СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 10853

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QUANTIZATION AND NONLOCAL FIELDS

1977

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E2 - 10853

Квантование и нелокальные поля

Рассмотрена связь постулата причинности с условиями квантования для случая скалярного поля.

Использован формализм внешних форм, заданных на пространстве функционалов. Получено выражение коммутатора токов для нелокальных полей в пространственно-подобной области.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1977

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E2 - 10853

Quantization and Nonlocal Fields

A relation between the postulate of causality and quantization conditions is considered for the scalar field.

A formalism of outer forms given on the functional space is used. A current commutator is obtained for nonlocal fields in the space-like region.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1977

Usually, in quantum field theory, the condition of local commutativity is used to consider as an independent postulate. It should be noted, however, that at least for free fields it is an immediate consequence of the covariant conditions of quantization[1]. Therefore, it is natural to try to find the connection between the quantization conditions and causality postulate. In attepmting to make this problem more clear, we shall use the formalism of outer forms given on a functional space.

Let the S-matrix be a function of the asymptotical in-fields $\phi(x)$.

Following the method by N.Bogolubov[2], suppose that the region of space-time, where $\phi(x)$ is nonzero, breaks into two subregions G_1 and G_2 such that all points of one of them (say, G_1) lie in the past relative to a certain time τ whereas all points of the other (G_2) in the future. Then

$$\delta S = \int_{x_0>\tau} \frac{\delta S}{\delta \phi(x)} \delta\phi(x) d^4 x , \tag{1}$$

where $\delta\phi(x)$ is an infinitesimal variation of $\phi(x)$ different from zero for $x_0>\tau$ only. Multiplying (1) by iS^+ we obtain the 1-form

$$\omega = i\delta S S^{+} = i \int_{x_0 > \tau} d^4x \frac{\delta S}{\delta \phi(x)} S^{+} \delta\phi(x) =$$

$$= \int_{x_0 > \tau} d^4x \, j(x) \, \delta\phi(x). \tag{2}$$

Taking the outer variational derivative of the 1-form (2) under the condition that $\delta\phi(x')$ differs from zero only for $x' \lesssim x$, we get

$$d_A \omega = \frac{1}{2} \int_{(x-x')^2 < 0} \left\{ \frac{\delta j(x)}{\delta\phi(x')} - \frac{\delta j(x')}{\delta\phi(x)} \right\} \times$$

$$\times [\delta\phi(x) \wedge \delta\phi(x')] d^4x d^4x'. \tag{3}$$

The expression in braces is identically written as

$$\left\{ \frac{\delta j(x)}{\delta\phi(x')} - \frac{\delta j(x')}{\delta\phi(x)} \right\} = i[\, j(x), j(x')]. \tag{4}$$

The causality condition implies /3/:

$$\left\{ \frac{\delta j(x)}{\delta\phi(x')} - \frac{\delta j(x')}{\delta\phi(x)} \right\} = 0, \qquad \text{for } (x-x')^2 < 0. \tag{5}$$

Consequently, using condition (5), expression (3) gives

$$d_A \omega \equiv 0. \tag{6}$$

Thus, the condition (6) is a consequence of the causality postulate. The field $\phi(x)$ which obeys the Klein-Gordon equation can be decomposed into the system of orthonormalized functions

$$\phi(x) = \sum_{\alpha} \{ f_{\alpha}(x) q^{\alpha} + f^{*}_{\alpha}(x) p_{\alpha} \}, \tag{7}$$

and the expansion coefficients can be regarded as canonical variables $\{q^{\alpha}, p_{\alpha}\}$ of a generalized phase space $\mathfrak{M}$. Then we can define the contravariant vector

$$\frac{\partial}{\partial \phi(x)} = \sum_{\alpha} \{ f^{*}_{\alpha}(x) \frac{\partial}{\partial q_{\alpha}} - f_{\alpha}(x) \frac{\partial}{\partial p_{\alpha}} \}. \tag{8}$$

Differentiating the expansion (7), the covariant vector is

$$d\phi(x) = \sum_{\alpha} \{ f_{\alpha}(x) dq^{\alpha} + f^{*}_{\alpha}(x) dp_{\alpha} \}. \tag{9}$$

Then, let us use the following expression for the outer derivative of certain 1-form ω given on $\mathfrak{M}$/4/ :

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]), \tag{10}$$

where X , Y are arbitrary contravariant vectors on $\mathfrak{M}$.

