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TOPOLOGIES ON THE ALGEBRA OF TEST FUNCTIONS

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Топологии на алгебре выборочных функций

Алгебраическая структура $\mathcal{S}_\otimes$ (тензорная алгебра над пространством Шварца $\mathcal{S}$) определяет две топологии, $\tau_\varphi, \tau_\otimes$. В данной работе исследуются некоторые свойства локально-выпуклых топологий, находящихся между τ_φ и $\tau_\otimes$, строится множество топологий, в которых конус K положительных элементов нормален, и рассматривается непрерывность функционалов Уайтмана свободных полей и квадрата Вика свободных полей и их производных в таких топологиях.

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Topologies on the Algebra of Test Functions

The algebraical structure of $\mathcal{S}_\otimes$ (tensor algebra over the Schwartz space $\mathcal{S}$) defines two topologies, $\tau_\varphi, \tau_\otimes$. We study some properties of the locally convex topologies situated between τ_φ and $\tau_\otimes$, construct a lot of topologies in which the cone K of positive elements is a normal one and regard the continuity of the Wightman functionals of the free fields and of the Wick squares in the free fields and their derivatives in such topologies.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

The motivation of this paper comes from quantum field theory and the study of the algebra of test functions $\mathcal{S}_\otimes$. This paper investigates the topologies on $\mathcal{S}_\otimes$. The algebraical structure of $\mathcal{S}_\otimes$ defines two topologies, firstly the topology of the direct sum $\tau_\otimes$ and secondly the topology τ_φ which is the restriction of the topology of the direct product to $\mathcal{S}_\otimes$. We study some properties of the locally convex topologies situated between $\tau_\otimes$ and τ_φ. Such topologies are, for instance, τ_∞, $\mathcal{N}$ studied in refs. /5,8,16/. In section 4 we investigate the normality of the cone K of positive elements with respect to these topologies and construct a lot of normal topologies. The results of this paper show that the topology $\tau_\otimes$ is a "good" one from view-point of the topological structure but a "bad" one from view-point of the order structure. In picture 9 we collect the results of sections 3 and 4. The topologies between τ_φ and $\tau_\otimes$ give a possibility of classifying the Wightman functionals with respect to the continuity in such topologies (section 5).

2. SOME NOTATIONS AND DEFINITIONS

The algebras of test functions in quantum field theory was introduced in refs./1,13/ and studied for instance in refs./2,15/. The elements of $\mathcal{S}_\otimes$ are finite sequences of the form

$$f = (f_0, f_1, \ldots, f_n, 0, \ldots)$$

with $f_0 \in C$, $f_k(x_1, \ldots, x_k) \in \mathcal{S}(R^{dk})$, where d is the space-time dimension, $\mathcal{S}(R^{dk})$ is the Laurent-Schwartz test function space /12/. It is $x_k = (x_k^0, x_k^1, \ldots, x_k^{d-1}) = (x_k^0, \vec{x}_k)$, $k = 1,2,\ldots$. We put $\mathcal{S}_k = \mathcal{S}(R^{dk})$. $\mathcal{S}_\otimes = \overset{\infty}{\underset{n=0}{\oplus}} \mathcal{S}_n$ is the topological direct sum of the spaces $\mathcal{S}_n, \mathcal{S}_0 = C$.

For $f, g \in \mathcal{S}_\otimes$ one defines the N-th component of f^*g by

$$(f^*g)_N(x_1, \ldots, x_N) = \sum_{n+m=N} \overline{f_n(x_n, \ldots, x_1)}\, g_m(x_{n+1}, \ldots, x_{n+m})$$

and the N-th component of $\lambda f + g$ by $(\lambda f + g)_N = \lambda f_N(x_1, \ldots, x_N) + g_N(x_1, \ldots, x_N)$, $\lambda \in C$. Thus $\mathcal{S}_\otimes$ becomes a $*$-algebra with identity $1 = (1, 0, \ldots)$. $K = \{\sum_{i<\infty} f^{(i)*} f^{(i)} ; f^{(i)} \in \mathcal{S}_\otimes\}$ is the cone of the positive elements. Some properties of K are investigated in /3,9,16/. Now there is the question about the topologies on $\mathcal{S}_\otimes$. All considerations of this paper are restricted to the case of locally convex topologies.

let ν_n be the well-known Schwartz space topology on $\mathcal{S}_n$, for instance, defined by

$$\|f_n\|_m = \sup_x \max_{r_i^j \leq m} \left| \prod_{i=1}^{n} \prod_{j=0}^{d-1} (1 + (x_i^j)^2)^m \left(\frac{\partial}{\partial x_i^j}\right)^{r_i^j} f_n(x_1, \ldots, x_n) \right| \tag{1}$$

$m = 0,1,\ldots,\ n = 1,2,\ldots$.

The algebraical structure of $\mathcal{S}_\otimes$ defines two topologies:

1) The topology of the direct sum $\tau_\otimes$ defined by the following system of semi-norms

$$p_{(\gamma_n)(\nu_n)}(f) = \sum_{n \geq 0} \gamma_n \|f_n\|_{\nu_n} \tag{2}$$

for all sequences (γ_n) of positive numbers and all sequences (ν_n) of natural numbers. It is $\|f_0\|_m = |f_0|$, $m = 0,1,\ldots$.

