ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА



E2 - 10860

V.G.Kadyshevsky, M.D.Mateev, R.M.Mir-Kasimov,
I.P.Volobuyev

# EQUATIONS OF MOTION FOR THE SCALAR AND THE SPINOR FIELDS IN FOUR-DIMENSIONAL NONEUCLIDEAN MOMENTUM SPACE

1977

E2 - 10860

V.G.Kadyshevsky, M.D.Mateev, R.M.Mir-Kasimov,
I.P.Volobuyev*

# EQUATIONS OF MOTION FOR THE SCALAR AND THE SPINOR FIELDS IN FOUR-DIMENSIONAL NONEUCLIDEAN MOMENTUM SPACE

*Submitted to ТМФ*

* *Institute of Nuclear Physics, MSU.*

Кадышевский В.Г. и др. **E2 - 10860**

Уравнения движения для скалярного и спинорного полей
в четырехмерном неевклидовом импульсном пространстве

Получены уравнения движения для скалярного и спинорного полей в четырехмерном неевклидовом импульсном пространстве. Они содержат в качестве параметра фундаментальную длину $\ell$ и переходят в обычные уравнения Клейна-Гордона и Дирака в пределе $\ell \to 0$.

В новом формализме важную роль играет "импульс вакуума" (это понятие принадлежит И.Е.Тамму).

Найденные уравнения остаются инвариантными при пространственном отражении в том случае, когда одновременно преобразуется импульс вакуума.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

**Препринт Объединенного института ядерных исследований. Дубна 1977**

Kadyshevsky V.G. et al. **E2 - 10860**

Equations of Motion for the Scalar and Spinor Fields in Four-Dimensional Noneuclidean Momentum Space

Equations of motion for scalar and spinor fields in a four-dimensional non-Euclidean momentum space are obtained. These equations incorporate as a parameter the fundamental length and coincide with the ordinary Klein-Gordon and Dirac equations in the limiting case $\ell \to 0$.

In the new formalism an important role is played by "vacuum momentum" (this notion was introduced by I.E.Tamm). The equations obtained remain invariant under the space inversion only if the vacuum momentum transforms simultaneously.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

**Preprint of the Joint Institute for Nuclear Research. Dubna 1977**

## 1. Introduction

In the papers[1-5] a new formulation of quantum field theory (QFT) has been put forward, in which the key part is assigned to a four-dimensional momentum space of constant curvature. A space like that can be realized as a second-order surface in an auxiliary flat 5-space with Cartesian coordinates ($p_0, \vec{p}, p_4$). Depending on the curvature sign, there arise two possibilities ($\hbar = c = 1$):

$$p_0^2 - \vec{p}^{\,2} + M^2 p_4^2 = M^2 \qquad \text{(curvature } \tfrac{1}{M^2}\text{)}, \tag{1.1}$$

$$p_0^2 - \vec{p}^{\,2} - M^2 p_4^2 = -M^2 \qquad \text{(curvature } -\tfrac{1}{M^2}\text{)}. \tag{1.2}$$

We call the new fundamental constant $M$ which appears here the "fundamental mass", the inverse quantity $\frac{1}{M} = \ell$ will be called the "fundamental length".

The spaces (1.1)-(1.2) are known in theoretical physics as De Sitter spaces. In the limiting case of small 4-momenta $|p| \ll M$ (formally, as $M \to \infty$, or $\ell \to 0$), the De Sitter geometry is undistinguishable from the flat pseudoeuclidean geometry. This fact is the basis of the correspondence principle between the new scheme and the usual theory in which a Minkowsky momentum space is employed. Free particles occupy in the $p$ -space the three-dimensional mass shells

$$p^2 - m^2 = 0, \tag{1.3}$$

and it is of no importance for them whether these surfaces are embedded in the flat ( $\ell$ =0) or curved ( $\ell \neq 0$) four-dimensional momentum space*).

However, when the interaction is introduced, the particles leave the surface (1.3) and can arrive at any region of the $p$ -space with arbitrary relative 4-momenta $p$ . The curvature of De Sitter momentum space becomes at large $|p| \gtrsim M$ such an essential factor that many relations of De Sitter geometry differ radically from their pseudoeuclidean analogues. Therefore, the laws of particle interaction in the region $|p| \gtrsim M$ (i.e., at small space-time intervals), which are prescribed respectively by the local QFT and the new scheme, will differ drastically from each other. As a result, a new physics at superhigh energies arises. This situation reminds qualitatively the transition from the non-relativistic mechanics to the relativistic one, many predictions of which differ radically from the conclusions of non-relativistic theory.

