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О свободном скалярном поле с нулевой массой в двухмерном пространстве-времени

Найдены решения уравнений квантовой теории свободных скалярных полей в двухмерном пространстве-времени. Показано, что эти решения не могут обращаться в нуль на пространственно-подобной бесконечности, что приводит к возникновению двух сохраняющихся операторов заряда. Изучены трансформационные свойства этих решений по отношению к двухмерной группе Лоренца.

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On the Free Scalar Massless Field in the Two-Dimensional Space-Time

In the present paper the solutions of the quantum field problem for the free scalar massless field in the two-dimensional space time are constructed. It is shown that the fields obtained cannot vanish at space-like infinity. The latter fact implies the existence of two conserved charge operators. The transformation properties of these solutions under the two-dimensional Lorentz-group are examined.

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INTRODUCTION

The scalar zero-mass quantum field in two space-time dimensions plays an essential role in the solution of the Thirring model. As is known [1], such a field does not exist in the conventional sense. Non-trivial operators for such a field can be intoduced only in an indefinite metric Hilbert space. However, as is seen from papers [2-4] this is not the only difference between the two-dimensional and the four-dimensional scalar fields. The existence of a non-zero charge, whose sense neighbours that of the topological charge, in two dimensions is the most essential difference from the case of four dimensions. The spontaneous breakdown of gauge invariance in the Thirring model is a corollary of the existence of such a charge. For the first time this breakdown was observed in papers [2-4], but these articles contain a mistake which did not permit their author to obtain one more charge (i.e., roughly speaking the vacuum is charged by two charges, rather than one as it is in papers [2-4]). In order to understand the situation we first define the quantum scalar field in two dimensions. Such a field satisfies the two-dimensional wave equation

$$\Box\phi = 0 \qquad (\Box = \partial_0^2 - \partial_1^2) \tag{1.1}$$

and the commutation relation

$$[\dot{\phi}(x), \phi(y)] = iD(x-y), \tag{1.2}$$

where D(x) is a commutation function

$$D(x) = \frac{1}{2\pi i} \int dp^2 \epsilon(p^0)\delta(p^2)e^{-ipx} =$$
$$= -\frac{1}{2}[\theta(x^1+x^0) - \theta(x^1-x^0)]. \tag{1.3}$$

The special form of the commutator function implies the following statement (see remark at the end of the paper).

For fields, satisfying (1.2), the quantity

$$Q = \int_{-\infty}^{\infty} dx^1 \partial \Phi(x^0, x^1) \tag{1.4}$$

cannot be equal to zero.

Indeed, if we differentiate first eq. (1.2) with respect to x^0 and then integrate with respect to x^1 in the region from $(-\infty, \infty)$ we obtain

$$[Q, \phi(y)] = -i. \tag{1.5}$$

We now introduce the field (we call it conjugate, for convinience):

$$\tilde{\phi}(x) = \int_{-\infty}^{x^1} dz^1 \partial_0 \phi(x^0, z^1) \tag{1.6}$$

If this integral exists it can easily be checked that the conjugate field satisfies eq. (1.1) and the commutation relation (1.2), and therefore the integral

$$\tilde{Q} = \int_{-\infty}^{\infty} dx^1 \, \partial_0 \phi(x^0, x^1) \tag{1.7}$$

should be non-zero too. It is the quantity (1.7) that is missed in papers /2,3/.

That is why in the present paper we shall try to find (at first classical) such solutions of equation (1.1), for which there exist finite non-zero charges (1.4) and (1.7). We insist that these are the solutions, which should be quantized. We do not impose the condition for finiteness of the integral (1.6) because it appears to confine unnecessary the class of possible solutions of (1.1). We define the conjugate field by the following equation:

$$\tilde{\phi}(x) = \int_{-\infty}^{x^1} dz^1 \partial_0 \phi(x^0, z^1) + R(x^0), \tag{1.8}$$

where $R(x^0)$ is a subtraction, necessary for the regularization of the integral in the R.H.S. of equations (1.8). We would suppose that $R(x^0)$ is such that between the fields $\phi(x)$ and $\tilde{\phi}(x)$ there takes place the differential relation

$$\partial_\mu \phi(x) + \epsilon_{\mu\nu} \partial^\nu \tilde{\phi}(x) = 0, \tag{1.9}$$

where $\epsilon_{01} = -\epsilon_{10} = 1$, $\epsilon_{00} = \epsilon_{11} = 0$ (the metric tensor in the two-dimensional space-time is $g_{00} = -g_{11} = 1$, $g_{01} = g_{10} = 0$). In a quantum theory the new definition of $\tilde{\phi}(x)$ (1.8) implies a change of the commutation relation between $\tilde{\phi}(x)$ and $\phi(x)$. That is why we now establish their form.