2) The topology $\tau\wp$ is the restriction of the topology of the direct product of the spaces $\mathcal{S}_n$ to $\mathcal{S}_\otimes$. $\tau\wp$ is defined by the following system of semi-norms

$$q_{n,m}(f) = \|f_n\|_m, \quad n,m = 0,1,2,\ldots \ . \tag{3}$$

If we restrict the set of sequences (γ_n) or the set of (ν_n) then we get a lot of topologies weaker than $\tau_\otimes$. Let Γ be the set of all sequences (γ_n) of positive numbers and N the set of all sequences (ν_n) of natural numbers. For each $\Gamma_1 \subset \Gamma$, $N_1 \subset N$ we define the topology $\tau(\Gamma_1, N_1)$ by

$$\{p_{(\gamma_n)(\nu_n)}(f) = \sum_{n>0} \gamma_n \|f_n\|_{\nu_n} ; (\gamma_n) \in \Gamma_1, (\nu_n) \in N_1\}. \tag{4}$$

Let

$$\begin{aligned} \Gamma_0 &= \{(\gamma_n) \in \Gamma ; \ \gamma_n \neq 0 \text{ for a finite number of } n \text{ only}\}, \\ N_0 &= \{(\nu_n) \in N ; \ \text{to every sequence } (\nu_n) \text{ there is a constant } m \text{ with } \nu_n \leq m,\ n = 0,1,\ldots\}. \end{aligned} \tag{5}$$

A simple consequence of the above given definitions is

Proposition 1: Let be

i) $\tau(\Gamma_0, N) = \tau_{\mathcal{P}}, \tau(\Gamma, N_0) = \tau_\infty, \tau(\Gamma, N) = \tau_\otimes$,

ii) $\tau(\Gamma_1, N_1) \prec \tau(\Gamma_2, N_2)$ for $\Gamma_1, \Gamma_2 \subset \Gamma, N_1, N_2 \subset N$ and $\Gamma_1 \subset \Gamma_2$, $N_1 \subset N_2$.

($\tau_1 \prec \tau_2$ means that the topology τ_2 is stronger (finer) than τ_1). τ_∞ is the important topology introduced in /5/.

Let us define a generalization of the topologies (4) which will be of some interest in section 4. Let $\mathfrak{M}$ be the set of all matrices of natural numbers with enumerable infinite many rows and columns, i.e., $\mathfrak{M} = \{(m_{ij})_{i,j=1,2,\ldots};$ m_{ij} is a natural number$\}$ and let

$$\|f_n\|_{(m_{nj})} = \sup_{s} \max_{r_i^j \leq m_{ni}} \left|(1+x_1^2)^{m_{n1}} \ldots (1+x_n^2)^{m_{nn}} \prod_{i=1}^{n} \prod_{j=0}^{3} \left(\frac{\partial}{\partial x_i^j}\right)^{r_i^j} f_n\right| \tag{6}$$

be for all $f_n \in \mathcal{S}_n$, $n = 1,2,\ldots$.
We define the topology $\lambda(\Gamma_1, \mathfrak{M}_1)$, $\Gamma_1 \subset \Gamma$, $\mathfrak{M}_1 \subset \mathfrak{M}$ by the following system of semi-norms

$$\{p_{(\gamma_n)(m_{nj})}(f) = \sum_{n\geq 0} \gamma_n \|f_n\|_{(m_{nj})}; (\gamma_n) \in \Gamma_1, (m_{nj}) \in \mathfrak{M}_1\} \tag{7}$$

$(\|f_0\|_{(m_{oj})} = |f_0|)$.

If $m_{n1} = m_{n2} = \ldots = \nu_n$, $n = 1,2,\ldots$, then $p_{(\gamma_n)(m_{nj})}(f) = p_{(\gamma_n)(\nu_n)}(f)$.

Let be

$$\mathfrak{M}_0 = \{(m_{ij}) \in \mathfrak{M} ; \text{ to every } (m_{ij}) \text{ there is a constant } c \text{ with } m_{ij} \leq c,\ i,j = 1,2,\ldots\}. \tag{8}$$

In analogy to Proposition 1 we have

Proposition 2:

Let be

i) $\lambda(\Gamma_0, \mathfrak{M}) = \tau_{\mathcal{P}}$, $\lambda(\Gamma, \mathfrak{M}_0) = \tau_\infty$, $\lambda(\Gamma, \mathfrak{M}) = \tau_\otimes$,

ii) $\lambda(\Gamma_1, \mathfrak{M}_1) \prec \lambda(\Gamma_2, \mathfrak{M}_2)$ for $\Gamma_1, \Gamma_2 \subset \Gamma, \mathfrak{M}_1, \mathfrak{M}_2 \subset \mathfrak{M}$ and $\Gamma_1 \subset \Gamma_2, \mathfrak{M}_1 \subset \mathfrak{M}_2$.

Another important topology is the topology $\mathfrak{N}$ introduced in /17/. $\mathfrak{N}$ is the strongest topology on $\mathcal{S}_\otimes$ such that the multiplication on $\mathcal{S}_\otimes$ is a jointly continuous bilinear mapping m: $\mathcal{S}_\otimes[\tau_\otimes] \times \mathcal{S}_\otimes[\tau_\otimes] \to \mathcal{S}_\otimes[\mathfrak{N}]$.

3. TOPOLOGICAL PROPERTIES

In this section we study the topological properties of $\mathcal{S}_\otimes[\tau]$ with $\tau_{\mathcal{P}} < \tau < \tau_\otimes$. The known results of $\tau_\infty, \mathfrak{N}, \tau_\otimes$ /2,8,15, 17/ settle down in the results of this section. We state the results in four lemmas. Further let τ be locally convex.

