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THE CONDITION DETERMINING NONRENORMALIZED ELECTROMAGNETIC CONSTANT $e^2 / (hc)$ IN QUANTUM ELECTRODYNAMICS

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Условие, определяющее величину неперенормированной константы связи в квантовой электродинамике

Рассматривается спинорная электродинамика в кулоновской калибровке. Показано, что неперенормированная функция Грина фотона $D(k;q,m,\ell)$ $(q=e^2/(4\pi^3)$, e – неперенормированный заряд электрона, m – масса электрона, ℓ – максимальный 3-импульс) имеет вид

$$D(k;q,m\ell)=\ell^{-2}[D'(k/\ell;q,0)+O((m/\ell)^2)],$$

где $D'(t;q,0)$ – функция, определенная уравнениями (7а), (8а), (6в). Для того, чтобы масса фотона была равна нулю, необходимо, чтобы функция $D'(t;q,0)$ имела полюс при $t^2=0$ (если бы эта функция имела полюс при $t^2=t_1^2\neq 0$, то фотон имел бы бесконечную массу $\ell|t_1|$ и был бы неподвижен). Это условие и определяет константу связи e, она оказывается не зависящей от m.

Настоящая работа основана на важном утверждении о том, что обычная процедура отбрасывания квадратично расходящихся $\sim\ell^2$ членов в операторе $D^{-1}(k;q,m,\ell)$ является неполной: условие равенства нулю квадратично расходящейся части позволяет определить величину константы e электромагнитного взаимодействия. Рассматриваемая в настоящей работе возможность определения константы связи существует в любой модели, где часть членов перенормировки запрещена (см. §5.3).

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The Condition Determining Nonrenormalized Electromagnetic Constant $e^2/(\hbar c)$ in Quantum Electrodynamics

The spinor electrodynamics in the Coulomb gauge is considered. It is shown that the nonrenormalized photon Green function $D(k;q,m,\ell)$ [$q=e^2/(4\pi^3)$, e is the nonrenormalized electron charge, ℓ is the cutoff momentum, as introduced by formula (3) (see also paragraph 2.2), m is the electron mass] has the form $D(k;q,m,\ell)=[D'(k/\ell;q,0)+O(m^2/\ell^2)]\,\ell^2$, where $D'(t;q,0)$ is the function, defined by (6b) and (7a), (8a), where one has to put $n=0$ (note that cutoff ℓ does not enter into these eqs.). For the photon mass to be zero it is necessary that the function $D'(t;q,0)$ has a pole at $t^2=0$ (if the pole were at $t^2=t_1^2\neq 0$, then the photon would have the physically inacceptable infinite mass $\sim\ell|t_1|$). The latter condition defines the value of the constant q, $q=q_0$. The constant q_0 does not depend on the electron mass m. This work rests entirely on the following important statement: the usual procedure of omitting the quadratically divergent terms ($\sim\ell^2$) of the function $D(k;q,m,\ell)^{-1}$ is incomplete; the condition for the omitted $\sim\ell^2$ part of the operator $D(k;q,m,\ell)^{-1}$ to be equal to zero enables one to find the value of the electromagnetic charge e. The discussed possibility of determining coupling constant exists in any model where one of the mass renormalization terms is forbidden (see paragraph 5.3).

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1. Consider a model of quantum electrodynamics describing the interacting spinor and electromagnetic fields. This model has the Hamiltonian /1/

$$H = H_{0e}+H_{0ph}+H_1+H_2+H_3 . \tag{1}$$

Here H_{0ph} and H_{0e} are the Hamiltonians of free transversal electromagnetic field and free spinor field $\vec{H}^{tr}$, e.g.,

$$H_{0e}=\int d^3x\,\psi^*(-i\vec{\alpha}\vec{\nabla}+\beta m)\psi . \tag{2}$$

The operator H_1,

$$H_1=e\int\psi^*(x)\vec{\alpha}\vec{A}^{tr}(x)\psi(x)\,d^3x, \tag{3}$$

describes the interaction of these fields;

$$\vec{A}^{tr}(\vec{x},t)=(2\pi)^{-3/2}\int_{|\vec{k}|<\ell} d^3k\,[a_{k\lambda}\vec{\epsilon}_{k\lambda}e^{i(\vec{k}\vec{x}-|\vec{k}|t)}+\text{c.c.}],$$

$$\vec{\epsilon}_{\vec{k}\lambda}\vec{\epsilon}_{\vec{k}\lambda'}=\delta_{\lambda\lambda'},\quad \vec{\epsilon}_{\vec{k}\lambda}\vec{k}=0,\ \lambda,\lambda'=1,2. \tag{4}$$

