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ON THE SUPERSYMMETRIC SINE-GORDON MODEL

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О суперсимметрической модели sine-Gordon

Модель sine-Gordon, как теория безмассового скалярного поля в одной пространственной и одной временной размерности, обобщается для случая скалярного суперполя. Показано, что решение является "суперсолитоном". Получены новые уравнения, смешивающие ферми- и бозе-поля, кроме того, в одном случае получается массовая модель Тирринга.

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On the Supersymmetric Sine-Gordon Model

The sine-Gordon model as the theory of a massless scalar field in one space and one time dimension with interaction Lagrangian density proportional to $\cos\beta\phi$ is generalized for scalar superfield and it is shown that the solution of the supercovariant sine-Gordon equation is the "supersoliton", it is the superfield, which has all ordinary fields in two dimensions as a type of the soliton solution. We also obtain the massive Thirring model and the new equation of motion coupling the Fermi field ψ and the Bose field ϕ. The notice about supersymmetric "SLAC-BAG" model is done.

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1. INTRODUCTION

The soliton theory introduced a new approach to the study of field theory of extended particles [1]. An instructive example of a single scalar field in one space and one time dimension is determined by the Lagrangian density [2]:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{a_0}{\beta^2}(\cos\beta\phi - 1). \qquad (1.1)$$

Here a_0 and β are real parameters where the physical meaning of these parameters is the following: a_0 is the "squared mass" of the minimum energy excitations and β is a parameter which measures the strength of the interaction between these small oscillations.

From the Lagrangian density (1.1) the equation of motion is the sine-Gordon equation:

$$\Box\phi = -\frac{a_0}{\beta}\sin\beta\phi. \qquad (1.2)$$

The static solution of the equation (1.2) is the soliton [3]:

$$\phi = \frac{4}{\beta}\,\mathrm{tg}^{-1}\exp(\sqrt{a_0}\,x). \qquad (1.3)$$

In such a theory the very surprising equivalence was shown by S.Coleman (see ref.[2]): the sine-Gordon equation is equivalent to the massive Thirring model in a sense that the perturbation series for the massive Thirring model is term-by-term identical with a series for the sine-Gordon equation, if the following identifications are held between the theories:

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi} \tag{1.4}$$

$$-\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu \phi = j^\mu \tag{1.5}$$

$$\frac{\alpha_0}{\beta^2}\cos\beta\phi = -m'\sigma . \tag{1.6}$$

The symbols in Eqs. (1.4-6) mean: g is the coupling constant in the Thirring model with the Lagrangian density

$$L = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2} g j^\mu j_\mu \tag{1.7}$$

and $j_\mu = \bar{\psi}\gamma_\mu\psi$. The symbol σ means a renormalized scalar density

$$\sigma = Z\bar{\psi}\psi ,$$

where Z is a cutoff-dependent constant. The massive Thirring model is formally derived by adding a term proportional to the Lagrangian density (1.7):

$$L \to L - m'\sigma \tag{1.8}$$

and the symbol m' is a real parameter: it is not to be identified with the mass of any presumed one-particle state.

This equivalence holds when β^2 is less than 8π. When $\beta^2 = 4\pi$, Eq. (1.4) implies $g = 0$ and the sine-Gordon equation is equivalent to a free massive Dirac theory is one spatial dimension.

In such a way S.Coleman established the close connections between fermion (massive Thirring field) and soliton (sine-Gordon field). Of course, it is surprising to see a Fermi field appearing as a coherent state of a Bose field, but fortunately there is no spin-statistics theorem in two dimensions (general remarks on spin and statistics in two dimensions are given in ref.[4]). Nevertheless the curious equivalence between fermion and boson raises the questions: What is the Fermi-Bose symmetry for solitons in two dimensions? What is the "supersoliton"? Do we obtain the solitary fields in the supermultiplet assuming the generalization of the sine-Gordon theory for the scalar superfield?

The solution of these questions is the main goal of this paper and the procedure is the following: In Sec. 2 we shall start from the four dimensional superspace where we use the technique of superfields in two Bose and two Fermi dimensions by straightforward adaptation of the usual technique in eight dimensional superspace. In Sec. 3 we construct the supersoliton theory. We generalize the sine-Gordon model for the scalar superfield and using the validity of the sine-Gordon equation for the ordinary scalar field in two dimensions, we obtain the special form for the Lagrangian density of Fermi fields. We show two independent ways for $\psi \neq 0$; the first gives the massive Thirring model and the Fermi field is identified with the sine-Gordon soliton; the second gives the special nonlinear Lagrangian density and from this the coupled set of nonlinear equations follows, one solution of these equation having the form as the soliton solution of the sine-Gordon model [5]. In both cases we get the Fermi and Bose fields as solitons in a certain sense. So, we obtain the solitons from the supersymmetric point of view - the supersoliton theory.