Further on, we shall use the system of units

$$\hbar = c = \ell = M = 1 \tag{1.4}$$

so that all the relations of the theory become dimensionless. Flat pseudoeuclidean limit corresponds in this system of units to the values

$$|p_0| \ll 1 \ , \ |\vec{p}| \ll 1. \tag{1.5}$$

---

*) In the case (1.1) the complementary restriction $m^2 \leq M^2$ must be fulfilled for the mass spectrum of the free particles. It is not burdensome provided the value $M$ is large enough.

Now we cannot choose with confidence between the possibilities (1.1)-(1.2). Each of them has, from methodical point of view, its own flaws and merits. It is the experiment that is to make the decision, if the hypothesis of the curved momentum space itself would be confirmed.

In the present paper we shall consider the $p$ -space to be with a negative curvature and described by equation (1.2). In the unit system (1.4) this equation reads:

$$p_0^2 - \vec{p}^2 - p_4^2 = -1 . \tag{1.6}$$

All the results derived below can be easily transferred into the scheme based on the equation (1.1).

## 2. Equation of Motion for the Scalar Field and the Vacuum Momentum

Let us consider free spinless particles of mass $m$ and introduce the notation (cf. /3/)

$$\begin{aligned} m &= \operatorname{sh}\mu \\ \sqrt{1+m^2} &= \operatorname{ch}\mu = m_4 . \end{aligned} \tag{2.1}$$

Then, owing to (1.6), the equation of the mass shell hyperboloid can be written in the form

$$(\operatorname{ch}\mu - p_4)(\operatorname{ch}\mu + p_4) = 0 . \tag{2.2}$$

There are two values of $p_4$ corresponding to any fixed $p$ on the surface (1.6), which differ only in sign. Therefore, any bracket in (2.2) can be equal to zero:

$$\operatorname{ch}\mu - p_4 = 0 \tag{2.3}$$

$$\operatorname{ch}\mu + p_4 = 0 . \tag{2.4}$$

Let us first assume that only the condition (2.3) is valid for free fields. This leads to the following "Klein-Gordon equation":

$$2(\operatorname{ch}\mu - p_4)\varphi(p, p_4) = 0 \tag{2.5}$$

$\varphi(p, p_4)$ being the scalar field describing our particles.

If we apply the operator $p_4/2$ to the left side of (2.5), we get:

$$(\operatorname{ch}^2\mu - p_4^2)\varphi(p, p_4) = (m^2 - p^2)\varphi(p, p_4) = 0. \tag{2.6}$$

Thus, the standard Klein-Gordon equation is a consequence of (2.5).

Let us remark that the Klein-Gordon equation can be obtained from (2.5) also in the flat limit $m^2, p^2 << 1$ :

$$(m^2 - p^2)\varphi(p, 1) = 0 \tag{2.7}$$

In so doing we imply $p_4 > 0$ because, due to (2.5),

$$\varphi(p, p_4) = 0 \qquad \text{when} \qquad p_4 < 0. \tag{2.8}$$

Now consider the equation

$$2(\operatorname{ch}\mu + p_4)\chi(p, p_4) = 0, \tag{2.9}$$

which is based on the relation (2.4). When multiplied by $-p_4/2$, this equation is also reduced to the ordinary Klein-Gordon equation:

$$(m^2 - p^2)\chi(p, p_4) = 0. \tag{2.10}$$

In the limit $m^2, p^2 << 1$ , putting:

$$\operatorname{ch}\mu \approx 1 + \frac{m^2}{2}$$

$$p_4 \approx -1 - \frac{p^2}{2}$$

we get, in complete analogy to (2.7), the equation:

$$(m^2-p^2)\,\chi(p,p_4)=0. \qquad (2.11)$$

Thus, a question arrises: what is the relation between the fields $\varphi$ and $\chi$ /6/? Owing to (2.6) and (2.10), they describe particles with equal masses. On the other hand, the $\varphi$ - and $\chi$ - particles correspond to different value of $p_4/|p_4|$. This quantity is a new quantum number, which has no analogue in Minkowsky $p$ -space.

