pi}} \ln\lambda . \qquad (3.7)$$

The absence of a term with $b^{\pm}(0)$ in eq. (3.6) and of a term with $a^{\pm}(0)$ in eq. (3.7), is due to the fact that such terms would violate the covariant sense of these relations with respect to space parity (since $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$, as well as $a^{\pm}(0)$ and $b^{\pm}(0)$ cannot be scalars or pseudoscalars simultaneously). Inserting eqs. (3.6) and (3.7) into (2.1), we obtain the following transformation law for the fields $\Psi(x)$:

$$\Psi(x) \xrightarrow{D\lambda} : e^{i \ln\lambda (\alpha L + \beta \gamma^5 S)} \Psi(\lambda x) : . \qquad (3.8)$$

Since this transformation is obviously not the standard one, it is meaningless to assign any conformal dimension to the field $\Psi(x)$, although such a dimension appears in the Green functions. This is a very important feature, since even in a standard formulation of the operators $\Psi(x)$ (i.e., in the Hilbert space) relation (3.8) remains valid.

Having defined the representations (3.1) and (3.2), on one side, and (3.6) and (3.7), on the other, the special conformal transformations for the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ are in general determined. The meaning of this statement we illustrate by the example of the fields $\phi^{\pm}(x)$.

We first define the commutators of the generators of the representations (3.1) and (3.6) with the fields $\phi^{\pm}(x)$. For this purpose we write down any transformation in the form

$$U_g^{-1} \phi(x) U_g = T_g \phi(x), \qquad (3.9)$$

where T_g stands for the R.H.S. of any of the relations (3.1), (3.2), (3.6), or (3.7). Differentiating both sides of eq. (3.9) with respect to the parameters of the transformation, multiplying by (-i) and evaluating the result at the unit element of the group, we obtain the necessary commutation relations. In particular,

$$[M_{\mu\nu}, \phi^{\pm}(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi^{\pm}(x) + \frac{i}{2\sqrt{2\pi}} \epsilon_{\mu\nu} b^{\pm}(0), \qquad (3.10)$$

$$[D, \phi^{\pm}(x)] = i x^\mu \partial_\mu \phi^{\pm}(x) + \frac{i}{2\sqrt{2\pi}} a^{\pm}(0), \qquad (3.11)$$

where $M_{\mu\nu}$ is the generator of the two-dimensional Lorentz group

$$M_{\mu\nu} = i\epsilon_{\mu\nu} \frac{\partial U_\chi}{\partial \chi}\Big|_{\chi=0} \qquad (3.12)$$

(U_χ is the representation of this group), while D is the generator of the dilatation transformations.

If we denote the R.H.S. of eqs.(3.10) and (3.11) by $\hat{M}^{\pm}_{\mu\nu}\phi^{\pm}$ and $\hat{D}^{\pm}\phi^{\pm}$ respectively, we see that $\hat{M}^{\pm}_{\mu\nu}$ and $\hat{D}^{\pm}$ can formally be written in the form

$$\begin{aligned} \hat{M}^{\pm}_{\mu\nu} &= \hat{M}^{\circ}_{\mu\nu} + i\,\hat{\Sigma}^{\pm}_{\mu\nu}\,, \\ \hat{D}^{\pm} &= D^{\circ} + i\hat{d}^{\pm}\,, \end{aligned} \tag{3.13}$$

where $M^{\circ}_{\mu\nu}$ and D° denote the differential parts of $\hat{M}^{\pm}_{\mu\nu}$ and $\hat{D}^{\pm}$ respectively, while

$$\begin{aligned} \hat{\Sigma}^{\pm}_{\mu\nu} &= \frac{\epsilon_{\mu\nu}}{2\sqrt{2\pi}}\, b^{\pm}(0)\frac{\partial}{\partial\phi^{\pm}} \\ \hat{d}^{\pm} &= \frac{a^{\pm}(0)}{2\sqrt{2\pi}}\;\frac{\partial}{\partial\phi^{\pm}} \end{aligned} \tag{3.14}$$

(the operation $\partial/\partial\phi^{\pm}$ has sence of ordinary differentiation with respect to $\phi^{\pm}$, respectively, the latter being treated as ordinary variables).