The analogous representation of the fermion operator $\psi(\vec{x}, t)$ also contains only momenta smaller than the cutoff momentum ℓ. The operator H_2,

$$H_2 = (e^2/2)\int \psi^*(\vec{x},t)\psi(\vec{x},t)\psi^*(\vec{y},t)\psi(\vec{y},t)\frac{d^3x\, d^3y}{|\vec{x}-\vec{y}|}, \tag{5}$$

describes the Coulomb interaction between electronics. The operator H_3,

$$H_3 = \delta m \int \bar{\psi}(\vec{x},t)\psi(\vec{x},t)\, d^3x, \tag{2a}$$

corresponds to the electron mass renormalization.

We stress that the Hamiltonian (1) has no term of photon mass renormalization

$$H_4 = \mathrm{const} \int (\vec{A}^{tr}(\vec{x},t))^2\, d^3x. \tag{1a}$$

2. Summing up the perturbation series gives the integral equations for determining the nonrenormalized photon and electron Green functions /2/ (see also /3/) and the vertex function (we are looking for the nonrenormalized Green functions so we do not include into the Hamiltonian (1) the terms of wave function renormalization):

$$\begin{aligned} G(p;q,m,\ell)^{-1} &= i\hat{p} + m + \delta m + \\ &+ q\int \Lambda_\mu(p,p-k;k;q,m,\ell)G(p-k;q,m,\ell)\times \\ &\times \gamma_\nu D_{\mu\nu}(k;q,m,\ell)d^4k, \end{aligned} \tag{6}$$

$$\begin{aligned} D_{\mu\nu}(k;q,m,\ell) &= D^{\circ}_{\mu\nu}(k) + qD_{\mu\sigma}(k;q,m,\ell)\times \\ &\times \mathrm{Sp}[\int G(p;q,m,\ell)\Lambda_\sigma(p,p-k;k;q,m,\ell)\times \\ &\times G(p-k;q,m,\ell)\gamma_\tau\, d^4p]\, D^{\circ}_{\tau\nu}(k), \end{aligned} \tag{7}$$

$$\begin{aligned} \Lambda_\sigma(p,p-s;s;q,m,\ell) &= \gamma_\sigma - \\ &- q\int \Lambda_\mu(p,p-k;k;q,m,\ell)G(p-k;q,m,\ell)\times \\ &\times \Lambda_\sigma(p-k,p-k-s;s;q,m,\ell)G(p-k-s;q,m,\ell)\times \\ &\times \Lambda_\nu(p-k-s,p-s;-k;q,m,\ell)D_{\mu\nu}(k;q,m,\ell)d^4k + \dots . \end{aligned} \tag{8}$$

Here

$$q = e^2/(4\pi^3);\ d^4p = dp_1 dp_2 dp_3 dp_0,\ k^2 = (\vec{k})^2 - k_0^2, \tag{9}$$

$$D^{\circ}_{\mu\nu}(k) = \begin{cases} [\delta_{\mu\nu} - k_\mu k_\nu/(\vec{k})^2]/k^2, & \mu,\nu = 1,2,3; \\ 0, & \mu = 0,\ \nu = 1,2,3; \\ & \nu = 0,\ \mu = 1,2,3; \\ 1/(\vec{k})^2, & \mu = \nu = 0, \end{cases} \tag{10}$$

is the photon Green function in the zero-order approximation ($D^{\circ}_{00}(k)$ arises from the Coulomb term (5)).

2.1. Equations (7) and (10) imply

$$\sum_{\nu=1}^{3} D_{\mu\nu}(k;q,m,\ell)k_\nu = 0, \ \mu = 1,2,3,$$
$$\sum_{\mu=0}^{3} D_{\mu\nu}(k;q,m,\ell)k_\mu = 0, \ \nu = 1,2,3,$$

so that $D_{33}(k;q,m,\ell) = 0$ ($\vec{e}_3$ is the unit vector in the direction $\vec{k}$, cf. (4)) and the matrix $D_{\mu\nu}(k;q,m,\ell)$, $\mu,\nu = 0,1,2,3$, has no inverse one. This matrix, however, has the inverse one in the space $\mu,\nu = 0,1,2$, just this latter space will be used in our work.