2. THE TWO DIMENSIONAL SUPERSYMMETRY THEORY

We shall now describe the formalism using the references [6]. We suppose 2-dimensional vector space **V** (over the complex field), generating a 4-dimensional Grassman algebra

$$\Lambda V = \Lambda^1 V + \Lambda^2 V ,$$

where we define $V \equiv \Lambda^1 V$ and $W \equiv \Lambda^2 V$. Since the Grassman algebra is graded commutative, the elements of **V** anticommute, and we shall denote $\theta_a (a=1,2)$ the elements of **V**. We suppose $\theta_a \in V$ to be Majorana spinor. The two dimensional quasi-Minkowski coordinate x^α (i.e., $(x^\alpha)^{n+1} = 0, \alpha = 1,2$) is the element of the space **W** and is constructed from two elements $\theta, \theta' \in V$:

$$x^\alpha = \bar{\theta}' \gamma^\alpha \theta , \qquad (2.1)$$

where

$$\gamma^\circ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The two dimensional superspace is as a matter of fact a fibre bundle $E = V \otimes W$ with the basis **V** and with the fibre **W** (for a detailed description see the last reference in ref. [6]). The supersymmetry transformation has the form:

$$x^\alpha \to x^\alpha + \bar{\epsilon}^a \Gamma^\alpha_a ,$$
$$\theta^a \to \theta^a + \epsilon^a ,$$

where $\Gamma^\alpha_a = i(\gamma^\alpha)_{ab} \theta^b$ is the connection form in the fibre bundle **E** and $\epsilon^a \in V$ is the infinitesimal change in the basis **V**. The covariant derivative and the scalar superfield are given by the usual expression [6]:

$$D_a = \partial_a - \Gamma^\alpha_a \partial_\alpha \qquad (2.2)$$

$$S(x,\theta) = \phi(x) + i\bar{\theta}\psi(x) + \frac{i}{2}\bar{\theta}\theta F(x). \qquad (2.3)$$

The supermultiplet $\{\phi(x), \psi(x), F(x)\}$ contains the Fermi field (Majorana spinor) $\psi(x)$ and the Bose fields $\phi(x)$ and $F(x)$. Using the covariant derivative on the scalar superfield $S(x,\theta)$ one can construct the multiplet

$$\frac{i}{2}\bar{D}DS(x,\theta) = F(x) + i\bar{\theta}\not{\partial}\psi(x) + \frac{i}{2}\bar{\theta}\theta\Box\phi(x) \qquad (2.4)$$

and obtain the free massless equation

$$\bar{D}DS(x,\theta) = 0. \qquad (2.5)$$

The usual massless free-field Lagrangian is the coefficient of $\bar{\theta}\theta$ (see Sec. 3) in the following expression:

$$\frac{i}{2}S(x,\theta)\bar{D}DS(x,\theta) = \ldots i\bar{\theta}\theta\left(\frac{1}{2}\phi\Box\phi + \frac{1}{2}F^2 - \frac{i}{2}\bar{\psi}\not{\partial}\psi\right). \qquad (2.6)$$

Now we shall construct the supersoliton theory.

3. THE SUPERSOLITON THEORY

From the relation (2.3) it follows that the scalar superfield $S(x,\theta)$ is equivalent to a set of ordinary fields in two dimensions. From this point of view we shall define the soliton superfield:

we shall call the scalar superfield $S(x,\theta)$ as the soliton superfield ("supersoliton"), if the supersymmetric generalization of the sine-Gordon equation for the scalar superfield will be fulfilled and if all ordinary fields in two dimensions will be as a solitary fields, exactly as a type of the soliton solution from the sine-Gordon equation.

We shall advance in the following way; at first, we construct the supercovariant sine-Gordon equation for the scalar superfield $S(x,\theta)$ using the supersymmetry Lagrangian technique. By analogy with the Lagrangian density (1.1) the dynamics of the supersoliton theory will be determined by the Lagrangian density, which is the coefficient of $\bar{\theta}\theta$ in the following expression:

$$\frac{i}{2} S(x,\theta)\bar{D}DS(x,\theta) + V(S(x,\theta)), \qquad (3.1)$$

where

$$V(S(x,\theta)) = -\frac{a}{b^2}(\cos bS(x,\theta)-1)$$

and a,b are parameters which will be specified later.