The formal expression (3.13) provides a method for obtaining a similar expression for the generators of the special conformal transformation. Having in mind the commutator

$$[\hat{P}_{\mu}\,, \hat{K}_{\nu}] = -2i[g_{\mu\nu}\hat{D} + \hat{M}_{\mu\nu}], \tag{3.15}$$

where $\hat{P}_{\mu} = i\partial_{\mu}$, we can write down K_{μ} in the form

$$\hat{K}^{\pm}_{\mu} = \hat{K}^{\circ}_{\mu} - 2ix_{\mu}\hat{d}^{\pm} - 2ix^{\nu}\hat{\Sigma}^{\pm}_{\nu\mu}\,, \tag{3.16}$$

where

$$\hat{K}^{\circ}_{\mu} = i(x^{2}\partial_{\mu} - 2x_{\mu}x^{\nu}\partial_{\nu}). \tag{3.17}$$

Therefore the commutators of the generators of the representation of the special conformal transformations can be written as

$$[K_\mu , \phi^\pm (x)] = \hat{K}^\pm_\mu \phi^\pm (x). \tag{3.18}$$

Inserting $\hat{K}^\pm_\mu$ from eqs. (3.16) and (3.17) taking into account eq. (3.14), we finally obtain

$$[K_\mu , \phi^\pm (x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu)\phi^\pm(x) - \\ - i\frac{a^\pm(0)}{\sqrt{2\pi}} x_\mu - \frac{ix^\nu \epsilon_{\nu\mu}}{\sqrt{2\pi}} b^\pm (0). \tag{3.19}$$

The corresponding commutator with the fields is obtained analogously

$$[K_\mu , \tilde{\phi}^\pm (x)] = i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu)\tilde{\phi}^\pm(x) + \\ + \frac{ib^\pm(0)}{\sqrt{2\pi}} x_\mu + i\frac{x^\nu \epsilon_{\nu\mu}}{\sqrt{2\pi}} a^\pm (0). \tag{3.20}$$

In order to reconstruct the global representations we note that the special conformal transformations are decomposed into two transformations. Let

$$x_+ = x^0 + x^1 \qquad \text{and} \qquad x_- = x^0 - x^1$$

then these coordinates transform as follows

$$x_+ \xrightarrow{K\delta} \frac{x_+}{\rho^+(\delta,x_+)} \qquad \text{and} \qquad x_- \xrightarrow{K\delta} \frac{x_-}{\rho^-(\delta,x_-)}, \tag{3.21}$$

where

$$\rho^\pm (\delta, x_\pm) = 1 + (\delta^0 \mp \delta^1)x_\pm \tag{3.22}$$

and δ^μ are the parameters of the special conformal transformations.

Since ρ^+ and ρ^- are transposed under space parity transformations, it is evident that the quantities

$$\ln|\rho^+\rho^-| \equiv \ln|\rho|\ ;\ \rho(\delta,x) = \rho^+(\delta,x)\rho^-(\delta,x) \tag{3.23}$$

and

$$\ln|\frac{\rho^+}{\rho^-}| \equiv \ln|\sigma|\ ;\ \sigma(\delta,x) = \frac{\rho^+(\delta,x_+)}{\rho^-(\delta,x_-)} \tag{3.24}$$

are a scalar and a pseudoscalar, respectively. Therefore the general form of the special conformal transformation for the fields $\phi^{\pm}(x)$ must be the following:

$$\phi^{\pm}(x) \xrightarrow{K_\delta} \phi\left(\frac{x_\mu+\delta_\mu x^2}{\rho(\delta,x)}\right) + c_1 a^{\pm}(0)\ln|\rho| + c_2 b^{\pm}(0)\ln|\sigma| \tag{3.25}$$

In view of eq. (3.19) the constants c_1 and c_2 are uniquely determined. For the fields $\tilde{\phi}^{\pm}(x)$ an analogous formula takes place, but one must interchange $a^{\pm}(0)$ and $b^{\pm}(0)$. Finally, we have

$$\phi^{\pm}(x) \xrightarrow{K_\delta} \phi^{\pm}\left(\frac{x_\mu+\delta_\mu x^2}{\rho(\delta,x)}\right) - \frac{a^{\pm}(0)}{2\sqrt{2\pi}}\ln|\rho(\delta,x)| + \frac{b^{\pm}(0)}{2\sqrt{2\pi}}\ln|\sigma(\delta,x)|, \tag{3.26}$$

$$\tilde{\phi}^{\pm}(x) \xrightarrow{K_\delta} \tilde{\phi}^{\pm}\left(\frac{x_\mu+\delta_\mu x^2}{\rho(\delta,x)}\right) + \frac{b^{\pm}(0)}{2\sqrt{2\pi}}\ln|\rho(\delta,x)| - \frac{a^{\pm}(0)}{2\sqrt{2\pi}}\ln|\sigma(\delta,x)|. \tag{3.27}$$

To accomplish the proof that the above transformations provide a representation of the two-dimensional conformal group, one must check the commutation relations between the generators. Since the latter is an elementary procedure we ommit it here.