2.2. According to eq. (4) and the remark after it, the integration regions in eqs. (6)-(8) are limited by the condition that spatial parts of all the 4-momenta are smaller than the cutoff momentum ℓ. As for the integration over zero-components p_0, k_0, these are to be performed in infinite limits (from $-\infty$ till $+\infty$). The latter rule follows from the usual transform of the three dimensional integral

$$(2\pi)^{-3} \int_{|\vec{k}|<\ell} \frac{d^3k}{2|\vec{k}|} \exp[i\vec{k}(\vec{x}-\vec{y}) - i|\vec{k}||x_0-y_0|](\delta_{\mu\nu} - k_\nu k_\mu/(\vec{k})^2),$$
$$\mu,\nu = 1,2,3$$

into the four-dimensional integral

$$i(2\pi)^{-4} \int_{-\infty}^{+\infty} dk_0 \int_{|\vec{k}|<\ell} \frac{d^3k}{k^2 - i\varepsilon} \exp[ik(x-y)](\delta_{\mu\nu} - k_\nu k_\mu/(\vec{k})^2).$$

2.3. Equations (6)-(8) imply the Ward identity [4]

$$(p_1-p_2)_\mu \Lambda_\mu(p_1,p_2;p_1-p_2;q,m,\ell) =$$
$$= G^{-1}(p_1;q,m,\ ) - G^{-1}(p_2;q,m,\ell), \tag{11}$$

thus facilitating the task of solving these eqs.

3. Let us produce in eqs. (6)-(8) the change of variables

$$m = \ell n, \quad \delta m = \ell \delta n,$$
$$p = \ell r, \quad k = \ell t, \quad s = \ell u,$$
$$G(p;q,m,\ell) = G'(r;q,n)/\ell, \tag{12}$$
$$\Lambda_\mu(p,p-k;k;q,m,\ell) = \Lambda'_\mu(r,r-t;t;q,n),$$
$$D_{\mu\nu}(k;q,m,\ell) = D'_{\mu\nu}(t;q,n)/\ell^2.$$

Then we get:

$$G'(r;q,n)^{-1} = i\hat{r} + n + \delta n +$$
$$+ q \int \Lambda'_\mu(r,r-t;t;q,n)\, G'(r-t;q,n) \times \tag{6a}$$
$$\times \gamma_\nu D'_{\mu\nu}(t;q,n) d^4t,$$

$$D'_{\mu\nu}(t;q,n) = D^{\circ}_{\mu\nu}(t) + qD'_{\mu\sigma}(t;q,n)\times$$

$$\times\int Sp[G'(r;q,n)\Lambda'_{\sigma}(r,r-t;t;q,n)\times \tag{7a}$$

$$\times G'(r-t;q,n)\gamma_{\tau})]D^{\circ}_{\tau\nu}(t)d^4t,$$

$$\Lambda'_{\sigma}(r,r-u;u;q,n) = \gamma_{\sigma} +$$

$$-q\int\Lambda'_{\mu}(r,r-t;t;q,n)G'(r-t;q,n)\times \tag{8a}$$

$$\times\Lambda'_{\sigma}(r-t,r-t-u;u;q,n)G'(r-t-u;q,n)\times$$

$$\times\Lambda'_{\nu}(r-t-u,r-u;-t;q,n)D'_{\mu\nu}(t;q,n)d^4t + \ldots$$

(here the spatial parts of all 4-momenta do not exceed unity: cf. paragraph 2.2).

4. The main advantage of the system (6a)-(8a) is that unlike (6)-(8) it does not contain explicitly the cutoff momentum ℓ.

5. Now it is easy to explain the idea of the present work. Equations (6a)-(8a) contain two constants $n+\delta n$ and q.

The solution of these eqs. must satisfy the following two physical conditions: the masses of electron and photon are to be equal to m and zero, respectively, so that the Green functions $G'(r;q,n)$ and $D'(t;q,n)$ would have poles at $r^2=-n^2$ and $t^2=0$. These

two conditions, in general, define completely both the constants at our disposal.

5.1. According to (12)

$$n = m/\ell, \tag{12a}$$

where m is the physical mass of electron. Since m does not depend on ℓ, eq. (12a) implies that n is an infinitesimal quantity: we are to solve the system (6a)-(8a) with $n=0$ and then look for the small corrections due to $n\neq 0$.