Lemma 3:

i) The restriction of any topology τ with $\tau_{\mathcal{P}} < \tau < \tau_\otimes$ to the subspaces $\mathcal{S}_n$ $(n = 1,2,\ldots)$ of $\mathcal{S}_\otimes$ is the well-known Schwartz space topology ν_n .

ii) $\tau_{\mathcal{P}}$ is the weakest topology on $\mathcal{S}_\otimes$ with this property but $\tau_\otimes$ the strongest.

The proof follows from the theory of the direct sum and the direct product /10/.

Let S_N be the projection from $\mathcal{S}_\otimes$ onto the subspace $\mathbb{C} \oplus \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_N$, i.e, $S_N(f) = (f_0, \dots, f_N, 0, \dots)$, $N = 0,1,\dots$, and let $\bar{M}^\tau$ be the closure of some set M with respect to the topology τ. We state

Lemma 4:

Let $\tau_{\mathcal{P}} < \tau < \tau_\otimes$, $M \subset \mathcal{S}_\otimes$ and $S_N M \subset M$, $N = 0,1,\dots$.

Then $\bar{M}^{\tau_{\mathcal{P}}} = \bar{M}^\tau = \bar{M}^{\tau_\otimes}$.

The proof of this lemma is in analogy to that of Theorem 6 of /17/.

Remark: After Lemma 6 we give an example of a set M with $S_N M \not\subset M$, $N = 0,1,\dots$, and $\bar{M}^\tau \neq \bar{M}^{\tau_\otimes}$ for some $\tau \neq \tau_\otimes$.

Lemma 5:

Let $\tau_\infty \prec \tau \prec \tau_\otimes$.

i) If there is a filter-base $\mathfrak{U}(\tau)$ of 0 of the topology τ with $S_N U \subset U$, $N = 0,1,\dots$, for all $U \in \mathfrak{U}(\tau)$, then $\mathcal{S}_\otimes[\tau]$ is complete.

ii) The τ-bounded sets are the same in all topologies τ. To every τ-bounded set there is a natural number m with $M \subset \mathbb{C} \oplus \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_m$.

Proof:

i) $\mathcal{S}_\otimes[\tau_\otimes]$ is an LF-space and thus is complete. $\mathcal{S}_\otimes[\tau_\infty]$ is complete too /5/. Let $\mathcal{F}$ be a τ-Cauchy filter in $\mathcal{S}_\otimes$. Then $\mathcal{F}$ is a τ_∞-Cauchy filter in $\mathcal{S}_\otimes$, too, and because of the completeness of $\mathcal{S}_\otimes[\tau_\infty]$ there is an element $f \in \mathcal{S}_\otimes$ with $\mathcal{F} \to f$ in τ_∞. Now let V be in $\mathfrak{U}(\tau)$ with $S_N V \subset V$, $N = 0,1,\dots$. Then $\mathcal{F}$ contains a set B small of the order $\frac{1}{2}V$. For any element g of B, $B \subset g + \frac{1}{2}V$. Because of $S_N V \subset V$, $N = 0,1,\dots$, and Lemma 4 V is τ_∞-closed. From the facts that f is in the τ_∞-closure of B and V is τ_∞-closed we get that f is an element of $g + \frac{1}{2}V$. Then g is an element of $f + \frac{1}{2}V$ and $B \subset f + \frac{1}{2}V + \frac{1}{2}V = f + V$. So we have $\mathcal{F} \to f$ with respect to the topology τ, too. This completes the proof of i).

ii) We prove that the bounded sets with respect to the topologies $\tau_\otimes$ and τ_∞ are the same. Then the assertion is true for all topologies τ with $\tau_\infty \prec \tau \prec \tau_\otimes$. Let M be τ_∞-bounded. Because there are sequences $(\gamma_n) \in \Gamma$ which grow arbitrary quickly there must be a natural number m with $M \subset \mathbb{C} \oplus \dots \oplus \mathcal{S}_m$. We get by Lemma 3 i) that the restrictions of $\tau_\otimes$ and τ_∞ to $C \oplus \dots \oplus \mathcal{S}_m$ are the same, thus M is $\tau_\otimes$-bounded, too.

Now let us give an example of a topology τ^* with $\tau_\infty \underset{+}{\prec} \tau^* \underset{+}{\prec} \tau_\otimes$ and $\mathcal{S}_\otimes[\tau^*]$ is not complete.

Example:

Let T be a $\tau_\otimes$-continuous linear functional on $\mathcal{S}_\otimes$ that is not τ_∞-continuous. The existence of such functionals will be stated in Remark 8 iii). Let τ^* be the topology described by the following system of seminorms $\{p_{(\gamma_n)(\nu_n)}(f); (\gamma_n) \in \Gamma, (\nu_n) \in N_0\}$ and $p_T(f) = |T(f)|$ (N_0 is defined in (5)). We have $\tau_\infty \underset{+}{\prec} \tau^* \underset{+}{\prec} \tau_\otimes$. But $S_N U \subset U$, $N = 0,1,\dots$, for $U = \{f \in \mathcal{S}_\otimes; |T(f)| \leq 1\}$. does not hold.