If the 1-form ω is determined by expression (2) and the contravariant vectors are determined by expression (8), then equation (10) will be of the following form

$$d_{A}\omega\{ \frac{\partial}{\partial \phi(x)} , \frac{\partial}{\partial \phi(x')} \} =$$

$$= \{ \frac{\partial j(x)}{\partial\phi(x')} - \frac{\partial j(x')}{\partial\phi(x)} \} - \omega([\frac{\partial}{\partial\phi(x)}, \frac{\partial}{\partial\phi(x')}]) . \quad (11)$$

Since from the condition (6) it follows that

$$d_A \omega \{ \frac{\partial}{\partial\phi(x)}, \frac{\partial}{\partial\phi(x')} \} = 0 \quad (12)$$

then from (11) and (4) we deduce the current commutator

$$[j(x), j(x')] = \omega([\frac{\partial}{\partial\phi(x)}, \frac{\partial}{\partial\phi(x')}]), \quad (13)$$

$$\text{for } (x-x')^2 < 0.$$

By using expression (8) the commutator $[\frac{\partial}{\partial\phi(x)}, \frac{\partial}{\partial\phi(x')}]$ can be given the other form

$$[\frac{\partial}{\partial\phi(x)}, \frac{\partial}{\partial\phi(x')}] = \sum_{\alpha,\beta} \{ f_\alpha(x) f^*_\beta(x') - f_\beta(x') f^*_\alpha(x) \} \times$$

$$\times [\frac{\partial}{\partial q^\alpha}, \frac{\partial}{\partial p_\alpha}] = \sum_{\alpha,\beta} \Delta_{\alpha\beta}(x,x') [\frac{\partial}{\partial q^\alpha}, \frac{\partial}{\partial p_\beta}], \quad (14)$$

where

$$\Delta_{\alpha\beta}(x,x') = \{ f_\alpha(x) f^*_\beta(x') - f_\beta(x') f^*_\alpha(x) \} .$$

From the quantization conditions for field $\phi(x)$ it follows

$$[\frac{\partial}{\partial q}, \frac{\partial}{\partial p}] \neq 0, \tag{15}$$

hence, expanding the commutator in the r.h.s. of (14) into the system of vectors $\{\frac{\partial}{\partial q^{\alpha}}, \frac{\partial}{\partial p_{\alpha}}\}$ we have

$$[\frac{\partial}{\partial q^{\alpha}}, \frac{\partial}{\partial p_{\alpha}}] = C^{\gamma}_{\alpha\beta} \frac{\partial}{\partial q^{\gamma}} - \bar{C}^{\gamma}_{\alpha\beta} \frac{\partial}{\partial p_{\gamma}}. \tag{16}$$

Without loss of generality, we may assume that

$$\begin{aligned} C^{\gamma}_{\alpha\beta} &= 0 \qquad && \text{for} \quad \alpha \neq \beta \\ C^{\gamma}_{\alpha\alpha} &= C^{\gamma}_{\alpha} \qquad && \text{for} \quad \alpha = 1, 2, \ldots \infty \end{aligned} \tag{17}$$

then the commutator (14) reads

$$\begin{aligned} &[\frac{\partial}{\partial \phi(x)}, \frac{\partial}{\partial \phi(x')}] = \\ &= \sum_{\alpha,\beta} \Delta_{\alpha\alpha}(x,x') \{ C^{\gamma}_{\alpha} \frac{\partial}{\partial q^{\gamma}} - \bar{C}^{\gamma}_{\alpha} \frac{\partial}{\partial p_{\gamma}} \}. \end{aligned} \tag{18}$$