If both sides of eq. (1.2) are differentiated with respect to x^0 and then integrated over x^1 from $-\infty$ to $+\infty$, having in mind (1.8), we obtain

$$[\tilde{\phi}(x),\phi(y)]-[R(x^0),\phi(y)]= = i\int_{-\infty}^{x^1-y^1} dz^1 \partial D(x^0-y^0,z^1). \tag{1.10}$$

The integral in the R.H.S. of the upper equation is convergent. If we introduce the function

$$\tilde{D}(x)=-\frac{1}{2\pi i}\int d^2p\,\epsilon(p)\delta(p^1)e^{-ipx}=-\frac{1}{2}[\theta(x^0+x^1)-\theta(x^0-x^1)] \tag{1.11}$$

(following papers [2,3]), then

$$\int_{-\infty}^{x^1} \partial_0 D(x^0,z^1)dz^1=\tilde{D}(x)-\frac{1}{2}. \tag{1.12}$$

It is easy to show that the commutator $[R(x^0),\phi(y)]$ is a constant. Indeed, having in mind translational invariance, we can write

$$[R(x^0),\phi(y)]=d(x^0-y^0). \tag{1.13}$$

(We certainly use the assumption that the commutators of two free fields are C-numbers). Using eq. (1.10) it is easy to obtain that

$$[\tilde{\phi}(x^0,x^1),\phi(y)]|_{x^1\to-\infty}=[R(x^0),\phi(y)], \tag{1.14}$$

and, therefore, instead of (1.13) we can write

$$d(x^0-y^0)=[\tilde{\phi}(x^0,x^1),\phi(y)]|_{x^1\to-\infty}. \tag{1.15}$$

Now, using relations (1.9) and (1.3), we can write down the following chain of equalities:

$$\partial_0^x d(x^0-y^0)=[\partial_0\tilde{\phi}(x^0,-\infty),\phi(y)]=$$

$$=\partial_1^x[\phi(x^0,x^1),\phi(y)]|_{x^1\to-\infty}=i\partial_1^x D(x^0-y^0,x^1-y^1)|_{x^1\to-\infty}=0.$$

Therefore

$$[R(x^0),\phi(y)=d=\text{const} \tag{1.16}$$

which proves our statement. Then it is implicit from (1.10) and (1.12) that

$$[\tilde{\phi}(x),\phi(y)]=i\tilde{D}(x-y)-\frac{i}{2}+d. \tag{1.17}$$

It is obvious, that the constant d is arbitrary and depends on the choice of arbitrariness of the regularization of the integral in the R.H.S. of eq. (1.8).

Now we shall show that the field $\tilde{\phi}(x)$ defined by (1.8) satisfies the commutation relation (1.2). Starting from eq. (1.17) and operating analogously to the way eq. (1.10) was obtained, we can derive

$$[\tilde{\phi}(x),\tilde{\phi}(y)]-[\tilde{\phi}(x),R(y^0)]=iD(x-y), \tag{1.18}$$

where we have made use of the relation

$$\int_{-\infty}^{x^1} dz^1\partial_0\tilde{D}(x^0,z^1)=D(x). \tag{1.19}$$

Now we shall show that

$$[\tilde{\phi}(x),R(y^0)]=0. \tag{1.20}$$

Indeed, since we have already proved eq.

(1.16), it is implicit that

$$[\tilde{\phi}(x), R(y^0)] - [R(x^0), R(y^0)] = 0 . \tag{1.21}$$

If we now use the equality

$$\partial_0 R(x^0) = \partial_1 \phi(x^0, x^1) |_{x^1 \to -\infty} \tag{1.22}$$

(which is the necessary condition for $R(x^0)$ in order that eq. (1.9) holds), it is easy to obtain that

$$[R(x^0), R(y^0)] = 0 .$$

Then eqs. (1.21) and (1.19) imply

$$[\tilde{\phi}(x), \tilde{\phi}(y)] = iD(x-y), \tag{1.23}$$

i.e., our statement is proved.

CLASSICAL SOLUTION

In this section we construct the classical solutions of eq. (1.1) with allowing for the existence of the non-zero charges (1.4) and (1.7). It is obvious from relations (1.9) that solutions, which vanish on the space-like infinity, are excluded. Indeed, if, for instance, the field $\phi(x)$ vanishes at $x \to \pm\infty$ then the quantity (1.7) should be equal to zero.

As it is known, the general solution of eq. (1.1) can be written in the form

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dp^1}{2|p^1|} A^+(p^1) e^{-ipx} + \frac{1}{\sqrt{2\pi}} \int \frac{dp^1}{2|p^1|} A^-(p^1) e^{ipx} + R , \tag{2.1}$$

where $p^0 = |p^1|$, while $A^+(p^1) = \overline{A^-(p^1)}$ are arbitrary functions (the line denotes complex conjugation) and R is a subtraction constant (the necessity of the latter will be seen in what follows). As is well known the decomposition into positive- and negative-frequency parts of the field $\phi(x)$ can be produced using the following formulae:

$$\phi^{\pm}(x) = -i \int dz^1 D(x-z) \overleftrightarrow{\partial}_0^z \phi(z), \qquad (2.2)$$

where $A \overleftrightarrow{\partial}_0 B = \partial_0 A . B - A \partial_0 B$ (see Appendix) and $D^{\pm}(x-z)$ are the frequency parts of the commutation functions $D(x-z)$. The decomposition onto $D^{\pm}(x-z)$, however, needs a regularization of integrals, and therefore,

$$D^{\pm}(x) = \pm \frac{1}{4\pi} \int \frac{dp^1}{|p^1|} [e^{\mp ipx} - \theta(\kappa - |p^1|)]_{p^0 = |p^1|}. \qquad (2.3)$$