Let $H = \{f \in \mathcal{S}_\otimes;\ T(f) = 1\}$. H is dense in $\mathcal{S}_\otimes[\tau_\infty]$ because T is not τ_∞-continuous. Thus there is a net $(f^{(a)})_{a \in A}$, A is a directed set of indices, with $f^{(a)} \in H$ and $f^{(a)} \to 0$ with respect to τ_∞. It is easy to see that $(f^{(a)})_{a \in A}$ is a Cauchy net with respect to τ^*. If $\mathcal{S}_\otimes[\tau^*]$ is complete then there should be a $g \in \mathcal{S}_\otimes$ with $f^{(\alpha)} \to g$ in τ^*. Because of $\tau^* \succ \tau_\infty$ it should be $f^{(\alpha)} \to g$ in τ_∞, too, i.e., $g=0$. But this contradicts $T(f^{(\alpha)}) = 1$, $\alpha \in A$. Thus $\mathcal{S}_\otimes[\tau^*]$ is not complete and $U = \{f \in \mathcal{S}_\otimes; \ |T(f)| \leq 1\}$ is $\tau_\otimes$-closed but not τ^*-closed.

Lemma 6:

i) $\mathcal{S}_\otimes[\tau_\otimes]$ is a barrelled space, but $\mathcal{S}_\otimes[\tau]$ is not a barelled one for all topologies $\tau_{\mathcal{P}} < \tau \underset{+}{<} \tau_\otimes$.

ii) $\mathcal{S}_\otimes[\tau_{\mathcal{P}}]$ and $\mathcal{S}_\otimes[\tau_\otimes]$ are bornological, but $\mathcal{S}_\otimes[\eta]$ is not bornological for all topologies $\tau_\infty < \eta \lneq \tau_\otimes$. Further there are topologies ξ with $\tau_{\mathcal{P}} \underset{+}{<} \xi \underset{+}{<} \tau_\infty$ and $\mathcal{S}_\otimes[\xi]$ is bornological.

Proof:

i) $\mathcal{S}_\otimes[\tau_\otimes]$ is an F-space and thus $\mathcal{S}_\otimes[\tau_\otimes]$ is barelled.

$$\mathfrak{U}(\tau_\otimes) = \{U \subset \mathcal{S}_\otimes; \ U = \{f; \sum_{n \geq 0} \gamma_n \|f_n\|_{\nu_n} \leq 1, (\gamma_n) \in \Gamma, (\nu_n) \in N\}\}$$

is a neighbourhood base of $\tau_\otimes$ containing $\tau_\otimes$-barrels only. It is $S_N U \subset U$, $N=0,1,\ldots$, $U \in \mathfrak{U}(\tau_\otimes)$ and thus U is τ-closed by Lemma 4. Hence the sets $U \in \mathfrak{U}(\tau_\otimes)$ are τ-barrels too. Because of $\tau \underset{+}{<} \tau_\otimes$ there should be a set $U_0 \in \mathfrak{U}(\tau_\otimes)$ which is not neighbourhood of 0 with respect to τ. Hence there are τ-barrels not being τ-neighbourhoods of 0 and thus $\mathcal{S}_\otimes[\tau]$ is not barrelled.

ii) $\mathcal{S}_\otimes[\tau_{\mathcal{P}}]$, respectively, $\mathcal{S}_\otimes[\tau_\otimes]$ are bornological because $\tau_{\mathcal{P}}$ is a metric, respectively, because $\mathcal{S}_\otimes[\tau_\otimes]$ is an F-space /10/. $\mathcal{S}_\otimes[\tau]$ is bornological if τ is the finest topology in the set of all topologies with the same bounded sets. This proves that $\mathcal{S}_\otimes[\eta]$, $\tau_\infty < \eta \underset{+}{<} \tau_\otimes$, is not bornological and that there are bornological topologies ξ with $\tau_{\mathcal{P}} \underset{+}{<} \xi \underset{+}{<} \tau_\infty$.

4. NORMALITY OF THE CONE K

In this section we discuss questions about the normality of the cone K in some topologies. We understand the concept of normality of a cone in the sense of ref. /10/. For instance the normality of K is of some interest in the theory of AO* -algebras /7,8,11/ and for the decomposition of linear functionals into positive ones. The normality of K with respect to τ_∞ and $\mathfrak{N}$ was proved in refs. /5,17/ and the non-normality with respect to $\tau_\otimes$ in ref. /6/.

In the following lemma we will construct a lot of topologies in which K will be normal, respectively, non-normal. We say Γ_1 has the form (A) if

i) $\Gamma_1 \subset \Gamma$,

ii) to each $(\gamma_n) \in \Gamma_1$ there is a $(\delta_n) \in \Gamma_1$ with $\delta_n > n^2 \gamma_n$, $n = 0,1,\ldots$,

iii) to each $(\gamma_n) \in \Gamma_1$ there is a $(\epsilon_n) \in \Gamma_1$ with $(8(s-1)\epsilon_{2s-1})^2 < \epsilon_{2s}$, $(8s\epsilon_{2s})^2 < \epsilon_{2s+2}$, $s = 1,2,\ldots$,

$\mathfrak{M}_1$ has the form (B) if

i) $\mathfrak{M}_1 \subset \mathfrak{M}$,

ii) if $(m_{ij}) \in \mathfrak{M}_1$ then $m_{ij} = m_{kj}$ for $i,j,k = 1,2,\ldots$,

iii) if $(m_{ij}) \in \mathfrak{M}_1$ and $i = 2s$ then $m_{i2} = m_{i2s}$, $m_{i2} = m_{i2s-1}, \ldots, m_{is} = m_{is+1}$ $(s = 1,2,\ldots)$ and if $(m_{ij}) \in \mathfrak{M}_1$ and $i = 2s+1$ then

$$m_{i1} = m_{i2s+1}, m_{i2} = m_{i2s}, \ldots, m_{is-1} = m_{is+1} \quad (s=0,1,\ldots), \qquad (B)$$

iv) $m_{i1} \leq m_{i2} \leq \dots \quad (i = 1,2,\dots)$,

and $\mathfrak{M}_2$ has the form (C) if

i) $\mathfrak{M}_2 \subset \mathfrak{M}$,

ii) there is an $(m_{ij}) \in \mathfrak{M}_2$ such that to every constant c there are indices i,j with $j \leq i$ and $c \leq m_{ij}$.

