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ON THE SUPERSYMMETRIC SINE-GORDON MODEL AND A TWO-DIMENSIONAL "BAG"

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О суперсимметрической модели sine-Gordon и о двухмерном "мешке"

В работе обсуждается модель sine-Gordon для случая скалярного суперполя в двухмерном пространстве. Получены новые уравнения, смешивающие ферми- и бозе-поля и их стационарные решения. Результаты обобщены на случай модели двухмерного "мешка".

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Hruby J. E2 - 10793

On the Supersymmetric Sine-Gordon Model and a Two-Dimensional "BAG"

The sine-Gordon model as the theory of a massless scalar field in one-space and one-field dimension with interaction Lagrangian density proportional to $\cos\beta\phi$ is generalized for scalar superfield. There are obtained exact solutions of the supercovariant coupled equations of motion. From this a "BAG" model is constructed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

Most of the physicists believe that quantum field theory based on quark fields is a good candidate for hadron dynamics. Despite the successes of the quark model, one is puzzled as to why we do not see quarks.

One of the possible answers is "quark confinement"/1/. The quarks may be "confined" in the "quark bags", for example the "SLAC-BAG" model/2/. This "quark bag" can be related to stable classical solutions of non-linear field equations, like "soliton".

An instructive example of a soliton field is determined by the Lagrangian density/3/:

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{a_0}{\beta^2}(\cos\beta\phi - 1). \qquad (1.1)$$

Here a_0 and β are real parameters, the physical meaning of which is the following: a_0 is the "squared mass" of the minimum energy excitations and β is a parameter which measures the strength of the interaction between these small oscillations.

From the Lagrangian density (1.1) the equation of motion is the sine-Gordon equation:

$$\Box\phi = -\frac{a_0}{\beta}\sin\beta\phi. \qquad (1.2)$$

The static solution of Eq. (1.2) is the soliton [1]:

$$\phi_S = \frac{4}{\beta} \mathrm{tg}^{-1} \exp \sqrt{\alpha_0}\, x . \tag{1.3}$$

The main goal of this work is the supersymmetric sine-Gordon model and a two dimensional "BAG" obtained from this. The procedure is the following: we shall start from the four dimensional superspace, using the technique of superfields in two-Bose- and two-Fermi-dimensions by straightforward adaptation of the usual technique in eight dimensional superspace. It means that the scalar superfield and the covariant derivative are given by the usual expressions [4]:

$$S(x_\alpha, \theta_a) = \phi(x) + i\bar{\theta}\psi(x) + \frac{i}{2}\bar{\theta}\theta F(x), \tag{1.4}$$

$$\bar{D}_a = \frac{\partial}{\partial\theta} + i\bar{\theta}\gamma\cdot\partial, \tag{1.5}$$

where $\alpha = 0,1$ and $a = 0,1$. The supermultiplet $\{\phi(x), \psi(x), F(x)\}$ contains the Fermi field (Majorana spinor) $\psi(x)$ and the Bose fields $\phi(x)$ and $F(x)$. Using the covariant derivative on the scalar superfield $S(x,\theta)$ one can construct the expression

$$\frac{i}{2}\bar{D}DS(x,\theta) = F(x) + i\bar{\theta}\not{\partial}\psi(x) + \frac{i}{2}\bar{\theta}\theta\Box\phi(x) \tag{1.6}$$

and obtain the free massless equation

$$\bar{D}DS(x,\theta) = 0. \tag{1.7}$$

The usual massless free-field Lagrangian is the coefficient of $\bar{\theta}\theta$ in the following expression:

$$\frac{i}{2} S(x,\theta)\bar{D}D\, S(x,\theta) = \ldots i\bar{\theta}\theta\left(\frac{1}{2}\phi\Box\phi + \frac{1}{2}F^2 - \frac{i}{2}\bar{\psi}\not{\partial}\psi\right) \tag{1.8}$$

In Sec. 2 we construct the supersymmetric sine-Gordon model and obtain two basic equations of motion coupling the Fermi field $\psi(x)$ and the Bose field $\phi(x)$. Using the soliton (1.3) as the input potential in the equation of motion for the Fermi field we obtain the exact stationary solution for ψ. In Sec. 3 we get from these equations the coupled semi-classical differential equations for the colourless quark field $\psi(x)$ of one flavour only and for the Higg's field $\phi(x)$. Such equations are obtained directly from the supersymmetric sine-Gordon model and are the starting point of the "BAG" model.