We recall that the Lagrangian density is obtained from the expression (3.1) applying the covariant operator D_a twice on a general superfield because in the four dimensional superspace E the power series in θ is finished for two θ. Such a construction of the Lagrangian density is in agreement with the construction of the supersymmetric Lagrangian in the work of A. Salam /6/. There is shown that the action is invariant under supersymmetry transformation (up to a variationally insignificant surface term) if it is independent of θ.

The supercovariant equation of motion is

$$\frac{i}{2}\bar{D}DS(x,\theta) = V'(S(x,\theta)), \qquad (3.2)$$

where

$$V'(S(x,\theta)) = -\frac{a}{b}\sin bS(x,\theta).$$

It is the supercovariant sine-Gordon equation. Now we shall show that the solution of the supercovariant sine-Gordon equation is the "supersoliton" in the sense of our definition. At first, we use the Taylor expansion of the cosine in the expression for the $V(S(x,\theta))$:

$$V(S(x,\theta)) = \frac{a}{b^2}(\cos bS(x,\theta)-1) = \qquad (3.3)$$

$$= \frac{a}{b^2}\left(-\frac{b^2S^2(x,\theta)}{2!} + \frac{b^4S^4(x,\theta)}{4!} - \frac{b^6S^6(x,\theta)}{6!}\ldots\right).$$

From the superfield theory /6/ we know that the product of superfields is again a superfield and therefore can be expanded in Taylor's expansion in θ, which is finished for $n=2$, because the θ-elements anticommute. Using this specific characteristic of the superfields, it is easy to express the cosine in the relation (3.3):

$$\frac{a}{b^2}\Big[-\frac{b^2}{2!}(\phi^2+2i\bar\theta\psi\phi+i\bar\theta\theta F\phi-\bar\theta\theta\bar\psi\psi)$$

$$+\frac{b^4}{4!}(\phi^4+4i\bar\theta\psi\phi^3+2i\bar\theta\theta F\phi^3-6\bar\theta\theta\bar\psi\psi\phi^2) \qquad (3.4)$$

$$-\frac{b^6}{6!}(\phi^6+6i\bar\theta\psi\phi^5+3i\bar\theta\theta F\phi^5-15\bar\theta\theta\bar\psi\psi\phi^4)\text{ etc.}\Big]$$

where we use $\bar\theta\psi=\bar\psi\theta$ and the symbolic expression for the fields. In the Lagrangian density, which is obtained from the relation (3.1), the elements with two θ only will play the role in the supersymmetric Lagrangian. So we get for ordinary fields Lagrangian density in two dimensional space the following expression:

$$L=\frac{1}{2}\phi\Box\phi+\frac{1}{2}F^2-\frac{i}{2}\bar\psi\not\partial\psi-\frac{a}{2}(F\phi+i\bar\psi\psi$$

$$-\frac{b^2}{3!}F\phi^3-i\frac{b^2}{2!}\bar\psi\psi\phi^2 \qquad (3.5)$$

$$+\frac{b^4}{5!}F\phi^5+i\frac{b^4}{4!}\bar\psi\psi\phi^4)\text{etc.}=$$

$$=\frac{1}{2}\phi\Box\phi+\frac{1}{2}F^2-\frac{i}{2}\bar\psi\not\partial\phi-\frac{a}{2b}F\sin b\phi-\frac{i}{2}a\bar\psi\psi\cos b\phi.$$

There is no kinetic term for **F**; hence this field can be eliminated by using the equation of motion which includes

$$F-\frac{a}{2b}\sin b\phi=0. \qquad (3.6)$$

Using Eq. (3.6) we get the Lagrangian density

$$L=\frac{1}{2}\phi\Box\phi-\frac{i}{2}\bar\psi\not\partial\psi-\frac{1}{8}\frac{a^2}{b^2}\sin^2 b\phi-\frac{ia}{2}\bar\psi\psi\cos b\phi \qquad (3.7)$$

and from this we obtain two basic equations of motion coupling the Fermi field ψ and the Bose field ϕ:

$$(i\not\partial+ia\cos b\phi)\psi=0, \qquad (3.8a)$$

$$\Box\phi-\frac{1}{4}\frac{a^2}{b}\sin b\phi\cos b\phi+\frac{i}{2}ab\bar\psi\psi\sin b\phi=0. \qquad (3.8b)$$

We can see that if we have no spinor field, i.e., $\psi=0$, then the equation (3.8b) is equivalent to the sine-Gordon equation (1.2) if these relations are held:

$$0=\Box\phi-\frac{a^2}{8b}\sin 2b\phi$$

$$\beta=2b$$

$$\frac{\alpha_0}{\beta}=-\frac{a^2}{8b}.$$

In this case we obtain the following expression for the parameters:

$b=\frac{\beta}{2}$, $a=2i\sqrt{\alpha_0}$.