We state

Lemma 7:

i) If Γ_1 has the form (A) and $\mathfrak{M}_1$ the form (B) then the cone K is $\lambda(\Gamma_1, \mathfrak{M}_1)$ - normal.

ii) $\mathfrak{N}$ is the strongest topology weaker than $\tau_\otimes$ in which K is a normal cone.

iii) If $\Gamma_2 \subset \Gamma$ and $\mathfrak{M}_2$ has the form (C) then K is non-normal with respect to $\lambda(\Gamma_2, \mathfrak{M}_2)$.

Proof:

i) Because of [4, Theorem 1] we have only to prove that each seminorm $p_{(\gamma_n)(m_{ij})}$ of a system describing the topology $\lambda(\Gamma_1, \mathfrak{M}_1)$ fulfils the relation

$$p_{(1)(m_{ij})}\Big(\sum_{k<\infty} f_r^{(k)*} f_s^{(k)}\Big) \leq p_{(1)(m_{ij})}\Big(\sum_{k<\infty} f_r^{(k)*} f_r^{(k)}\Big)^{1/2} \quad (9)$$
$$p_{(1)(m_{ij})}\Big(\sum_{k<\infty} f_s^{(k)*} f_s^{(k)}\Big)^{1/2}$$

with any $f^{(k)} = (f_0^{(k)}, \dots, f_n^{(k)}, 0, \dots) \in \mathcal{S}_\otimes$. By the Cauchy-Schwarz inequality and (A) we have

$$p_{(1)(m_{ij})}\Big(\sum_{k<\infty} f_r^{(k)*} f_s^{(k)}\Big) =$$

$$= \sup_x \max_{\ell_j \leq m_{r+s\,j}} \Big| \prod_{j=1}^{r+s} (1+x_j^2)^{m_{r+s\,j}} D_j^{\ell_j} \sum_{k<\infty} f_r^{(k)*} f_s^{(k)} \Big| \leq$$

$$\leq \sup_x \max_{\ell_j \leq m_{r+s\,j}} \Big\{ \sum_{k<\infty} \Big| \prod_{j=1}^{r+s} (1+x_j^2)^{m_{r+s\,j}} D_j^{\ell_j} \bar{f}_r^{(k)}(x_1, \dots, x_r) \Big|^2 \times$$

$$\times \sum_{k<\infty} \Big| \prod_{j=1+r}^{r+s} (1+x_j^2)^{m_{r+s\,j}} D_j^{\ell_j} f_s^{(k)}(x_{r+1}, \dots, x_{r+s}) \Big|^2 \Big\}^{1/2} =$$

$$= \Big\| \sum_{k<\infty} f_r^{(k)*} f_r^{(k)} \Big\|_{(m_{2r\,j})}^{1/2} \Big\| \sum_{k<\infty} f_s^{(k)*} f_s^{(k)} \Big\|_{(m_{2s\,j})}^{1/2} \times$$

$$\times \Big(D_j^{\ell} = \prod_{\lambda=0}^{3} \Big(\frac{\partial}{\partial x_j^{\lambda}}\Big)^{\ell^\lambda}, \; \ell = (\ell^0, \dots, \ell^3)\Big).$$

But this is (9) and thus we have i).

ii) We give the proof in the concept of 0*-topologies [7,8]. Let η be the strongest topology weaker than $\tau_\otimes$ in which the cone K is η-normal. Then η has to be the strongest 0*-topology weaker than $\tau_\otimes$ because, on the one hand, K is normal in each 0*-topology [8,11] and, on the other hand, the corresponding uniform operator-topology $\tau_{\mathfrak{T}}$ of the universal representation to a normal topology is stronger [8,11]. But $\mathfrak{N}$ is the greatest 0*-topology on $\mathcal{S}_\otimes$ weaker than $\tau_\otimes$ too [8]. Thus we have $\eta = \mathfrak{N}$. A direct proof of the fact $\eta = \mathfrak{N}$ is also possible.

iii) The proof is in analogy to the proof of ref. [6, Theorem 5].

Remark 8:

i) τ_∞ is a special case of Lemma 7 i) and $\tau_\otimes$ of Lemma 7 iii). Thus we have again the τ_∞-normality and the $\tau_\otimes$-non-normality of K.

ii) Because of Lemma 5 ii) we can describe the topology $\mathfrak{N}$ by the base of neighbourhoods

$$\mathfrak{U}(\mathfrak{N}) = \{ U = [V] ;\ V \in \mathfrak{U}(\tau_\otimes) \}$$

with $[V] = (V+K) \cap (V-K)$, the K-saturated hull of V, and $\mathfrak{U}(\tau_\otimes)$ is a base of neighbourhoods of $\tau_\otimes$.

iii) Let $T = (T_0, T_1, T_2, \dots)$ be a linear functional on $\mathcal{S}_\otimes$ defined by $T_0 f_0 = f_0$,

$$T_n f_n = \int \dots \int \prod_{\nu=1}^{n} \prod_{\lambda=0}^{3} (1 + (x_\nu^\lambda)^2) f_n dx_1 \dots dx_n .$$

By a simple estimation we get that T is $\tau_\otimes$-continuous, but, for instance, not τ_∞-continuous. Further examples show that between $\tau_{\mathcal{P}}$ and $\tau_\otimes$ there are a lot of several dualities.