The exact solution of the coupled equations is derived in Sec. 4. So, we obtain the "BAG" describing strongly bound quarks in two dimensions from the supersymmetric sine-Gordon point of view.

2. THE SUPERSYMMETRIC SINE-GORDON MODEL

We shall proceed as follows: first, we construct the supercovariant sine-Gordon equation for the scalar superfield $S(x,\theta)$ using the supersymmetry Lagrangian technique [5]. By analogy with the Lagrangian density (1.1) the dynamics of the supersymmetric sine-Gordon model will be determined by the Lagrangian density which is the coefficient of $\bar{\theta}\theta$ in the following expression:

$$\frac{i}{2} S(x,\theta)\bar{D}D\, S(x,\theta) + V(S(x,\theta)), \tag{2.1}$$

where

$$V(S(x,\theta)) = \frac{a}{b^2}(\cos bS(x,\theta) - 1),$$

and a, b are parameters which will be specified later. The supercovariant equation of motion is

$$\frac{i}{2}\bar{D}D\,S(x,\theta) = V'(S(x,\theta)), \qquad (2.2)$$

where

$$V'(S(x,\theta)) = -\frac{a}{b}\sin bS(x,\theta).$$

It is the supercovariant sine-Gordon equation. Now, we obtain the coupled equation of motion from the Lagrangian density (2.1) for ordinary fields in two dimensions.

First, we use the Taylor expansion of the cosine in the expression for the $V(S(x,\theta))$:

$$V(S(x,\theta)) = \frac{a}{b^2}(\cos bS(x,\theta) - 1) \qquad (2.3)$$

$$= \frac{a}{b^2}\left(- \frac{b^2 S^2(x,\theta)}{2!} + \frac{b^4 S^4(x,\theta)}{4!} - \frac{b^6 S^6(x,\theta)}{6!} \ldots \right).$$

From the superfield theory/5/ we know that the product of superfields is again a superfield, and therefore, can be expanded in Taylor's expansion in θ which is finished for two θ because the θ-elements anticommute. This specific characteristic of superfields allows one to express the cosine in the relation (2.3):

$$\frac{a}{b^2}\Big[-\frac{b^2}{2!}(\phi^2 + 2i\bar{\theta}\psi\phi + i\bar{\theta}\theta F\phi - \bar{\theta}\theta\bar{\psi}\psi)$$

$$+\frac{b^4}{4!}(\phi^4 + 4i\bar{\theta}\psi\phi^3 + 2i\bar{\theta}\theta F\phi^3 - 6\bar{\theta}\theta\bar{\psi}\psi\phi^2) \qquad (2.4)$$

$$-\frac{b^6}{6!}(\phi^6 + 6i\bar{\theta}\psi\phi^5 + 3i\bar{\theta}\theta F\phi^5 - 15\bar{\theta}\theta\bar{\psi}\psi\phi^4)\,\text{etc.}$$

where we use $\bar{\theta}\psi = \bar{\psi}\theta$ and the symbolic expression for the fields. In the Lagrangian density, which is obtained from the relation (2.1), the elements with two θ only are important in the supersymmetric Lagrangian, and so, we get for ordinary fields Lagrangian density in two dimensional space the following expression:

$$L = \frac{1}{2}\phi\Box\phi + \frac{1}{2}F^2 - \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{a}{2}(F\phi + i\bar{\psi}\psi$$

$$- \frac{b^2}{3!}F\phi^3 - \frac{ib^2}{2!}\bar{\psi}\psi\phi^2$$

$$+ \frac{b^4}{5!}F\phi^5 + \frac{ib^4}{4!}\bar{\psi}\psi\phi^4)\,\text{etc.}$$

$$= \frac{1}{2}\phi\Box\phi + \frac{1}{2}F^2 - \frac{i}{2}\bar{\psi}\partial\psi - \frac{a}{2b}F\sin b\phi - \frac{i}{2}a\bar{\psi}\psi\cos b\phi. \qquad (2.5)$$

There is no kinetic term for F; hence, this field can be eliminated by using the equation of motion which includes:

$$F - \frac{a}{2b} \sin b\phi = 0. \qquad (2.6)$$

Using Eq. (2.6) we get the Lagrangian density

$$L = \frac{1}{2}\phi\Box\phi - \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{8}\frac{a^2}{b^2}\sin^2 b\phi - \frac{ia}{2}\bar{\psi}\psi\cos b\phi \qquad (2.7)$$

and hence, we obtain two basic equations of motion coupling the Fermi field ψ and the Bose field ϕ:

$$(i\partial\!\!\!/ + ia\cos b\phi)\psi = 0, \qquad (2.8a)$$

$$\Box\phi - \frac{1}{4}\frac{a^2}{b}\sin b\phi\cos b\phi + \frac{i}{2}ab\bar{\psi}\psi\sin b\phi = 0. \qquad (2.8b)$$