Equation (3.8b) also will be equivalent to the sine-Gordon equation (1.2) if the relation

$$\left(\frac{1}{4}\frac{a^2}{b}\cos b\phi - \frac{i}{2}\, ab\bar{\psi}\psi\right)\sin b\phi = -\frac{\alpha_0}{\beta}\sin\beta\phi . \qquad (3.9)$$

is held.

Now we have two independent possibilities to fulfill the relation (3.9):

a) We suppose $b=\beta$ and we obtain from the relation (3.9) the expression for the cosine:

$$\cos\beta\phi = -4\frac{\alpha_0}{a^2} + 2i\frac{\beta^2}{a}\bar{\psi}\psi . \qquad (3.10)$$

From Eq. (3.8a) the Lagrangian density follows:

$$L = \frac{1}{2}\left(i\bar{\psi}\partial\!\!\!/\psi + ia\cos b\phi\,\bar{\psi}\psi\right) \qquad (3.11)$$

and using the relation (3.10), we get

$$L = \frac{i}{2}\bar{\psi}\,\partial\!\!\!/\psi - 2i\frac{\alpha_0}{a}\bar{\psi}\psi - \beta^2(\bar{\psi}\psi)^2 . \qquad (3.12)$$

The Lagrangian density (3.12) will be the Lagrangian density (1.8) of the massive Thirring model, if the following condition will be satisfied:

$$2i\frac{\alpha_0}{a}\bar{\psi}\psi = m'\sigma , \qquad (3.13a)$$

$$\beta^2 = \frac{g}{2} . \qquad (3.13b)$$

From the condition (3.13a) it follows the expression for the parameter a:

$$a = \frac{2i\alpha_0}{m'Z} . \qquad (3.14)$$

The massive Thirring model is equivalent to the sine-Gordon model (when β^2 is less than 8π and identifications (1.4-6) are held) and in this sense our Fermi field ψ of the multiplet is connected with the sine-Gordon soliton (see also Sec. 4). In such a way both fields ϕ and ψ are solitons and the scalar superfield $S(x,\theta)$ is really the "super-soliton".

b) We suppose

$$\frac{1}{4}\frac{a^2}{b}\cos b\phi = -i\frac{ab}{2}\bar{\psi}\psi \qquad (3.15)$$

and from the relation (3.9) the expression for the cosine follows:

$$\cos b\phi = -2i\frac{b^2}{a}\bar{\psi}\psi \qquad (3.16)$$

and it formally looks like S.Coleman's condition (1.6). Using Eq. (3.15) in the relation (3.9), we obtain from Eq. (3.8a) the sine-Gordon equation:

$$\Box\phi = \frac{1}{4}\frac{a^2}{b}\sin 2b\phi = -\frac{\alpha_0}{\beta}\sin\beta\phi \qquad (3.17)$$

and from this equation the parameters follow:

$$a = \pm i\sqrt{2\alpha_0}$$

$$b = \frac{\beta}{2}$$

If we put the expression (3.16) into Eq. (3.11), we get the special nonlinear Lagrangian density

$$L = \frac{1}{2}\left(i\bar{\psi}\partial\!\!\!/\psi + \frac{\beta^2}{2}(\bar{\psi}\psi)^2\right) . \qquad (3.18)$$

Then the equation of motion has the form

$$(i\partial\!\!\!/ - \beta^2 \bar{\psi}\psi)\psi = 0. \qquad (3.19)$$

Writing $\psi = \binom{u}{v}$, we get static equations

$$\dot{u} = -\beta^2 (u^2 - v^2) v \qquad (3.20a)$$

$$\dot{v} = -\beta^2 (u^2 - v^2) u . \qquad (3.20b)$$

In terms of polar coordinates $\psi = \begin{pmatrix} r\cos\omega \\ r\sin\omega \end{pmatrix}$ the equations (3.20) become:

$$\dot{r} = -\beta^2 r^3 \sin 2\omega \cos 2\omega = -\frac{\beta^2 r^3}{2} \sin 4\omega \qquad (3.21a)$$

$$\dot{\omega} = -\beta^2 r^2 \cos^2 2\omega . \qquad (3.21b)$$

The solutions of this couples set of the special nonlinear equations are amazingly simple (see ref./5/):

$$r^2 = r_0^2 \cosh[\mp \beta^2 r_0^2 (x - x_0)] \qquad (3.22a)$$

$$\omega = \pm\, tg^{-1} \exp[\mp \beta^2 r_0^2 (x - x_0)] - \frac{\pi}{4} . \qquad (3.22b)$$

The formula for ω is readily recognizable as the soliton solution for the sine-Gordon equation. In this sense the connection between fermions and solitons exists and the scalar superfield is really the "supersoliton".