We collect the results on the foregoing lemmas in the following Picture 9. This picture shows that $\tau_\otimes$ has "good" properties from the point of view of topological spaces but it is "bad" adapted to the order structure induced by K. Conversely τ_∞, $\mathfrak{N}$ and the other normal topologies are "good" ones from viewpoint of the order structure but "bad" ones from viewpoint of topological spaces.

5. CONTINUITY OF WIGHTMAN FUNCTIONALS

In this section we prove the continuity of Wightman functionals of scalar fields with respect to some topologies. Let τ_f be the topology defined by the norm $p(f) = \sum_{n \geq 0} n^2 \|f_n\|_2$. We have $\tau_{\mathcal{P}} < \tau_f < \tau_\infty$.

Picture 9:

Let τ be a locally convex topology on $\mathcal{S}_\otimes$	$\tau_{\mathcal{P}} \prec \tau_\infty \prec \tau_\otimes$
The restriction of τ to $\mathcal{S}_n$	$\tau/\mathcal{P}_n = \sigma_n$
The closure of a set $M \subset \mathcal{S}_\otimes$	$\bar{M}^{\tau_{\mathcal{P}}} = \bar{M}^{\tau} = \bar{M}^{\tau_\otimes}$, if M fulfils $S_N M \subset M$, $N = 0,1,\dots$
Is $\mathcal{S}_\otimes[\tau]$ complete ?	Yes, if there is a + . + . + . + . + . + . + . + filter base $\mathfrak{U}(\tau)$ with $S_N U \subset U$, $N = 0,1,\dots$, $U \in \mathfrak{U}(\tau)$
The τ-bounded sets	the same bounded sets ++++++++++++++++++++++++++
Is $\mathcal{S}_\otimes[\tau]$ barreled?	no . . . .+
Is $\mathcal{S}_\otimes[\tau]$ bornological?	+ . + .+. . . . .+
Is K τ-normal?	. + . +.+.+.+.$\underset{\mathfrak{N}}{+}$. . . . .

(+ means "yes", but . means "no").

Lemma 10:

i) The Wightman functionals are $\mathfrak{N}$-continuous.

ii) The Wightman functionals of the free fields are τ_f-continuous.

iii) The Wightman functionals of the Wick polynomials to the power $\ell = 2$ in the free fields with mass $m>0$ and their derivatives are τ_∞-continuous.

Proof:

i) The Wightman functionals W of the scalar fields are hermitean linear functionals on $\mathcal{S}_\otimes$ and have to fulfil $W(\bar{K}^{\tau\otimes}) \geq 0$. But each such functional is $\mathfrak{N}$-continuous ref.$^{/17,\text{ Theorem } 5/}$.

ii) The Wightman functionals of the free fields are of the form $W = (W_0, W_1, \ldots)$ with

$$W_0 = 1,\ W_{2s+1} = 0,\ s = 0,1,\ldots,$$

$$W_2(x_1,x_2) = (2\pi)^{-3} \int e^{ip(x_1-x_2)}\theta(p^\circ)\delta(p^2-m^2)dp,$$

$$x_i = (x_i^\circ, x_i^1, x_i^2, x_i^3) = (x_i^\circ, \vec{x}_i),\ i=1,2,\quad p = (p^\circ, p^1, p^2, p^3) = (p^\circ, \vec{p}),$$

$$dp = dp^\circ dp^1 dp^2 dp^3,\quad d\vec{p} = dp^1 dp^2 dp^3,$$

$$V_m^+ = \{ p \in R^4;\ p^\circ = (p^2+m^2)^{1/2} \}$$

$W_{2s} = \sum_{(i,j)} \prod_{\nu=1}^{n} W_2(x_{i_\nu}, x_{j_\nu})$ and the sum runs over all participations of the indices $(1,2,\ldots,2n)$ in tupel $(i_1,j_1),\ldots,(i_n,j_n)$ $i_\nu < j_\nu$, $s=1,2,\ldots,\nu=1,\ldots,n$.

We estimate for some $f_2 \in \mathcal{S}_2$ and mass $m \geq 0$

$$|W_2(f_2)| = (2\pi)^{-3}\, 2^{-1} \Big| \int_{R^3} \tilde{f}_2((\vec{p}^{\,2}+m^2)^{1/2}, \vec{p}; -(\vec{p}^{\,2}+m^2)^{1/2}, -\vec{p})d\vec{p}\Big|$$

$$\leq 2^{-1}(2\pi)^{-3} \sup_{p\in V_m^+} |(1+p^2)^2 \tilde{f}_2(p,-p) \int_{R^3} (\vec{p}^{\,2}+m^2)^{-1/2}(1+\vec{p}^{\,2})^{-2}d\vec{p}$$

$$\overset{*}{\leq} 2^{-1}(2\pi)^{-3} \sup_p |(1+p^2)^2 \tilde{f}_2(p,-p)| \overset{**}{\leq} ||f_2||_2 .$$

* follows from $\int_{R^3} (\vec{p}^{\,2}+m^2)^{-1/2}(1+\vec{p}^{\,2})^{-2}d\vec{p} \leq \pi$.