At any rate, we can assert that the functions $\varphi(p,p_4)$ and $\chi(p,p_4)$ are connected through a discrete transformation

$$\chi(p,p_4)=\hat{I}\,\varphi(p,p_4) \qquad (2.12)$$

containing the reflection of the coordinate $p_4$ :

$$p_4 \to -p_4. \qquad (2.13)$$

Let us obtain this transformation. Our following reasoning essentially employs the notion of vacuum 4-momentum $V_\mu$ , which has been introduced in QFT by I.E.Tamm /2/. He proposed to measure all 4-momenta from a certain vector $V_\mu$ rather than from zero. It corresponds to the transformation

$$p_\mu \to p_\mu - V_\mu. \qquad (2.14)$$

As a result, the equations of the theory become formally covariant under the whole 10-parameter motion group of the momentum 4-space (Poincaré group)

$$p'=Lp+k. \qquad (2.15)$$

For example, the Klein-Gordon equation now reads

$$[m^2-(p-V)^2]\,\varphi'(p)=0, \qquad (2.16)$$

where we use the notation

$$\varphi'(p) = \varphi(p - V). \tag{2.17}$$

This equality demonstrates the transformation law of the field $\varphi(p)$ under the translations in Minkowsky $p$ -space.

Let us emphasize the following point: the vacuum 4-momentum $V_\mu$ may be interpreted in the case of charged particles as a constant (unobservable) vector-potential $eA_\mu$ of electromagnetic field, the quantity $p_\mu - V_\mu$ being analogous to the generalized momentum $p_\mu - eA_\mu$.

The theory based on the curved momentum space (1.6) demands that we should use the De Sitter group $SO$ (4,1)

$$p' = (Lp)(+)k \tag{2.18}$$

instead of Poincaré group (2.15). The symbol (+) in (2.18) denotes "translations" of the space (1.6)[2,3]:

$$p'_\mu = (p(+)k)_\mu = p_\mu + k_\mu\left(p_4 + \frac{(pk)}{1+k_4}\right), \quad \mu = 0,1,2,3. \tag{2.19}$$
$$p'_4 = (p(+)k)_4 = p_4 k_4 + (pk), \quad (pk) = p_0 k_0 - \vec{p}\cdot\vec{k}.$$

These transformations are De Sitter rotations in the planes ( $p_\mu$ , $p_4$ ). In the limit (1.5) they are equivalent, up to some discrete transformations, to the pseudoeuclidean translations $p + k$ .

The scalar product of any two vectors $p_L$ and $k_L$ ( $L$ =0,1,2,3,4)

$$(pk) - p_4 k_4 = g^{LM} p_L k_M \equiv [pk] \tag{2.20}$$

is an invariant of the group $SO$ (4,1).
If these vectors belong to the space (1.6)
$$[p]^2 = [k]^2 = -1$$

then,

$$\pm 2[pk] = [p \pm k]^2 + 2 \tag{2.21}$$

or, taking into account (2.19),

$$\pm 2[pk] = \mp 2(p(-)k)_4 . \tag{2.22}$$

It is clear that the vacuum momentum in the new scheme is a vector of the De Sitter space (1.6):

$$V_0^2 - \vec{V}^2 - V_4^2 = -1, \tag{2.23}$$

the transformation (2.14) being generalized as follows:

$$p' \to p(-)V. \tag{2.24}$$

Therefore, due to (2.22), equation (2.5) can be rewritten in the $SO$ (4,1)-covariant form (cf. (2.16))

$$2(ch\mu + [pV])\varphi'(p,p_4) = 0, \tag{2.25}$$

where

$$\varphi'(p,p_4) = \varphi(p(-)V, (p(-)V)_4). \tag{2.26}$$

The relation (2.26) represents the transformation law of the field $\varphi$ under translations of De Sitter $p$ -space (cf.(2.17)).

It is well known that the continuous motion group of spaces of constant curvature can also include such transformations, which are reduced in the flat limit to reflections, i.e., to unproper operations /7/. This fact has direct relation to our further discussion.