We can see that if we have no spinor fields, i.e., $\psi = 0$, then (2.8b) is equivalent to the sine-Gordon equation (1.2) under the conditions:

$$\Box\phi - \frac{a^2}{8b}\sin 2b\phi = 0$$

$$\beta = 2b$$

$$\frac{\alpha_0}{\beta} = -\frac{a^2}{8b}.$$

In this case we obtain the following expression for the parameters: $b = \frac{\beta}{2}$, $a = 2i\sqrt{\alpha_0}$.

The static solution of Eq. (1.2) is the soliton (1.3). We shall now consider the soliton (1.3) as the potential in Eq. (2.8a) and we obtain for the static solution ψ the following equation:

$$i\gamma^1\partial_1\psi = -ia\cos b\left(\frac{4}{\beta}\mathrm{tg}^{-1}\exp\sqrt{\alpha_0}\,x\right)\psi. \qquad (2.9)$$

We want to derive the stationary solution of Eq. (2.9):

$$i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = 2\sqrt{\alpha_0}\cos 2\,\mathrm{tg}^{-1}\exp\sqrt{\alpha_0}\,x\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where we denote $\partial_1\psi = \psi'$. So, we have the equations:

$$+i\psi_2' = 2\sqrt{\alpha_0}\cos 2\,\mathrm{tg}^{-1}\exp\sqrt{\alpha_0}\,x\,\psi_1, \qquad (2.10a)$$

$$-i\psi_1' = 2\sqrt{\alpha_0}\cos 2\,\mathrm{tg}^{-1}\exp\sqrt{\alpha_0}\,x\,\psi_2. \qquad (2.10b)$$

Now, we denote $\psi(x) = \chi(x)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and from Eqs. (2.10a,b) we get

$$\frac{u_2}{u_1} + \frac{u_1}{u_2} = 0$$

$$u_1 = u \qquad (2.11)$$

$$u_2 = \pm i$$

and we have to solve one equation

$$\pm\chi' = 2\sqrt{\alpha_0}\cos 2\,\mathrm{tg}^{-1}\exp\sqrt{\alpha_0}\,x\,\chi. \qquad (2.12)$$

The solution of Eq. (2.12) has the amazingly simple form

$$\chi = C\,(\cosh\sqrt{a_0}\,x)^{\pm 2}.$$

Since the condition

$$\int |\psi|^2 < \infty$$

must be fulfilled, we obtain the stationary solution

$$\psi(x) = C\,(\cosh\sqrt{a_0}\,x)^{-2}\begin{pmatrix}1\\-i\end{pmatrix},$$

where C is the constant. Such a solution is a fermion like the soliton.

3. A TWO-DIMENSIONAL "BAG" MODEL

Starting with Eq. (2.8a,b), we shall now study small oscillations about the ground state $\phi=0$. We can expand the cosine and sine in power series in Eq. (2.8a,b), and we obtain:

$$i\partial\!\!\!/\,\psi + ia\psi = \frac{iab^2}{2}\phi^2\psi, \qquad (3.1a)$$

$$\Box\phi - \frac{1}{4}a^2\phi + \frac{a^2b^2}{6}\phi^3 = -\frac{iab^2}{2}\phi\bar{\psi}\psi. \qquad (3.1b)$$

We shall specify the parameters a, b with usual notation (see ref. /1/):

$$g = i\frac{ab^2}{2},$$

$$\mu^2 = -\frac{a^2}{4},$$

$$\Lambda = -\frac{a^2b^2}{6},$$

where Λ and μ^2 are the positive parameters in the familiar ϕ^4 theory and g is the coupling constant. Now, the equations (3.1a, b) become:

$$i\partial\!\!\!/\,\psi - 2\mu\psi = g\phi^2\psi, \qquad (3.2a)$$

$$\Box\phi + \mu^2\phi - \Lambda\phi^3 = -g\phi\bar{\psi}\psi. \qquad (3.2b)$$

These equations are the starting point of the "BAG" model describing strongly bound quarks in a simple two dimensional case.