Now we insert (2.1) into (2.2) (for definiteness we choose sign + in formula (2.2)). After simple calculations, we obtain

$$\phi^+(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dp^1}{2|p^1|} A^+(p^1) e^{-ipx} + \qquad (2.4)$$

$$+ \frac{1}{4\sqrt{2\pi}} \int \frac{dp^1}{|p^1|} \theta(\kappa - |p^1|)[A^-(0) - A^+(0)] + \frac{1}{2} R.$$

It is evident from this result, that if $A^-(0) - A^+(0) \neq 0$ * the integrals in formulae

*The case $A^-(0) - A^+(0) = 0$ is excluded. Indeed in the latter $R = 0$, the integrals (2.1) are convergent, and it is easy to show that the quantity Q (see (1.4)) equals zero.

(2.1) are logarithmically divergent, and therefore, a regularization is needed. In this case

$$R = -\frac{1}{2\sqrt{2\pi}}\int\frac{dp^1}{|p^1|}\theta(\kappa-|p^1|)[A^+(0)+A^-(0)], \quad (2.5)$$

and instead of (2.4) we can write

$$\phi^{\pm}(x) = \frac{1}{\sqrt{2\pi}}\int\frac{dp^1}{2|p^1|}[A^{\pm}(p^1)e^{\mp ipx} - A^{\pm}(0)\theta(\kappa-|p^1|)]. \quad (2.6)$$

Formula (2.6) can be used to obtain the asymptotic behaviour of $\phi^{\pm}(x)$ (and therefore the field $\phi(x)$ at the space-like infinity. For this purpose in the first term of the R.H.S. of eq. (2.6) we insert

$$A^{\pm}(p^1) = A^{\pm}(p^1) - A^{\pm}(0) + A^{\pm}(0) \quad (2.7)$$

and after rearrangement, we obtain

$$\phi^{\pm}(x) = \frac{1}{2\sqrt{2\pi}}\int\frac{dp^1}{|p^1|}[A^{\pm}(p^1) - A^{\pm}(0)]e^{\mp ipx} \pm \pm\sqrt{2\pi}\,A^{\pm}(0)D^{\pm}(x). \quad (2.8)$$

It is easy to show that the integral in the R.H.S. of eq. (2.8) is finite at infinity, and therefore, the leading asymptotic behaviour of the field $\phi^{\pm}(x)$ is given by the second term in the R.H.S. of these equations:

$$\phi^{\pm}(x)|_{x^1\to\pm\infty} \sim -\frac{A^{\pm}(0)}{2\sqrt{2\pi}}\ln\mu^2|x^1|^2 \quad (2.9)$$

(μ - is a renormalization constant associated with the constant κ from eq. (2.3): $\mu = e^{-\Gamma'(1)}\kappa$).

It is obvious that in order to obtain eq. (2.4) we have supposed that the quantities $A^{\pm}(0)$ exist, thus choosing such solutions of eq. (1.1) which have the weakest growth at the space-like infinity. We note that such a growth still allows the existence of formula (2.2).

Having in mind eq. (2.6) it is easy to obtain the form of the corresponding conjugate field. For this purpose we use the differential relation (1.9) and obtain

$$\tilde{\phi}^{\pm}(x) = \frac{-1}{\sqrt{2\pi}} P \int \frac{dp^1}{2p^1} A^{\pm}(p^1) e^{\mp ipx} + \tilde{R}^{\pm}. \quad (2.10)$$

If $A^{\pm}(0)$ is finite (as we have assumed), then the integral (2.7) is convergent at $p^1 = 0$, if $A^{\pm}(p^1)$ is a smooth function at this point. If however $A^{\pm}(p^1)$ jumps at $p^1 = 0$ a regularization is needed and $\tilde{R}^{\pm}$ is the regularization constant.

It can be shown, that in the case when $A^{\pm}(p^1)$ is a smooth function at $p^1 = 0$ then the integral (1.7) equals zero. Therefore, we set

$$A^{\pm}(p^1) = a^{\pm}(p^1) + \epsilon(p^1) b^{\pm}(p^1) \quad (2.11)$$

and shall suppose

$$A^{\pm}(0) = a^{\pm}(0). \quad (2.12)$$

Here $\epsilon(p^1) = 1$ if $p^1 > 0$ $\epsilon(p^1) = -1$ if $p^1 < 0$. At times, it is convenient to assume that $\epsilon(0) = 0$. Moreover we shall assume that the functions $a^{\pm}(p^1)$ and $b^{\pm}(p^1)$ are independent and smooth at $p^1 = 0$. Inserting (2.11) in (2.10) we see that $\tilde{R}^{\pm}$ should have the following form :

$$\tilde{R}^{\pm} = \frac{b(0)}{2\sqrt{2\pi}} \int \frac{dp^1}{|p^1|} \theta(\epsilon - |p^1|) \quad (2.13)$$

(following from the equality $p^1 \epsilon(p^1) = |p^1|$).