** follows from $\sup |(1+p^2)^2 \tilde{f}_2(p,-p)| \leq$

$$\leq \sup_{p_1,p_2} \Big|(2\pi)^{-8} \iint e^{-i(p_1x_1+p_2x_2)} \prod_{i=1}^{2}\prod_{j=0}^{3} (1+(\frac{\partial}{\partial x_i^j})^2) f_2(x_1,x_2)dx_1\,dx_2\Big|$$

$$\leq (2\pi)^{-8} \sup_{x_1,x_2} \Big|\prod_{i=1}^{2}\prod_{j=0}^{3} (1+(x_i^j)^2)(1+(\frac{\partial}{\partial x_i^j})^2) f_2(x_1,x_2)\Big| \times$$

$$\times \iint (1+(x_i^j)^2)^{-1}dx_1 dx_2 \leq ||f_2||_2 .$$

Then we get for $f_{2n} \in \mathcal{S}_{2n}$

$$|W_{2n}(f_{2n})| = \Big|\sum_{(i,j)} \prod_{\nu=1}^{n} W_2(x_{i_\nu}, x_{j_\nu}) f_{2n}(x_1,\ldots,x_{2n})dx_1\ldots dx_n\Big| \leq$$

$\leq 2n^2 ||f_{2n}||_2$. This proves ii).

iii) At first we describe the structure of the Wightman functionals of the Wick polynomials in the free fields. We have

$$(:D^{\alpha^1}\phi D^{\alpha^2}\phi \dots D^{\alpha^\ell}\phi:(g)\Phi)^{(n)}(\xi_1,\dots,\xi_2) =$$

$$= \frac{\pi^{\ell/2}}{(2\pi)^{2(\ell-1)}} \sum_{j=0}^{\ell} \left[\frac{(n-\ell+2j)}{n!}\right]^{1/2} \int\dots\int \left(\prod_{k=1}^{j} d\Omega_j\right) \sum_{k_1<\dots<k_{\ell-j}=1}^{n} (j!)^{-1} \times$$

$$\times \sum_{P} P((-i\eta_1)^{\alpha^1}\dots(-i\eta_j)^{\alpha^j}(i\xi_{k_1})^{\alpha^{j+1}}\dots(i\xi_{k_{\ell-j}})^{\alpha^\ell}\tilde{g}(\sum_{r=1}^{j}\eta_r - \sum_{r=1}^{\ell-j}\xi_{k_r}))\times$$

$$\times \Phi^{(n-\ell+2j)}(\eta_1,\dots,\eta_j,\xi_1,\dots,\hat{\xi}_{k_1},\dots,\hat{\xi}_{k_{\ell-j}},\dots,\xi_n),$$

where $\tilde{g}$ is the Fourier transform of g, $d\Omega_j = d\vec{\eta}_j(m^2+\vec{\eta}_j^{\,2})^{-1/2}$, m is the mass of the corresponding free field, the sum $\sum\limits_{P}$ is over all permutations of the variables $\eta_1\dots\eta_j(-\xi_{k_1})\dots(-\xi_{k_{\ell-j}})$ and ^ over a symbol means to omit it /14/. If $n\ell/2$ is an integer then $\mathfrak{R}_{n\ell}$ denotes the set of all $(n, n\ell/2)$ - matrices R with elements $1, -1, 0$ only and in every row are ℓ numbers not equal to 0 and in every column is exact one 1 and exact one -1 and the -1 stand over the 1. Let $R = (r_{\nu\mu})_{\nu=1,\dots,n}^{\mu=1,\dots,jn/2}$, $\kappa_k^\nu\ (k = 1,2,\dots,\ell)$ denote the numbers of the columns which the ℓ elements of the ν-th row stand which differ

from 0 and k_ν the number of the elements 1 in the ν-th row.

Let $\sigma_{(\alpha^1,\dots,\alpha^\ell)} : \mathfrak{R}_{n\ell} \times \mathcal{S}_1^{\otimes n} \to \mathbb{R}$ be a map defined by

$$\sigma_{(\alpha)}((r_{\nu\mu}), g^{(n)}\dots g^{(1)}) =$$

$$= C_{n\ell}\int\dots\int_{V_m^+} \left(\prod_{\mu=1}^{\ell n/2}(d\Omega_\mu)\right)\prod_{\nu=1}^{n}\tilde{g}^{(\nu)}\left(\sum_{\mu=1}^{\ell n/2} r_{\nu\mu}x_\mu\right) \times$$

$$\times \sum_{\pi\in P_\ell}\left(\prod_{k=1}^{\ell}(ir_{\nu\kappa_k^\nu}x_{\kappa_k^\nu})^{\alpha^{\pi(k)}}\right)$$

with $\mathcal{S}^{\otimes n} = \mathcal{S}_1 \otimes \mathcal{S}_1 \dots \otimes \mathcal{S}_1$ (n -times),

$$C_{n\ell} = \pi^{1/2}\ell^{1/2}(2\pi^{2(\ell-1)}\prod_{\nu=2}^{n}k_\nu!\prod_{\mu=2}^{n-1}(\mu\ell-2k_2-\dots-2k_\mu-1)!)^{-1},$$

$$d\Omega_\mu = d\vec{x}_\mu(m^2+\vec{x}_\mu^{\,2})^{-1/2}$$

and P_ℓ is the group of permutations of the numbers $\{1,2,\dots,\ell\}$. There it follows

Proposition 11:

The Wightman functional of the field $(:D^\alpha\phi\dots D^\alpha\phi:)$ is $W_n(f_n) = \sum\limits_{R\in\mathfrak{R}_{n\ell}} \sigma_{(\alpha^1,\dots,\alpha^\ell)}(R, f_n)$, $n = 1,2,\dots$.