In such a way in both cases we have obtained the supersoliton theory.

4. COMMENTS

We have discussed how in two dimensional supersymmetry theory the soliton Fermi-Bose symmetry is realized. Using the sine-Gordon equation for the scalar field ϕ in the supermultiplet, we have obtained the new nonlinear equation for fermions like solitons.

The theory for $\psi \neq 0$ and with the conditions (3.9) gives two ways in which the parameters a and b are determined. In both the ways the parameter a has mass dimension and the parameter b has the same dimension as the parameter β. In every way the parameters are different.

The first way gives the massive Thirring model from the supercovariant sine-Gordon equation for the scalar superfield $S(x, \theta)$. The connection in this first way between fermion and soliton is not only due to massive Thirring model, but also due to nonlinear equation of motion which follows from the Lagrangian density (3.12):

$$i\partial\!\!\!/\psi + 4\left(i\frac{\alpha_0}{a} + \beta^2 \bar{\psi}\psi\right)\psi = 0. \qquad (4.1)$$

Using the same procedure as in the second case on the equation (3.19), we obtain the following solutions:

$$r^2 = r_0^2 \cosh[4(\tfrac{1}{2} m'Z + \beta^2 r_0^2)(x - x_0)] \qquad (4.2a)$$

$$\omega = \pm\, tg^{-1} \exp[4(\tfrac{1}{2} m'Z + \beta^2 r_0^2)(x - x_0)] - \frac{\pi}{4} \qquad (4.2b)$$

where we used

$$a = 2i \frac{\alpha_0}{m'Z} .$$

We can see that the formula for ω is again readily recognizable as the soliton solution for the sine-Gordon equation and in this sense we have again the supersoliton theory as in the second case.

Of course in both cases the Bose field ϕ is not equal to the Fermi field ψ. The relations between ϕ and ψ are given via Eq. (3.10) and Eq. (3.16), respectively. From the relations (4.2a,b) and (3.22a,b) we can obtain the field ϕ, or using the expression (1.3) for the soliton we can get the field ψ. Finally we want indicate briefly a possible physical interpretation of the supersymmetric sine-Gordon model. For this purpose from the aforesaid we have to use only the derivation of Eq. (3.8a,b) - the most interesting part of this paper. If we shall now study small oscillations about the ground state $\phi=0$, we can expand the cosine and the sine in power series in Eq. (3.8a,b), and we obtain:

$$i\partial\!\!\!/\psi + ia\psi = \frac{iab^2}{2}\phi^2\psi \; . \tag{4.3a}$$

$$\Box\phi - \frac{1}{4}a^2\phi + \frac{a^2b^2}{6}\phi^3 = -\frac{iab^2}{2}\phi\bar{\psi}\psi \; . \tag{4.3b}$$

We denote

$$g = \frac{iab^2}{2}$$

$$\mu^2 = -\frac{a^2}{4}$$

$$\Lambda = -\frac{a^2b^2}{6}$$

and from Eq. (4.3a,b) it follows:

$$i\partial\!\!\!/\psi - 2\mu\psi = g\phi^2\psi \; , \tag{4.4a}$$

$$\Box\phi + \mu^2\phi - \Lambda\phi^3 = -g\phi\bar{\psi}\psi \; . \tag{4.4b}$$

These equations are the starting point of the supersymmetric "SLAC-BAG" model, describing strongly bound quarks in a simple two dimensional example. The fields of this model consist of a colourless quark field ψ of only one flavour, a Higg's field ϕ, but no gluons. The supersymmetry invariance changes the "SLAC-BAG" model described by the field equations /7/:

$$i\partial\!\!\!/\psi = g\phi\psi \tag{4.5a}$$

$$\Box\phi + \mu^2\phi - \Lambda\phi^3 = -g\bar{\psi}\psi \; . \tag{4.5b}$$

The realization and the physical aspects of the supersymmetric "SLAC-BAG" model is in preparation.

In such a way we have obtained the surprising metamorphosis of fermions into bosons from the two dimensional supersymmetric point of view.

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