Thus, we finally obtain the following solutions of eq. (1.1):

$$\phi^{\pm}(x) = \frac{1}{2\sqrt{2\pi}} \int dp^1 \{ \frac{a^{\pm}(p^1) e^{\mp ipx} - a^{\pm}(0)\, \theta(\kappa - |p^1|)}{|p^1|} + \qquad (2.14)$$

$$+ P \frac{b^{\pm}(p^1)}{p^1} e^{\mp ipx} \}$$

and analogously for the conjugate field

$$\tilde{\phi}^{\pm}(x) = \frac{-1}{2\sqrt{2\pi}} \int dp^1 \{ \frac{b^{\pm}(p^1) e^{\mp ipx} - b^{\pm}(0)\,\theta(\kappa - |p^1|)}{|p^1|} + P \frac{a^{\pm}(p^1)}{p^1} e^{\mp ipx} \} . \qquad (2.15)$$

Now it is easy to calculate the charges (1.4) and (1.7). Doing the proper substitutions we obtain after simple calculations

$$Q = i \sqrt{\frac{\pi}{2}} [a^-(0) - a^+(0)] ,$$

$$\tilde{Q} = -i \sqrt{\frac{\pi}{2}} [b^-(0) - b^+(0)] . \qquad (2.16)$$

QUANTIZATION

In this section we introduce the second quantization of the fields (2.14) and (2.15). For this purpose it is convenient to use formula (2.6). From the latter it is easy to write down the inverse Fourier transform in the following form:

$$A^{\pm}(p^1) = \pm i \sqrt{\frac{2}{\pi}} \int \partial_0 \phi^{\pm}(x)\, e^{\pm ipx}\, dx^1 \qquad (3.1)$$

and in view of eq. (1.2) we obtain the following commutator for $A^+(p^1)$ and $A^-(q^1)$

$$[A^+(p^1), A^-(q^1)] = 2|p^1|\delta(p_1^1 - q^1). \tag{3.2}$$

But, it is easy to see that if we now calculate once more the commutator (1.2), we obtain under the integral the quantity

$$\frac{[A^+(p^1), A^-(q^1)]}{|p^1||q^1|}. \tag{3.2}$$

It is obvious that we cannot use eq. (3.2) in order to determine it, since the equation

$$|p^1||q^1| f(p^1, q^1) = 0 \tag{3.3}$$

has nontrivial solutions. Indeed, suppose that $f(p^1, q^1)$ is a solution of (3.3). Then in general, we have

$$\frac{[A^+(p^1), A^-(q^1)}{|p^1||q^1|} = \frac{2}{|p^1|}\delta(p^1 - q^1) + f(p^1, q^1). \tag{3.4}$$

So, we see that we need not formula (3.2) but formula of the type (3.4), and its determination is equivalent to the determination of the function $f(p^1, q^1)$. We first of all notice that an arbitrary solution of equation (3.3) has the form

$$f(p^1, q^1) = h\delta(p^1)\delta(q^1) + g_1(q^1)\phi(p^1) + g_2(p^1)\delta(q^1). \tag{3.5}$$

Therefore, from (3.4) we can uniquely determine the commutator of $A^+(p^1)$ and $A^-(0)$ (or $A^+(0)$ and $A^-(q^1)$). Namely,

$$\frac{1}{|p^1|}[A^+(p^1), A^-(0)] = 2\delta(p^1). \tag{3.6}$$

In order to obtain the function $f(p^1, q^1)$ from (3.5) we first calculate the integral

$$\int \phi^+(x)\, e^{ipx}\, dx^1 .$$

Inserting $\phi^+(x)$ from (2.6), it is easy to obtain

$$\int \phi^+(x)\, e^{+ipx}\, dx^1 = \sqrt{\frac{\pi}{2}} \left[\frac{A^+(p^1)}{|p^1|} - A^+(0)\, \delta(p^1) \int \theta(\kappa - |q^1|) \frac{dq^1}{|q^1|} \right] \qquad (3.7)$$

or

$$\frac{A^+(p^1)}{|p^1|} = \sqrt{\frac{2}{\pi}} \int \phi^+(x)\, e^{ipx}\, dx^1 + A^+(0)\, \delta(p^1) \int \theta(\kappa - |q^1|) \frac{dq^1}{|q^1|} . \qquad (3.8)$$

Using formulae (3.1) we can also calculate the commutator of $A^-(p^1)$ and $\sqrt{\frac{2}{\pi}} \int \phi^+(x)\, e^{ipx}\, dx^1$. As a result we have

$$\left[\sqrt{\frac{2}{\pi}} \int \phi^+(x) e^{ipx}\, dx ,\, A^-(q^1) \right] = 2\delta(p^1 - q^1) . \qquad (3.9)$$

Now we are ready to find the explicit form of (3.2). For that purpose we insert $\frac{A^+(p^1)}{|p^1|}$ from eq. (3.8). Using relations (3.6) and (3.9) we finally obtain

$$\frac{[A^+(p^1), A^-(q^1)]}{|p^1|\,|q^1|} = \frac{2}{|q^1|} \delta(p^1 - q^1) + 2\delta(p^1) \Delta(q^1) \int \frac{dk^1}{|k^1|} \theta(\kappa - |k^1|) . \qquad (3.10)$$