Substituting expressions (18) and (2) into (13) and taking into account the orthogonality of the basis phase space

$$dq_{\alpha}(\frac{\partial}{\partial q^{\beta}}) = \delta^{\alpha}_{\beta}, \quad dp_{\alpha}(\frac{\partial}{\partial p_{\beta}}) = \delta^{\beta}_{\alpha} \tag{19}$$

we find

$$[j(x), j(x')] =$$

$$= \int d^4y\, j(y) \sum_{\alpha,\gamma} \Delta_{\alpha\alpha}(x,x')\{f_\gamma(y) C_\alpha^\gamma - f^*_\gamma(y) \bar{C}_\alpha^\gamma\} \qquad (20)$$

for $(x-x')^2 < 0$.

In the limit of plane waves

$$f_\alpha(x) \to f_k(x) \sim \frac{1}{\sqrt{k_0}} e^{ikx} \qquad (21)$$

the expression (20) reads

$$[j(x), j(x')] =$$

$$= \iint d^4p\, d^4k \left\{ \frac{\tilde{j}(k)}{\sqrt{k_0}} C_k(p) - \frac{\tilde{j}^+(k)}{\sqrt{k_0}} \bar{C}_k(p) \right\} \epsilon(p_0)\, \delta(p^2 - m^2), \qquad (22)$$

for $(x-x')^2 < 0$,

where $\tilde{j}(k) = \int d^4k\, j(x)\, e^{ikx}$.

If the Hilbert space $\mathcal{H}$ is defined through the scalar product

$$<\phi, \psi> = \int \phi(p) \psi^*(p)\, d\Omega_m(p), \qquad (23)$$

where

$$\Omega_m(p) = \frac{d^3p}{\sqrt{\vec{p}^2 + m^2}}; \qquad \phi, \psi \in \mathcal{H},$$

then the orthogonal transformation J on $\mathcal{H}$

$$J: \ \phi \to i\ \epsilon(p_0)\phi(p) \tag{24}$$

determines the complex structure of $\mathcal{H}$ and allows one to determine the skew-symmetric form[1]

$$B\{\phi,\psi\} = -\langle J\phi,\psi\rangle =$$
$$= -i \int_{p^2=m^2} \phi(p)\psi^*(p)\,\epsilon(p_0)\,d\Omega_m(p). \tag{25}$$

In terms of definitions (23) and (25) the current commutator is $[j(x), j(x')] =$

$$= i \int_{p^2=m^2} \{\tilde{j}(k)B[C_k\ \chi_k(x-x')] -$$
$$-\tilde{j}^+(k)B[\bar{C}_k\ \bar{\chi}_k(x-x')]\}d_m(k) = \tag{26}$$
$$= \mathrm{Im}\langle j,\ B(x-x')\rangle, \quad \text{for } (x-x')^2 < 0,$$

where

$$\chi_k(x-x') = \exp ik(x-x').$$

Thus, finally, the current commutator acquires the following form:

$$[j(x), j(x')] = \mathrm{Im}\langle j, B(x-x')\rangle,$$
$$\text{for} \quad (x-x')^2 < 0. \tag{27}$$

For the $\phi(x)$ local, the coefficients

$$C_k(p) \equiv C_k(p^2) \tag{28}$$

are analytic functions of p and, consequently,

$$B[C_k \chi_k(x-x')] = \int C_k(m)\Delta(m, x-x')dm, \qquad (29)$$

where $\Lambda(m, x-x')$ is the standard commutation function for a free scalar field. The properties of this function give rise to the following condition:

$$[j(x), j(x')] = 0, \qquad \text{for } (x-x')^2 < 0. \qquad (30)$$

That is just the usual condition of local commutativity.

The author is grateful to D. Elokhintsev, V. Kadyshevsky and V. Suslenko for interest in the work and to E. Ivanov for useful discussions.

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Received by Publishing Department
on July 13, 1977.