Equations (3.2a,b) are derived from the supersymmetric Lagrangian density (2.7) for small oscillations:

$$L = \frac{1}{2}\phi\Box\phi - \frac{1}{8}a^2\phi^2 + \frac{1}{24}a^2b^2\phi^4 - \frac{i}{2}\bar{\psi}\partial\!\!\!/\,\psi - \frac{ia}{2}\bar{\psi}\psi + \frac{iab^2}{4}\phi^2\bar{\psi}\psi$$

or in usual notation

$$L = \frac{1}{2}\phi\Box\phi + \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\Lambda\phi^4 - \frac{i}{2}\bar{\psi}\partial\!\!\!/\,\psi + \mu\bar{\psi}\psi + \frac{g}{2}\phi^2\bar{\psi}\psi. \qquad (3.3)$$

There are trivial classical solutions of Eqs. (3.2a,b)

$$\psi = 0, \qquad \phi^{\pm}_{cl} = \pm\frac{\mu}{\sqrt{\Lambda}} \qquad (3.4)$$

corresponding to the minimum of the "potential energy" $V(\phi)$:

$$V(\phi) = \frac{\Lambda}{4} \int (\phi^2(x) - \frac{\mu^2}{\Lambda})^2 d^1x$$

which is normalized to zero. We can see that the energy degeneracy of the classical ground state solutions (3.4) implies spontaneous symmetry breaking.

Expanding around the stable vacuum solutions (3.4):

$$\phi(x) = \tilde{\phi}(x) \pm \frac{\mu}{\sqrt{\Lambda}}$$

leads to

$$[-i\partial\!\!\!/ - (2\mu + g\frac{\mu^2}{\Lambda})]\psi \pm 2\frac{\mu}{\sqrt{\Lambda}} g\tilde{\phi}\psi - g\tilde{\phi}^2\psi = 0 \qquad (3.5a)$$

$$(-\Box - 2\mu^2)\tilde{\phi} \pm 3\mu\sqrt{\Lambda}\,\tilde{\phi}^2 - \Lambda\tilde{\phi}^3 + g(\tilde{\phi} \pm \frac{\mu}{\sqrt{\Lambda}})\bar{\psi}\psi = 0 \qquad (3.5b)$$

thus determining the classical mass parameters of the quark field

$$m_q = 2\mu + \frac{g\mu^2}{\Lambda} \qquad (3.6)$$

and of the Higg's field

$$m_H = \sqrt{2}\,\mu \,. \qquad (3.7)$$

From Eqs. (3.2a,b) we can see there is another stationary solution, with $\psi = 0$ of the equation

$$(\frac{d^2}{dx^2} + \mu^2 - \Lambda\phi^2)\phi(x) = 0 \qquad (3.8)$$

namely, the "kink solution":

$$\phi(x) = \frac{\mu}{\Lambda} \tanh\frac{\mu x}{\sqrt{2}} \,. \qquad (3.9)$$

This kink solution has some obvious characteristic features of the classical solutions:

i) $\phi(x)$ is classically stable,

ii) $\phi(x)$ connects the two degenerate vacuum solutions:

$$\phi(x) \to \pm\frac{\mu}{\sqrt{\Lambda}} \qquad \text{for } x \to \pm\infty \,;$$

iii) The total energy of $\phi(x)$ according to (3.4) is finite (see also ref. /2/)

$$E(\phi) = \frac{\sqrt{8}}{3}\frac{\mu^3}{\Lambda} \qquad (3.10)$$

and the energy density is concentrated in a finite region around $x = 0$;

iv) The kink solution $\phi(x)$ is singular for $\Lambda \to 0$, i.e., "non-perturbative".

The solution of non-linear field equations with the properties i)-iv) is the soliton. In our approach we have obtained such a soliton directly from the supersymmetric sine-Gordon model.

Of course the soliton $\phi(x)$ is stable, because it is the lowest energy solution with the conserved topological number

$$T = \frac{\sqrt{\Lambda}}{2\mu}(\phi(\infty) - \phi(-\infty)).$$

The topological number is conserved because the spatial asymptotic behaviour of solutions $\phi(x,t)$ of the field equation (3.2b) remains fixed. The topological property of the 2-dimensional spacetime allows one to relate T to a conserved current:

$$B^{\mu}(x,t) = \frac{\sqrt{\Lambda}}{2\mu}\,\epsilon^{\mu\nu}\,\partial_{\nu}\,\phi(x,t)$$

$$T = \int_{-\infty}^{+\infty} B_0(x,t)\, d^1x \qquad (3.11)$$

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}, \qquad \epsilon^{01} = 1 .$$

Now, we present the exact solution to our coupled field equations (3.2a,b) in 1 space, 1 time dimension.