5.1.1. The quantity δn in eq. (6a) is determined by the condition that the Green function $G'(r;q,n)$ has a pole at $r^2=-n^2$. So, for the case $n=0$ eq. (6a) is to be rewritten in the form:

$$G'(r;q,0)^{-1} = i\hat{r} + q\int\Lambda'_{\mu}(r,r-t;t;q,0)\times$$

$$\times G'(r-t;q,0)\gamma_{\nu}D'_{\mu\nu}(t;q,0)d^4t - \tag{6b}$$

$$-q\int\Lambda'_{\mu}(0,-t;t;q,0)G'(-t;q,0)\gamma_{\nu}D'_{\mu\nu}(t;q,0)d^4t;$$

here the latter term is δn; in eqs. (7a), (8a) one has to put $n=0$.

5.2. Let $q\neq q_0$ (see Abstract). Then the Green function $D'(t;q,0)$ would have a pole at some value $t^2=t_1^2\neq 0$ of the parameter t^2, so that the photon Green function $D(k;q,m,\ell)$ would have a pole at $k^2=k_1^2\ell^2$. This pole corresponds to the physically inacceptable photon mass $\sim\ell|k_1|$. (In particular, there would be no interaction between such "photons").

5.2.1. Let $n\neq0$; then the electron mass would be infinite $(\sim\ell n)$, and there would be no interaction between "electrons".

5.2.2. So, eqs. (6)-(8) have the physically admissible solution only for some definite value $q=q_0$ of the electromagnetic coupling constant q.

5.3. One can make the photon mass to be zero by including into the Hamiltonian the additional term (1a). Such an inclusion, however, is inadmissible one for the reason that it spoils the initial Maxwell eqs. (unlike Hamiltonian (1)).

5.4. For any value of n, the condition of paragraph 5.2 (the function $D'(t;q,n)$ has a pole at $t^2=0$) defines q as a function of n:

$$q = f(n).$$

The infinitesimal variation of n due to $m\neq0$ gives only an infinitesimal increase in the value of q_0,

$$\delta q_0 \sim m/\ell.$$

5.4.1. So, the value of electromagnetic charge does not depend on the electron mass m.

6. Our work rests entirely on the following important statement: the usual procedure (see, e.g., refs. /3,4/) of omitting the quadratically divergent part $\sim\ell^2$ of operator $D(k;q,m,\ell)^{-1}$ (see paragraph 2.1) is incomplete: the condition for the omitted $\sim\ell^2$ part of $D(k;q,m,\ell)^{-1}$ to be equal to zero enables one to determine the value of the electromagnetic charge e. The argument for omitting is that this part contradicts the gauge invariance. The latter statement is wrong for the following reason. a) The formulation of electrodynamics we use (eqs. (1)-(5)) evidently is gauge invariant throughout. b) The $\sim\ell^2$ part of the operator $D(k;q,m,\ell)^{-1}$ is nonzero in the lowest $\sim e^2$ order of perturbation theory. This is evident when we take into account the eqs. (3) and (4).

6.1. We stress that our consideration is gauge invariant throughout. The variables which were subjected to the gauge transformation were removed in deriving the Hamiltonian (1)-(5) /1/. The operators $\vec{A}^{tr}(\vec{x},t)$ and $\psi(\vec{x},t)$ in eqs. (1)-(5) are gauge invariant*.

6.2. The ℓ^2 divergent part of the operator $D(k;q,m,\ell)^{-1}$ does not arise if one uses the gauge invariant Pauli-Villars regularization

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* $\vec{A}^{tr}(x) = \vec{A}(x) - \Delta^{-1}\,\mathrm{grad}\,\mathrm{div}\vec{A}$,

$\psi(x) = \exp[ie\Lambda_0(x)]\psi_0(x)$, $\Lambda_0(x) = \Delta^{-1}\mathrm{div}\vec{A}$,

where $\psi_0(x)$ is the function in the initial Lagrangian $\delta\mathfrak{L} = -ie\bar{\psi}_0\gamma_\mu\psi_0 A_\mu(x)$; Δ is the Laplace operator.

(4). This regularization, however useful practically, cannot be considered as satisfactory in principle, for it means, e.g., the breakdown of the Hamiltonian (1)-(5) and Schrödinger equations.

7. In the present work we use the noncovariant cutoff. For this reason we have to use noncovariant counter-terms of wave function renormalization /4/.

8. The possibility of the coupling constant determination, noted in this paper, exists evidently, in any other model where the mass renormalization term of type (1a) is forbidden (e.g., the scalar electrodynamics and massless Yang-Mills model)*.

9. The problem of coupling constant determination we discuss is considered also in two (at least) groups of works /5,6/ and /7/ (see also references in /7/). All these works do not take into account the $\sim\ell^2$ part of the operator $D(k;q,m,\ell)^{-1}$.

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*See also a model, defined by Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right) - g\phi^4.$$

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