Let us consider the translation (2.24) in the case $V_0 = 0$. Owing to (2.23), $V_1^2 + V_2^2 + V_3^2 + V_4^2 = 1$

and we can set:

$$V_4 = \cos\theta \quad , \quad 0 \le \theta \le \pi$$
$$\vec{V} = \sin\theta\, \vec{n} \; , \quad \vec{n}^2 = 1 .$$

Equations (2.19) give

$$\begin{aligned} &(p(-)V)_0 = p_0 \\ &\overrightarrow{(p(-)V)} = \vec{p} - \vec{n}(\vec{p}.\vec{n})(1-\cos\theta) - \vec{n}\, p_4 \sin\theta . \end{aligned} \tag{2.27}$$

Provided $\theta$ =0, the vacuum momentum is

$$V_L = (0,0,0,0,1), \tag{2.28}$$

and the translation (2.27) is reduced to the identity transformation of the group $SO$ (4,1): $(p(-)V) = p$. If $\theta = \pi$ ,

$$V_L = (0,0,0,0,-1). \tag{2.29}$$

In this case from (2.27), we obtain

$$\begin{aligned} &(p(-)V)_0 = p_0 \\ &\overrightarrow{(p(-)V)} = \vec{p} - 2\vec{n}(\vec{p}.\vec{n}). \end{aligned} \tag{2.30}$$

The second line represents the reflection of the 3-vector $\vec{p}$ in the plane orthogonal to the vector $\vec{n}$ . In particular, when $\vec{n} = (0,0,1)$

$$\overrightarrow{(p(-)V)} = (p_1, p_2, -p_3). \tag{2.31}$$

Thus, continuous transformations in De Sitter space really enable us to make a reflection of an odd number of spatial axes.

Supposing that the conditions (2.29) and (2.31) are fulfilled, one can easily see that equation (2.25) gets the form:

$$2(\operatorname{ch}\mu + p_4)\psi(p_0, p_1, p_2, -p_3, -p_4) = 0. \tag{2.32}$$

Having compared (3.32) and (2.9) we have the right to conclude that the fields $\chi(p, p_4)$ and $\varphi(p, p_4)$ are connected through a reflection operation, which contains together with (2.13), an inversion of an odd number of spatial components of the vector $p_\mu$. In particular, we may set (cf. (2.12)):

$$\chi(p, p_4) = \hat{I}\,\varphi(p, p_4) = \varphi(p_0, -\vec{p}, -p_4). \tag{2.33}$$

We shall consider the relation (2.33) to be the definition of the space reflection in the new scheme. The fact that this definition is quite natural becomes evident in the case of spinor fields where the transformation (2.33) corresponds to multiplying a spinor by the matrix $\gamma^0$ (see (3.12)).

Let us stress that the phase factor in (2.33) equals unity since the reflection (2.33) can be continuously connected with the identity transformation in $SO(4,1)$. Thus, the field $\varphi(p, p_4)$ is a scalar field. Pseudoscalar fields correspond in this formalism to the "fourth components" of 5-vector fields.

Now we can sum up our considerations concerning equations (2.5) and (2.9), and the vector of vacuum momentum $V_L$:

1. Equations (2.5) and (2.9) are connected through the transformation of space reflection, and, therefore, in fact we need only one of them, for example eq. (2.5).

2. To the vacuum momentum, even in the "physical gauge" $V_\mu = 0$, there correspond two 5-vectors (2.28) and (2.29), which turn into one another under the transformation of space reflection.

## 3. Dirac Equation

Let us first remind (see, for example, [8,9]) that an arbitrary matrix of the four-dimensional (spinor) representation of the group $SO(4,1)$ is determined by the relation

$$S^{-1}\Gamma^M S = \Gamma^L \Lambda_L{}^M \tag{3.1}$$

$\|\Lambda_L{}^M\|$ being a 5x5-matrix of De Sitter rotation in the space ( $p_0, p_1, p_2, p_3, p_4$ ).

$\Gamma^M = (\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4)$ denote five fourth-order anticommuting matrices:

$$\{\Gamma^M, \Gamma^N\} = \Gamma^M\Gamma^N + \Gamma^N\Gamma^M = 2g^{MN}$$
$$g^{MN} = \mathrm{diag}\,(+----). \tag{3.2}$$

Owing to (3.2) and (2.20), the following formula is valid for arbitrary 5-vectors $p_L$ and $k_L$ :

$$\{(p_L\Gamma^L), (k_M\Gamma^M)\} = \{[p\Gamma], [k\Gamma]\} = 2[pk]$$
$$[p\Gamma]^2 = [p]^2. \tag{3.3}$$