4. THE "KINK WITH TRAPPED QUARK"

Equation (3.2b), as was shown in Sec. 3, has for the "kink solution" (3.9):

$$\phi_K(x) = \frac{\mu}{\sqrt{\Lambda}} \tanh \frac{\mu x}{\sqrt{2}} .$$

If we now consider this solution as the input potential in Eqs. (3.2a), we obtain:

$$i\gamma^1 \partial_1 \psi - 2\mu\psi = g\frac{\mu^2}{\Lambda} \tanh^2 \frac{\mu x}{\sqrt{2}} \psi . \qquad (4.1)$$

We want to obtain the stationary solution of Eq. (4.1):

$$i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = (2\mu + \frac{g\mu^2}{\Lambda} \tanh^2 \frac{\mu x}{\sqrt{2}}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

where we denote $\partial_1 \psi = \psi'$. So, we have the equations:

$$+ i\psi_2' = (2\mu + \frac{g\mu^2}{\Lambda} \tanh^2 \frac{\mu x}{2})\psi_1 , \qquad (4.2a)$$

$$- i\psi_1' = (2\mu + \frac{g\mu^2}{\Lambda} \tanh^2 \frac{\mu x}{2})\psi_2 . \qquad (4.2b)$$

Using the same method as in Sec. 2 we have to solve one equation:

$$\pm \chi' = (2\mu + \frac{g\mu^2}{\Lambda} \tanh^2 \frac{\mu x}{\sqrt{2}})\chi . \qquad (4.3)$$

The solution of Eq. (4.3) has the form:

$$\chi = C \exp \pm [(2\mu + \frac{g\mu^2}{\Lambda})x - \frac{g\mu\sqrt{2}}{\Lambda} \tanh \frac{\mu x}{\sqrt{2}}] . \qquad (4.4)$$

Using the relation (3.6) for the quark mass and the relation (2.11), we can write

$$\psi(x) = C \exp \pm (m_q x - \frac{g\mu\sqrt{2}}{\Lambda} \tanh \frac{\mu x}{\sqrt{2}}) \begin{pmatrix} 1 \\ \mp i \end{pmatrix} . \qquad (4.5)$$

Since the condition

$$\int |\psi|^2 < \infty$$

must be fulfilled, we obtain the stationary solution of Eq. (4.1)

$$\psi_{tr}(x) = C \exp -(m_q x - \frac{g\mu\sqrt{2}}{\Lambda} \tanh \frac{\mu x}{\sqrt{2}}) \begin{pmatrix} 1 \\ +i \end{pmatrix} \quad \text{for } x > 0$$

$$\psi_{tr}(x) = C \exp +(m_q x - \frac{g\mu\sqrt{2}}{\Lambda} \tanh \frac{\mu x}{\sqrt{2}}) \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{for } x < 0 . \qquad (4.6)$$

The solution (3.9) (the kink field $\phi_K(x)$) and the solution (4.6) (the trapped quark field $\psi_{tr}(x)$) provide exact solutions of the coupled Eqs. (3.2a,b).

5. COMMENTS

We have discussed the realization of the two-dimensional supersymmetry theory in the sine-Gordon model. We have obtained the coupled field equations in two dimensions and also their exact stationary solutions:

$$\phi(x) = \frac{4}{\beta}\,\mathrm{tg}^{-1}\ \exp\sqrt{a_0}\,x\ ,$$

$$\psi(x) = C\,(\cosh\sqrt{a_0}\,x)^{-2}\begin{pmatrix}1\\ -i\end{pmatrix}.$$

We have studied how from the two-dimensional supersymmetric sine-Gordon model the quark confinement is obtained . By analogy with the "SLAC-BAG" model we have obtained "kink with trapped quark" in our "BAG" model.

The trapping of the quark does not play role in the kink energy because

$$\bar{\psi}\psi = \psi^{+}\gamma^{0}\psi = \chi\,(1, \mp i)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ \pm i\end{pmatrix}\chi = 0.$$

The integral for the mass term vanishes, and we obtain the same expression (3.10) for the classical energy.

Thus, the state "kink with trapped quark" described by the classical solutions (3.9) and (4.6) represents a "field theoretical bound state" with strong binding and may describe hadrons with confined quarks. In contrast with "SLAC-BAG" in the ref. /2/ our approach is obtained directly from the supersymmetric sine-Gordon model.

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