Let $W_{2s+1} = 0$, $s = 0,1,\dots,$ for odd ℓ. ($\sigma_{(\alpha)}$ is defined by continuity for all $f_n \in \mathcal{S}_n$).

Now we prove the Lemma 10 iii), i.e., $\ell = 2$, in the case $(\alpha_1,\alpha_2) = (0,0)$. The proof is analogous in the other cases. We have to estimate

$$\sigma_{(0,0)}(R, g^{(n)}\dots g^{(1)}) = 2^n C_{n2}\int_{V_m^+}\dots\int_{V_m^+} d\Omega_1\dots d\Omega_n \prod_{\nu=1}^{n}\tilde{g}^{(\nu)}\sum_{\mu=1}^{n} r_{\nu\mu}x_\mu)$$

with $R = (r_{\nu\mu}) \in \mathfrak{R}_{n2}$. Because of $\sum\limits_{j=1} r_{j\mu} = 0$ for we get $\mathrm{Det}(R) = 0$. We define the matrix $\tilde{R} = (\tilde{r}_{\nu\mu})$ by $\tilde{r}_{\nu\mu} = r_{\nu\mu}$, $\nu = 1, \ldots, n-1$, $\mu = 1, \ldots, n$, and $\tilde{r}_{n\kappa_2^n} = r_{n\kappa_2^n}$, $r_{nj} = 0$, for $j \neq \kappa_2^n$) and a linear transformation of variables by

$$\begin{matrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{matrix} = \tilde{R} \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix}$$

Let $\mathrm{Det}(\tilde{R}) \neq 0$. We estimate

$$|\sigma_{(0,0)}(R, g^{(n)} \ldots g^{(1)})| \leq$$

$$\leq 2^n C_{n2} \sup_{x_{\kappa_1^n}, x_{\kappa_2^n} \in V_m^+} |\tilde{g}^{(n)}(x_{\kappa_1^n} + x_{\kappa_2^n})(1 + (x_{\kappa_2^n})^2)^2| \times$$

$$\times m^{-n} \int_{R^3} \ldots \int_{R^3} d\vec{x}_1 \ldots d\vec{x}_n (1 + (\vec{x}_{\kappa_2^n})^2)^{-2} \times$$

$$\times \prod_{\nu=1}^{n-1} \tilde{g}^{(\nu)}((\sum_{\mu=1}^{n} r_{\nu\mu}(m^2 + \vec{x}_\mu^{\,2})^{1/2}, \sum_{\mu=1}^{n} r_{\nu\mu} \vec{x}_\mu) \leq$$

$$\leq 2^n m^{-n} C_{n2} (\mathrm{Det}(R))^{-1} \sup_{x \in R^4} |(1 + x^2)^2 \tilde{g}^{(n)}(x)| \times$$

$$\times \int \ldots \int d\vec{\xi}_1 \ldots d\vec{\xi}_n | (1 + \vec{\xi}_n^{\,2})^{-2} \times$$

$$\times \prod_{\nu=1}^{n-1} \tilde{g}^{(\nu)}((1 + \vec{\xi}_\nu^{\,2})^{1/2}, \vec{\xi}_\nu) \leq$$

$$\leq 2^n m^{-n} C_{n2} (\mathrm{Det}(R))^{-1} | \prod_{\nu=1}^{n} \sup_{\xi_\nu \in R^4} (1 + (\xi_\nu)^2)^2 \tilde{g}^{(\nu)}(\xi_\nu) | \times$$

$$\times \int \ldots \int d\vec{\xi}_1 \ldots d\vec{\xi}_n \prod_{\nu=1}^{n} (1 + \vec{\xi}_\nu^{\,2})^{-2} \leq \qquad (11)$$

$$\leq 2^n \pi^{2n} m^{-n} C_{n2} (\mathrm{Det}(R))^{-1} \| g^{(n)} \ldots g^{(1)} \|_2 \, .$$

$(\xi_\nu)^2$ stands for $\sum_{\lambda=0}^{3} (\xi_\nu^\lambda)^2$ and $\vec{\xi}_\nu^{\,2}$ for $\sum_{\lambda=1}^{3} (\xi_\nu^\lambda)^2$. Because (11) is also right for sums of elements of $\mathcal{S}_1^{\otimes n}$ and these sums are dense in $\mathcal{S}_n$, (11) holds for arbitrary $f_n \in \mathcal{S}_n$.Applying Proposition 11 this proves Lemma 10 iii) in the case $a_1 = a_2 = 0$. The index 2 of the norm $\| g^{(n)} \ldots g^{(1)} \|_2$ is a consequence of $a_1 = a_2 = 0$. We get other indices for other a_1, a_2.

Lemma 10 demonstrates how one can classify Wightman functionals with respect to the continuity in the topologies τ, $\tau_{\mathcal{P}} \prec \tau \prec \tau_\otimes$.

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