Eq. (3.8) gives also the possibility to calculate the commutator of $A^+(0)$ and $A^-(0)$. Indeed the following equality is readily verified:

$$\frac{[A^{+}(p^{1});A^{-}(0)]}{|p^{1}|} =$$

$$= 2\delta(p^{1}) + [A^{+}(0),A^{-}(0)]\delta(p^{1}) \int \frac{dk^{1}}{|k^{1}|}\,\theta(\kappa - |k^{1}|). \qquad (3.11)$$

Comparing the latter with formula (3.6), we see that

$$[A^{+}(0),\ A^{-}(0)] = 0. \qquad (3.12)$$

In order to accomplish the quantization of the fields $A^{\pm}(p^{1})$ we must add also that the equal frequency fields are commuting. For the analogous quantization of the fields $\tilde{\phi}(x)$ we use the following formal procedure. If we write the field (2.15) in the form

$$\tilde{\phi}^{\pm}(x) = -\frac{1}{2\sqrt{2\pi}} \int \frac{dp^{1}}{|p^{1}|} [B^{\pm}(p^{1})e^{\mp ipx} - B^{\pm}(0)\,\theta(\kappa - |p^{1}|)], \qquad (3.13)$$

then comparing (3.13), (2.11) and (2.12) we see that

$$B^{\pm}(p^{1}) = b^{\pm}(p^{1}) + \epsilon(p^{1})\,a^{\pm}(p^{1}) = \epsilon(p^{1})A^{\pm}(p^{1}). \qquad (3.14)$$

Since $\epsilon(p^{1})$ enters the integrals by means of principal value (i.e., with $p^{1}=0$ point dropped) we must formally set

$$\epsilon(p^{1})\,\delta(p^{1}) = 0. \qquad (3.15)$$

On the other hand, as the generalized function $\epsilon^{2}(p^{1})$ does not differ from unity, therefore the expression $\epsilon(p)\epsilon(q)\,\delta(p)\,\delta(q)$ must be formally defined as

$$\epsilon(p)\,\delta(p)\,\epsilon(q)\delta(q) = \epsilon(p)\epsilon(q)\delta(p)\delta(p-q) = \epsilon^{2}(p)\delta(p)\delta(q) =$$
$$= \delta(p)\delta(q). \qquad (3.16)$$

Now starting from eq. (3.10) and multiplying it by $\epsilon(p^1)$ and $\epsilon(q^1)$, with taking into account relations (3.14), (3.15) and (3.16), we can obtain the following commutators

$$\frac{1}{|p^1||q^1|}[B^+(p^1), B^-(q^1)] = \frac{2}{|q^1|}\delta(p^1-q^1) + 2\delta(p^1)\delta(q^1)\int\theta(\kappa-|k|)\frac{dk}{|k|}, \quad (3.17)$$

$$\frac{1}{|p^1||q^1|}[A^+(p^1), B^-(q^1)] = \frac{1}{|p^1||q^1|}[B^+(p^1), A^-(q^1)] = P\frac{2}{p^1}\delta(p^1-q^1).$$

From the latter we can easily obtain the rest of the commutators

$$\frac{1}{|q^1|}[A^+(0), B^-(q^1)] = \frac{1}{|p^1|}[A^+(p^1), B^-(0)] = 0$$
$$\frac{1}{|p^1|}[B^+(p^1), B^-(0)] = 2\delta(p^1). \quad (3.18)$$

As in the previous case the equal frequency fields are commuting, so are the operators $A^{\pm}(0)$ and $B^{\pm}(0)$ which in view of their definitions (2.12) and (3.14) ($B^{\pm}(0) = b^{\pm}(0)$) are independent. Relations (3.17) and (3.18), obtained above, show that in the R.H.S. of eq. (1.17) one must take $d = i/2$, and therefore, instead of that equation we must have

$$[\tilde{\phi}(x), \phi(y)] = i\tilde{D}(x-y). \quad (3.19)$$

Using the relations (2.11) and (3.14) we can easily write down all the commutation relations in terms of the operators $a^{\pm}(p^1)$ and $b^{\pm}(p^1)$. In doing this we must consider that all operators $a^{\pm}(p^1)$ commute with all operators $b^{\pm}(p^1)$. Since there are no principal difficulties in such a procedure, we ommit it here.