Using eqs. (2.1), (3.26) and (3.27) we can obtain the corresponding global transformations for the field $\Psi(x)$:

$$\Psi(x) \xrightarrow{K_\delta} :|\rho(\delta,x)|^{i(\alpha L+\beta\gamma^5 S)}\,|\sigma(\delta,x)|^{i(\alpha S+\beta\gamma^5 L)}\ \Psi\left(\frac{x_\mu+\delta_\mu x^2}{\rho(\delta,x)}\right): \tag{3.28}$$

A representation similar to (3.26) has been considered in paper /7/, but there the term with $\ln|\sigma(\delta,x)|$ (in our notation) has been missed.

4. THE COVARIANCE OF THE THIRRING EQUATION

The representations of the two-dimensional conformal group, constructed in the previous section, have the remarkable feature that the Thirring equation (2.2) is covariant with respect to the corresponding infinitesimal transformations. In this section we prove this fact, and find those representations of the universal covering group of the two-dimensional conformal group for which the Thirring equation is globally covariant.

The Lorentz covariance of eq. (2.2) as well as its covariance with respect to the dilatation transformations (3.8) are not only infinitesimal but global also. This can be checked straightforwardly. However, we will proceed in a different manner. It is sufficient to convince oneselves, that the commutation relations (1.5) and (1.6) between the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ are invariant with respect to the given representations. Since eq. (1.1) is also covariant we come to the conclusion that the transformed fields $(\phi^{\pm}(x))'$ and $(\tilde{\phi}^{\pm}(x))'$ satisfy the same equations as the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$. Then according to the second section, the quantity $(\Psi(x))'$ constructed out of $(\phi^{\pm}(x))'$ and $(\tilde{\phi}^{\pm}(x))'$, i.e., the transformed Thirring field, satisfies eq. (2.2) also. The latter proves the covariance of this equation under the Lorentz and scale transformations.

We apply the same method to prove the covariance with respect to the special conformal transformations. Namely, if we prove that the transformed fields $(\phi^{\pm}(x))'$ and $(\tilde{\phi}^{\pm}(x))'$ satisfy the same equations (1.1)-(1.6) that are satisfied by $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$, then according to the statement of the second section, the transformed field $(\Psi(x))'$ (it is obtained from (2.1) by substituting the transformed fields $(\phi^{\pm}(x))'$ and $(\tilde{\phi}^{\pm}(x))'$) satisfies equation (2.2) also. And this again proves its covariance.

Equation (1.1) is covariant with respect to the global transformations (3.26) and (3.27). Indeed, we note the following equalities

$$\ln|\rho(\delta,x)| = \ln|\rho^{+}(\delta,x_{+})| + \ln|\rho^{-}(\delta,x_{-})|,$$
$$\ln|\sigma(\delta,x)| = \ln|\rho^{+}(\delta,x_{+})| - \ln|\rho^{-}(\delta,x_{-})| \quad (4.1)$$

and therefore

$$\Box \ln|\rho(\delta,x)| = \Box \ln|\sigma(\delta,x)| = 0. \quad (4.2)$$

Thus the transformed fields satisfy eq. (1.1).

Let us pass to the commutators (1.5) and (1.6). We must note that they are not invariant with respect to the global transformations (3.26) and (3.27). Therefore we consider the infinitesimal transformations

$$(\phi^{\pm}(x))' = \phi^{\pm}(x) - i\alpha^{\mu}[K_{\mu}, \phi^{\pm}(x)], \quad (4.3)$$

$$(\tilde{\phi}^{\pm}(x))' = \tilde{\phi}^{\pm}(x) - i\alpha^{\mu}[K_{\mu}, \tilde{\phi}^{\pm}(x)]. \quad (4.4)$$

We now evaluate the commutators of the transformed fields. We have, f.i.

$$[(\phi^{+}(x))', (\phi^{-}(x))'] = [\phi^{+}(x), \phi^{-}(y)] -$$
$$- i\alpha^{\mu}[[K_{\mu}, \phi^{+}(x)], \phi^{-}(y)] - i\alpha^{\mu}[\phi^{+}(x), [K_{\mu}, \phi^{-}(y)]]. \quad (4.5)$$

Making use of eq. (3.19) and of the identity

$$\{(x^2 - y^2)(\alpha,\partial) - 2[(\alpha x)(x.\partial) - (\alpha.y)(y.\partial)]\}D^{+}(x-y) = \frac{\alpha.(x+y)}{2\pi}. \quad (4.6)$$

We see that the second and third terms in the R.H.S. of equality (4.5) cancel and therefore the commutator is invariant. One can analogously prove the invariance of all other commutators. In the case of the commutator $[\tilde{\phi}^{+}(x), \phi^{-}(y)]$ one must use the analogous to eq. (4.6) identity for the $\tilde{D}^{+}(x-y)$ function, which has the form

$$\{(x^2-y^2)(\alpha.\partial)-2[((\alpha x)(x.\partial)-(\alpha y)(y\partial)]\}\tilde{D}^+(x-y)=\frac{\alpha^\mu}{2\pi}\epsilon_{\mu\nu}(x+y)^\nu : \quad (4.7)$$

Here as in the identity (4.6) the following notations are current $(\alpha\beta)=\alpha^\mu\beta_\mu$, and ∂_μ is the differentiation with respect to the full argument of the D-functions.