The explicit form of the $\Gamma$ -matrices is chosen to be (cf. /8,9/)

$$\Gamma^0 = \gamma^0 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

$$\Gamma^m = \gamma^m = \begin{pmatrix} 0 & \sigma_m \\ -\sigma_m & 0 \end{pmatrix}, \quad m = 1,2,3 \tag{3.4}$$

$$\Gamma^4 = \gamma^5 = -i\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix},$$

where $\sigma_m$ are the Pauli matrices, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Obviously

$$(\Gamma^M)^+ = g^{MM}\Gamma^M = \gamma^0\Gamma^M\gamma^0 \tag{3.5}$$

The matrix $S$ can be written in the exponential parametrization:

$$S = e^{\frac{i}{2}\omega_{NM}M^{NM}} \quad (N,M = 0,1,2,3,4), \tag{3.6}$$

where $\omega_{MN} = -\omega_{NM}$ is an angle of De Sitter rotation in the $(M,N)$ -plane, and

$$M^{KL} = \frac{i}{4}\left(\Gamma^K\Gamma^L - \Gamma^L\Gamma^K\right) \tag{3.7}$$

are the corresponding generators. For example, the translation (2.19) by a time-like vector $k_\mu$ can be expressed, due to (3.6), in the form

$$S(k) = \exp\left(-\frac{\varkappa}{2\sqrt{k^2}}\, k_\mu \gamma^\mu \gamma^5\right), \quad \varkappa = \operatorname{arsh}\sqrt{k^2}. \tag{3.8}$$

Formulae (3.5) and (3.6) produce

$$\gamma^0 S^+ \gamma^0 = S^{-1}. \tag{3.9}$$

Let $\Psi(p, p_4)$ be a spinor field defined in De Sitter $p$ - space (1.6). It is clear that this field transforms with respect to the group $SO$ (4,1) as follows:

$$\Psi'(p', p_4') = S(\omega)\Psi(p, p_4). \tag{3.10}$$

In particular, under the translations (2.19) we have

$$\Psi'(p(+)k, (p(+)k)_4) = S(k)\,\Psi(p, p_4). \tag{3.11}$$

If, for example,

$$p_L' = (p_0, -\vec{p}, -p_4),$$

the matrix $S(\omega)$ appears to be (up to a sign):

$$S(\omega) = e^{\frac{\pi}{2}\Gamma^1\Gamma^2}\, e^{\frac{\pi}{2}\Gamma^3\Gamma^4} = \gamma^0.$$

Therefore,

$$\Psi'(p_0, -\vec{p}, -p_4) = \gamma^0\Psi(p_0, \vec{p}, p_4). \tag{3.12}$$

We have obtained the spinor representation of the operation of space reflection $\hat{I}$ (cf. (2.33)). As in the case of the scalar field, the phase factor in (3.11) is real.

Owing to (3.10) and (3.9),

$$\bar{\Psi}'(p', p_4') = \Psi^{+}(p', p_4')\gamma^0 = \bar{\Psi}(p, p_4) S^{-1}(\omega). \tag{3.13}$$

Hence, the Dirac quadratic form is an invariant of the De Sitter group $SO(4,1)$:

$$\bar{\Psi}(p, p_4)\Psi(p, p_4) = inv. \tag{3.14}$$

It is also evident that the quantities $\bar{\Psi}\Gamma^M\Psi$ and $\bar{\Psi}\Gamma^M\Gamma^N\Psi$ transform under the $SO(4,1)$-group as a 5-vector and a 5-tensor respectively.

An analogue of the Dirac equation for the spinor $\Psi(p, p_4)$ can be obtained by the procedure of "extracting square root" from the wave operator of the scalar equation (2.5). Using (2.21) and (2.25), we first represent the scalar equation in the form:

$$\left(4 sh^2 \tfrac{\chi}{2} - [p - V]^2\right)\varphi'(p, p_4) = 0. \tag{3.15}$$

Further, taking into account (3.3) we get:

$$\left(4 sh^2 \tfrac{\chi}{2} - [p - V]^2\right) = \left\{2 sh \tfrac{\chi}{2} + [(p - V)\Gamma]\right\}\left\{2 sh \tfrac{\chi}{2} - [(p - V)\Gamma]\right\}. \tag{3.16}$$