ONE-PARTICLE STATES

Consider the "one-particle" states associated with the fields $\phi(x)$ and $\tilde{\phi}(x)$. The vacuum is defined as the state for which

$$\phi^+(x)|0\rangle = \tilde{\phi}^+(x)|0\rangle = 0, \qquad \langle 0|0\rangle = 1. \tag{4.1}$$

Having in mind (3.1) and analogous formulae for the creation and annihilation operators $B^{\pm}(q^1)$ associated with the field $\tilde{\phi}(y)$ eq. (4.1) means that

$$A^+(p^1)|0\rangle = B^+(q^1)|0\rangle = 0. \tag{4.2}$$

Operating with test functions that temper the field operators, one must be very carefull in view of the infrared divergency (see ref. /1/). Suppose F belongs to the space of complex rapidly decreasing infinitely differentiable functions of two arguments $S(R^2)$. Consider its restriction on the light-cone (the space of such functions we denote by $S(C_+)$):

$$f(p^1) \equiv F(|p^1|, p^1). \tag{4.3}$$

In order that the single particle state has a finite norm, a regularization is needed

$$|f\rangle = \int \frac{dp^1}{|p^1|}[A^-(p^1)f(p^1) - A^-(0)f(0)\theta(\kappa - |p^1|)]|0\rangle. \tag{4.4}$$

Indeed

$$<f|f> = 2\int \frac{dp^1}{|p^1|}[|f(p^1)|^2 - |f(0)|^2\,\theta(\kappa - |p^1|)] \qquad (4.5)$$

and obviously $|<f|f>| < \infty$. It is necessary to note that the second term in the R.H.S. of the commutator (3.10) garantees the convergency of the integral (4.5).

If $f(0) = 0$ (the subspace of $S(C_+)$ for which that takes place, we denote by $S_0(C_+)$) then

$$<f|f> \geq 0. \qquad (4.6)$$

The completion by the norm

$$||f||^2 = <f|f> = 2\int \frac{dp^1}{|p^1|}|f(p^1)|^2 \qquad (4.7)$$

of the pre-Hilbert space $S_0(C_+)$ is in fact the Hilbert space $L^2(C_+, \frac{dp^1}{|p^1|})$ of quadratically integrable with respect to the invariant measure $dp^1/|p^1|$ functions $f(p^1)$. We note that the condition $f(0) = 0$ is equivalent to the condition $A^+(0)|f> = 0$ since (see ref. /2/)

$$A^+(0)|f> = 2\int dp^1\delta(p^1)\, f(p^1)\, |0> = 2f(0)|0>. \qquad (4.8)$$

It is evident that for functions from $S_0(C_+)$ the second term in the R.H.S. of (3.10) is zero and the commutators obtain their usual form.

LORENTZ TRANSFORMATION PROPERTIES

Here we study the transformation properties of the fields $\phi(x)$ and $\tilde{\phi}(x)$ under transformations belonging to the Lorentz group $O(1,1)$, which in the two dimensional space time, where no space rotations exist, are just hyperbolic rotations in the (x^0, x^1) plane:

$$\Lambda_a = \begin{pmatrix} \mathrm{ch}\, a & \mathrm{sh}\, a \\ \mathrm{sh}\, a & \mathrm{ch}\, a \end{pmatrix}. \tag{5.1}$$

Note that

$$(\Lambda_a x)^0 \pm (\Lambda_a x)^1 = e^{\pm a} (x^0 \pm x^1) \tag{5.2}$$

and for the case $p^0 = |p^1|$

$$(\Lambda_a p)^1 = \begin{cases} e^a p^1, & p^1 > 0 \\ e^{-a} p^1, & p^1 < 0 \end{cases} . \tag{5.3}$$

We first of all note, that in view of the commutators (3.17) and (3.18) we have

$$[\tilde{\phi}^{\pm}(x), \phi^{\mp}(y)] = \tilde{D}^{\pm}(x-y), \tag{5.4}$$

$$\tilde{D}^{\pm}(x) = \mp \frac{P}{4\pi} \int \frac{dp^1}{p^1} e^{\mp ipx} = \mp \frac{1}{4\pi} \ell n \frac{x^0 - x^1 + i0}{x^0 + x^1 + i0}. \tag{5.5}$$

From the latter formulae it is easy to obtain the following identity:

$$\tilde{D}^{\pm}(\Lambda_a x) = \tilde{D}^{\pm}(x) \mp \frac{a}{2\pi}. \tag{5.6}$$

Taking into account eq. (5.4), this implies that the fields $\tilde{\phi}^{\pm}(x)$ and $\phi^{\pm}(x)$ are not scalars. Moreover, the Lorentz transformations of these fields cannot be homogeneous. Therefore, eq. (2.2) defines the frequency parts of the field operators only up to certain additive constant operators. The source of this ambiguity is the principal value term in the R.H.S. of eqs. (2.11) and (2.12). Indeed, if we introduce the notation

$$\phi^{\pm}_{R}(x) = \frac{1}{2\sqrt{2\pi}} \int \frac{dp^1}{|p^1|} [a^{\pm}(p^1) e^{\mp ipx} - a^{\pm}(0)\theta(\kappa - |p^1|)],$$

$$\tilde{\phi}^{\pm}_{R}(x) = -\frac{1}{2\sqrt{2\pi}} \int \frac{dp^1}{|p^1|} [b^{\pm}(p^1) e^{\mp ipx} - b^{\pm}(0)\theta(\kappa - |p^1|)],$$

$$\phi^{\pm}_{p}(x) = \frac{1}{2\sqrt{2\pi}} P \int \frac{dp^1}{p^1} b^{\pm}(p^1) e^{\mp ipx}, \qquad (5.7)$$

$$\tilde{\phi}^{\pm}_{p}(x) = -\frac{1}{2\sqrt{2\pi}} P \int \frac{dp^1}{p^1} a^{\pm}(p^1) e^{\mp ipx},$$