Thus, we have proved the covariance of eq. (2.2) with respect to the infinitesimal transformations of the two-dimensional conformal group.

The method used to prove the previous statement shows how to obtain the global transformations of the fields, under which eq. (2.2) would be covariant. For this purpose we must note that the transformations of the quantities x_+ and x_- (3.21) and (3.22) belong to the SL(2, R) group, and that they must be treated as the limits of the corresponding afine transformations of the lower complex half-plane, when they tend to the real axis. Then the function

$$\mathfrak{T}(z)=-\frac{1}{4\pi}\ln(-\mu^2 z_+ z_-) \quad (4.8)$$

defined on the lower complex half-plane of the complex variables z_+ and z_- transforms under these transformations as

$$\mathfrak{T}'(z)\equiv\mathfrak{T}(z')=\mathfrak{T}(z)+\frac{1}{4\pi}\ln\rho^+(\delta,z^+)\rho^-(\delta,z^-). \quad (4.9)$$

When $\mathrm{Im}\, z_\pm\to 0$ we have

$$D(z)\to D^+(x^\circ+i0,x^1) \quad \text{and} \quad D'(z)\to D^{+\prime}(x^\circ+i0,x^1) \quad (4.10)$$

and therefore

$$D^{\pm\prime}(x^\circ+i0,x^1)=D^+(x^\circ+i0,x^1)+\frac{1}{4\pi}\ln\rho(x^\circ+i0,x^1,\delta).(4.11)$$

We have analogously

$$\tilde{D}^{+\prime}(x^\circ+i0,x\;)=\tilde{D}^+(x^\circ+i0,x\;)+\frac{1}{4\pi}\ln\sigma(\delta,x^\circ+i0,x\;),(4.12)$$

where $\rho(\delta, x)$ and $\sigma(\delta, x)$ have been defined in the previous section.

Equalities (4.11) and (4.12) show that if in formulae (3.26), (3.27) and (3.28) we write $\sigma(\delta, x^0 + i0, x^1)$ and $\sigma(\delta, x^0 + i0, x^1)$ instead of $|\rho(\delta, x^0, x^1)|$ and $|\sigma(\delta, x^0, x^1)|$, respectively, we shall obtain just these global transformations under which equation (2.2) is covariant. It is easy to show that the so-obtained transformations belong to the universal covering group of the two-dimensional conformal group.

In conclusion the authors express their deep gratitude to Prof. I.T.Todorov for his interest in this work and for the numerous usefull discussions. We are grateful to Drs. M.I.Palchik and P.Koulish for interest in this paper as well as in the work /1/. Thanks are also to the participants of the seminar of the Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research.

Table of Commutation Functions

$$D(x) = \frac{1}{2\pi i} \int d^2p\, \epsilon(p^0)\delta(p^2)\, e^{-ipx} =$$

$$= -\frac{1}{2}[\theta(x^1 + x^0) - \theta(x^1 - x^0)],$$

$$D^{\pm}(x) = \pm \frac{1}{4\pi} \int \frac{dp^1}{|p^1|}[e^{\mp ipx} - \theta(\kappa - |p^1|)]_{p^0 = |p^1|} =$$

$$= \mp \frac{1}{4\pi} \ln(-\mu^2 x^2 \pm i0x^0) =$$

$$= \mp \frac{1}{4\pi} \ln|\mu^2 x^2| - \frac{i}{4}\epsilon(x^0)\theta(x^2),$$

(Cont.)

$$\tilde{D}(x) = -\frac{1}{2\pi i}\int d^2p\,\epsilon(p^1)\delta(p^2)e^{-ipx} =$$

$$= -\frac{1}{2}[\theta(x^1+x^\circ) - \theta(x^1 - x^\circ)],$$

$$\tilde{D}^{\pm}(x) = \mp\frac{1}{4\pi}\mathcal{P}\int\frac{dp^1}{p^1}(e^{\mp ipx})_{p^\circ = |p^1|} =$$

$$= \pm\frac{1}{4\pi}\ln\frac{x^\circ - x^1 \mp i0}{x^\circ + x^1 \mp i0} =$$

$$= \pm\frac{1}{4\pi}\ln\left|\frac{x^\circ - x^1}{x^\circ + x^1}\right| - \frac{i}{4}\epsilon(x)\theta(-x^2).$$

Here $\mathcal{P}$S denotes the principal value of the integral, and

$$\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \qquad \epsilon(x) = \theta(x) - \theta(-x).$$

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Received by Publishing Department on October 7, 1977.