This relation enables us to write a "covariant" Dirac equation for the spinor field in De Sitter space:

$$\left\{2 sh \tfrac{\chi}{2} - [(p - V)\Gamma]\right\}\Psi'(p, p_4) = 0, \tag{3.17}$$

where

$$\Psi'(p, p_4) = S(V)\Psi(p(-)V, (p(-)V)_4). \tag{3.18}$$

Let us now put $V_L = (0,0,0,0,1)$ and pass over to the notation (3.4). Then we get the following equation for $\Psi(p, p_4)$ :

$$[2\,\mathrm{sh}\,\mu/2 - p_\mu \gamma^\mu - (p_4-1)\gamma^5]\,\Psi(p,p_4) = 0. \qquad (3.19)$$

The multiplication of equation (3.19) by the operator $p_\mu\gamma^\mu + (p_4-1)\gamma^5$ results evidently in the scalar equation (2.5) for $\Psi(p, p_4)$ :

$$2\,(\mathrm{ch}\,\mu - p_4)\,\Psi(p,p_4) = 0. \qquad (3.20)$$

It is also clear that the flat limit (1.5) of equation (3.19) coincides with the ordinary Dirac equation.

The requirement that the system (3.19) for the functions $\Psi_\alpha(p,p_4)$ $(\alpha = 1,2,3,4)$ possess a non-trivial solution gives:

$$\det\,[2\,\mathrm{sh}\,\mu/2 - p_\mu\gamma^\mu - (p_4-1)\gamma^5] = (2\,\mathrm{ch}\,\mu - 2p_4)^2 = 0. \qquad (3.21)$$

Taking into account (1.6) and (2.1) we obtain the usual expression for the energy spectrum of a Dirac particle:

$$p_0 = \pm\sqrt{\vec{p}^{\,2} + m^2}\ .$$

Let us go back once again to the "covariant" equation (3.17) and set $V_L = (0,0,0,\sin\theta,\cos\theta)$ . Further we shall vary the parameter $\theta$ . When $\theta = \pi$ , from formulae (2.31) and (3.6)-(3.7) we get:

$$\{2\,\mathrm{sh}\,\mu/2 - p_\mu\gamma^\mu - (p_4+1)\gamma^5\}\,\gamma^3\gamma^5\Psi(p_0,p_1,p_2,-p_3,-p_4) = 0.$$

This equation can further be transformed by rotation in the plane (1.2) by angle $\pi$ to:

$$\{2\,\mathrm{sh}\,\mu/2 - p_\mu\gamma^\mu - (p_4+1)\gamma^5\}\,\gamma^0\,\Psi(p_0,-\vec{p},-p_4) = 0 \qquad (3.22)$$

Evidently, (3.22) coincides with the initial Dirac equation (3.19) up to the substitution $\vec{p} \to -\vec{p}$ , $p_4 \to -p_4$ . On the other hand, it is exactly the equation for the spinor wave-function which has undergone the transformation of reflection (3.12):

$$\{ 2\,\mathrm{sh}\,\chi/_2 - p_\mu \gamma^\mu - (p_4 + 1)\gamma^5 \} \psi'(p_0, \vec{p}, p_4) = 0. \qquad (3.23)$$

One can easily see that equations (3.23) and (3.19) do not coincide. This is a direct consequence of the fact that the vacuum momentum is not invariant under the space reflection (see the end of the previous paragraph). One may say that the pair of the Dirac equations (3.19) and (3.23) is equivalent to the scalar pair (2.5) and (2.9) in what concerns their behaviour under the operation of space reflection.

Now we suppose

$$\psi(p, p_4) = u(k)\,\theta(k_0)\,\delta^4(p-k) \qquad (3.24)$$

$$k_4 = m_4\,,\ k_0 > 0$$

and insert (3.24) into equation (3.19). Taking into account the relations

$$p_4 - 1 = 2\,\mathrm{sh}^2\,\chi/_2$$

$$p_\mu \gamma^\mu = 2\,\mathrm{sh}\,\chi/_2\,\mathrm{ch}\,\chi/_2\, n_\mu \gamma^\mu\,, \quad n^2 = 1$$

we get

$$(1 - \mathrm{ch}\,\chi/_2\, n_\mu \gamma^\mu - \mathrm{sh}\,\chi/_2\, \gamma^5)\, u(p) = 0. \qquad (3.25)$$

One can easily verify that the translation (3.8) with $\varkappa = \chi/_2$ reduces equation (3.25) to the standard Dirac equation:

$$(1 - n_\mu \gamma^\mu)\, u'(p) = 0, \qquad (3.26)$$

$$u'(p) = \exp\left(-\frac{\mu}{4}\, n_\mu \gamma^\mu \gamma^5\right) u(p). \tag{3.27}$$

Let us stress that the argument of the exponent in (3.27) is a pseudoscalar. It means that the equivalence of the new and old Dirac equations fails when the transformation of space reflection is taken into account.