then it is easy to confirm that, for instance the integrals

$$\int dz^1 D^{\pm}(x-z) \overleftrightarrow{\partial}{}^{z}_{0} \phi^{\pm}_{p}(z), \qquad \int dz^1 D^{\pm}(x-z) \overleftrightarrow{\partial}{}^{z}_{0} \phi^{+}_{p}(*)$$

as well as the analogous integrals for $\tilde{\phi}^{\pm}_{p}(x)$ are not determined. This arbitrariness can be fixed by means of the following identities:

$$-i \int dz^1 D^{\pm}(x-z) \overleftrightarrow{\partial}{}^{z}_{0} \phi^{\pm}_{p}(z) =$$

$$= -i \int dz^1 \tilde{D}^{\pm}(x-z) \overleftrightarrow{\partial}{}^{z}_{0} \tilde{\phi}^{\pm}_{R}(z) + A^{\pm}(\infty) - A^{\pm}(-\infty)$$

$$-i\int dz^1 D^{\pm}(x-z)\overleftrightarrow{\partial}{}_0^z \phi_p^{\pm}(z) =$$
$$= -i\int dz^1 D^{\pm}(x-z)\overleftrightarrow{\partial}{}_0^z \phi_R^{\pm}(z) + A^{\pm}(\infty) - A^{\pm}(-\infty), \qquad (5.8)$$

where

$$A^{\pm}(z) = iD^{\pm}(x-z)\left[\phi_R^{\pm}(z) \pm \sqrt{2\pi}\, D^{\pm}(z) b^{\pm}(0)\right]$$

(see Appendix). Now instead of eq. (2.2) we can write analogous but unambiguous equalities

$$\phi^{\pm}(x) = -i\int dz^1 D^{\pm}(x-z)\overleftrightarrow{\partial}{}_0^z \phi_R^{\pm}(z) - i\int dz^1 \tilde{D}^{\pm}(x-z)\overleftrightarrow{\partial}{}_0^z \tilde{\phi}_R^{\pm}(z) +$$
$$+ A^{\pm}(\infty) - A^{\pm}(-\infty) \qquad (5.9)$$

as well as

$$\tilde{\phi}^{\pm}(x) = -i\int dz^1 D^{\pm}(x-z)\overleftrightarrow{\partial}{}_0^z \tilde{\phi}_R^{\pm}(z) - i\int dz^1 \tilde{D}^{\pm}(x-z)\overleftrightarrow{\partial}_0 \phi_R^{\pm}(z) +$$
$$+ A^{\pm}(\infty) - A^{\pm}(-\infty). \qquad (5.10)$$

Having in mind the above relations it is easy to obtain the transformation laws for the field $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$ with respect to the Lorentz group.

We note first that $\phi_R^{\pm}(x)$ and $\tilde{\phi}_R^{\pm}(x)$ are scalars, i.e.,

$$U_a^{*1} \phi_R^{\pm}(\Lambda_a^{-1} x) U_a = \phi_R^{\pm}(x),$$
$$U_a^{-1} \phi_R^{\pm}(\Lambda_a^{-1} x) U_a = \phi_R^{\pm}(x),$$

where U_a is the representation of the two-dimensional Lorentz group in the definition space of the field operators. Then using eq. (5.6), we obtain the following transformation law for the field under consideration

$$U_a^{-1}\phi^{\pm}(\Lambda_a^{-1}x)U_a = \phi^{\pm}(x) - \frac{a}{2\sqrt{2\pi}}b^{\pm}(0)$$

$$U_a^{-1}\tilde{\phi}^{\pm}(\Lambda_a^{-1}x)U_a = \tilde{\phi}^{\pm}(x) + \frac{a}{2\sqrt{2\pi}}a^{\pm}(0)$$

APPENDIX

Here we shall prove the identity (5.8) and explain its sense. We first introduce the following notation:

$$L_p^{\pm}(x) = \phi_p^{\pm}(x) \pm \sqrt{2\pi}\,\tilde{D}^{\pm}(x)\,b^{\pm}(0), \qquad (A.1)$$

$$\tilde{L}_p^{\pm}(x) = \tilde{\phi}_p^{\pm}(x) \mp \sqrt{2\pi}\,\tilde{D}^{\pm}(x)a^{\pm}(0), \qquad (A.2)$$

$$L_R^{\pm}(x) = \phi_R^{\pm}(x) + \sqrt{2\pi}\,D^{\pm}(x)\,a^{\pm}(0), \qquad (A.3)$$

$$\tilde{L}_R^{\pm}(x) = \tilde{\phi}_R^{\pm}(x) \pm \sqrt{2\pi}\,D^{\pm}(x)\,b^{\pm}(0). \qquad (A.4)$$

where $\phi_p^{\pm}(x)$, $\tilde{\phi}_p^{\pm}(x)$, $\phi_R^{\pm}(x)$ and $\tilde{\phi}_R^{\pm}(x)$ are defined by eq. (5.7). It is easy to prove, that arbitrary $L^{\pm}$ from (A.1) (A.4) is vanishing at $x^1 \to \pm\infty$. As an example we prove this statement for $L_p^{\pm}(x)$. Inserting $\phi_p^{\pm}(x)$ and $\tilde{D}^{\pm}(x)$ in the R.H.S. of (A.1)

$$L_p^{\pm}(x) = \frac{1}{2\sqrt{2\pi}} P\int \frac{dp^1}{p^1}[\,b^{\pm}(p^1) - b^{\pm}(0)\,]\,e^{\mp ipx}. \qquad (A.5)$$

From the latter there immediately follows our statement, if we have in mind that

$$P\,\frac{e^{\pm ipx}}{p^1}\Big|_{x^1\to\pm\infty} \sim \delta(p^1). \qquad (A.6)$$

The quantities L introduced by (A.1)-(A.4) satisfy the following differential equations

$$\begin{aligned} &\partial_\mu L^\pm_P(x) + \epsilon_\mu^{\ \nu}\,\partial_\nu \tilde{L}^\pm_R(x) = 0\,,\\ &\partial_\mu L^\pm_R(x) + \epsilon_{\mu\nu}\,\partial^\nu \tilde{L}^\pm_p(x) = 0\,. \end{aligned} \qquad (A.7)$$

Now we can pass to the proof of identity (5.8). Consider the integral

$$J = -i\int dz\; D^\pm(x-z)\,\overleftrightarrow{\partial}{}^z_0\, L^\pm_p(z). \qquad (A.8)$$

Writing its explicit form by using the first of eq. (A.7) and the analogous identity

$$\partial_\mu D^\pm(x) + \epsilon_\mu^{\ \nu}\,\partial_\nu \tilde{D}^\pm(x) = 0 \qquad (A.9)$$

we can bring the integral (A.8) in the form

$$J = -i\int dz^1\,\partial^z_1\,\tilde{D}^\pm(x-z)\,L^\pm_p(z) + i\int dz\,D\,(x-z)\,\partial^z_1\,\tilde{L}^\pm_R(z).$$

Integrating by part and subsequently making use once more of the identities (A.7) and (A.9) we obtain

$$\begin{aligned} &-i\int dz^1 D^\pm(x-z)\,\overleftrightarrow{\partial}{}^z_0\, L^\pm_p(z) =\\ &= -i\int dz^1\,\tilde{D}^\pm(x-z)\,\overleftrightarrow{\partial}{}^z_0\,\tilde{L}^\pm_R(z) + A^\pm(\infty) - A^\pm(-\infty)\,, \end{aligned} \qquad (A.10)$$

where we have introduced the notation

$$A^{\pm}(z^1) = iD^{\pm}(x-z)\,\tilde{L}^{\pm}_R(z). \qquad (A.11)$$

Now in order to obtain finally identity (5.8) we must insert $L^{\pm}_p(z)$ and $\tilde{L}^{\pm}_R(z)$ from (A.1) and (A.4), respectively in (A.10), having in mind the obvious equality

$$-i\int dz^1 D^{\pm}(x-z)\overleftrightarrow{\partial}^z_0 \tilde{D}^{\pm}(z) = \qquad (A.12)$$

$$= -i\int dz^1 \tilde{D}^{\pm}(x-z)\overleftrightarrow{\partial}^z_0 D^{\pm}(z) = \tilde{D}^{\pm}(x).$$

It follows from (A.11) that the constants $A^{\pm}(\infty)$ and $A^{\pm}(-\infty)$ are determined by the asymptotic behaviour of $\tilde{L}^{\pm}_R(z)$ and are nonzero if and only if the following asymptotic condition takes place:

$$\tilde{L}^{\pm}_R(z)\Big|_{|z^1|\to\infty} \sim \frac{A}{\ln|z^1|}. \qquad (A.13)$$

If we now refer to the definition of $\tilde{L}^{\pm}_R$ (equality (A.4)) remembering that the quantities $\mp\sqrt{2\pi}\,D^{\pm}(x)\,b^{\pm}(0)$ are the leading terms in the asymptotic behaviour of $\tilde{\phi}^{\pm}_R(x)$ it is implicite that the constants $A(\infty)$ and $A(-\infty)$ are determined by the next terms in the asymptotic expansion of the given fields. The additive arbitrariness, which was discussed in the last section, is due to these constant operators.

That is why fixing these constant operators together with eqs. (5.9) and (5.10) we determine completely the fields $\phi^{\pm}(x)$ and $\tilde{\phi}^{\pm}(x)$, after which their transformation properties with respect to the two-dimensional Lorentz group take the form (5.12).

At last we note, that the results of this Appendix do not contradict those of the second section where in fact the basic role in obtaining eq. (2.6) is plaid by the components $\phi_R^{\pm}(x)$ of the field $\phi^{\pm}(x)$, for which formula (2.2) remains valid.

Remark: In fact eq. (1.5) follows directly from the equal times canonical commutation relations

$$[\partial_p \phi(x), \ \phi(y)]_{x^0=y^0} = -i\delta(\vec{x} - \vec{y})$$

not only in the case of two-dimensional space-time. Eq. (1.5) is obtained by integrating the upper equality over the space-like surface $x^0 = \text{const}$.

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