## 4. Electromagnetic Concept of the Vector of Vacuum Momentum in the New Scheme

It has been mentioned in § 2 that the vector of vacuum momentum can be treated in the ordinary theory as a constant vector-potential of the electromagnetic field. If this concept is transferred to the theory with curved momentum space, we can arrive at the following conclusion:

1. The vector-potential of electromagnetic field should be a unit 5-vector /10/.

2. If we perform a substitution in equation (3.17)

$$\begin{aligned} V_\mu &\to -V_\mu \\ V_4 &\to V_4 \end{aligned} \tag{4.1}$$

it should be equivalent to a certain transformation of "charge conjugation" of the wave-function $\psi(p, p_4) \to \psi^c(p, p_4)$, the new function $\psi^c(p, p_4)$ again satisfying equation (3.17). It can be demonstrated that such an operation exists and coincides with the ordinary one:

$$\psi^c(p, p_4) = C\psi(-p, p_4). \tag{4.2}$$

The proof of this statement essentially employs the following property of the translations $S(V)$ under the complex conjugation:

$$S^{*}(V) = (-i\gamma^2)\, S(-V)\, (-i\gamma^2). \tag{4.3}$$

## 5. Conclusion

The importance of the free field equations of motion, in particular the Dirac equation, is well known in the existent QFT. The generalization of these equations containing the fundamental length is important in the new formulation of the theory developed by us.

The obtained equations lead to modified expressions of the propagators, and therefore, to a new description of the virtual particles in the region of superhigh energy-momenta (small 4-distances). Extremely intriguing is the fact that the new equations of motion need redefinition of the space reflection operation.

The authors are sincerely grateful to N.N.Bogolubov, S.S.Gerstein, A.D.Donkov, V.Ia.Fainberg, D.A.Kirzhnits, A.A.Komar, A.A.Logunov, V.A.Matveev, V.A.Meshcheryakov, A.N.Sissakyan, L.A.Slepchenko, L.D.Soloviev, A.N.Tavkhelidze, I.T.Todorov for fruitful discussions.

## References

1. Kadyshevsky V.G. Preprint JINR, P2-5717, Dubna, 1971.
2. Kadyshevsky V.G. Quantum Field Theory and Momentum Space of Constant Curvature, in Problems of Theoretical Physics, dedicated to the memory of I.E.Tamm, Moscow, Nauka, 1970.
3. Donkov A.D., Kadyshevsky V.G., Mateev M.D., Mir-Kasimov R.M. Bulg. Journ. of Phys., 1, 58, 150, 233 (1974), 2, 3 (1975)

and Proceedings of the Mathematical Institute V.A.Steklov, vol. CXXXVI, p. 85-129, Nauka, Moscow, 1975.

4. Donkov A.D., Kadyshevsky V.G., Mateev M.D., Mir-Kasimov R.M. Proceedings of the XVII International Conference on High Energy Physics, p.1-267, London, 1974 and JINR preprint E2-7936, Dubna, 1974.
5. Donkov A.D., Kadyshevsky V.G., Mateev M.D., Mir-Kasimov R.M. Proceedings of the XVIII International Conference on High Energy Physics, Tbilisi, p. A5-1, D1,2-10400, Dubna, 1977.
6. Donkov A.D., Kadyshevsky V.G., Mateev M.D., Mir-Kasimov R.M. In: "Nonlocal, Nonlinear, Nonrenormalizable Field Theories", D2-9788, Dubna, 1976.
7. Felix Klein, Vorlesungen über nicht-euklidische geometrie, Verlag von Julius Springer, Berlin, 1928.
8. Kadyshevsky V.G. JETP, 41, 1885, 1961.
9. Volobuyev I.P., Mir-Kasimov R.M. JINR preprint, P2-10676, Dubna, 1977.
10. Kadyshevsky V.G. In: "Nonlocal, Nonlinear and Nonrenormalizable Field Theories", D2-7161, Dubna, 1973.

Received by Publishing Department
on July 15, 1977.