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ДУБНА

E2 - 10749

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Submitted to ТМФ

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Метод фазовых функций для квазипотенциального уравнения в терминах быстрот

В релятивистском конфигурационном представлении получены уравнения фазового типа для парциальной и полной амплитуды рассеяния двух частиц. Отправным пунктом является квазипотенциальное уравнение в терминах быстрот.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1977

Amirkhanov I.V., Grusha G.V., Mir-Kasimov R.M. **E2 - 10749**

The Phase Function Approach to the Quasipotential Equation in Terms of Rapidities

In the relativistic configurational representation the phase type equations for the partial and total amplitudes of two-particle scattering are obtained. The scheme is based on the quasipotential equation in terms of rapidities.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1977

1. Introduction

The present paper is aimed at the relativistic generalization of the phase function method. We proceed from the quasipotential equation (QPE) in terms of rapidities[/1/], reducing in the relativistic configurational representation[/2/] (see also reviews[/3/])[1)] to the differential one.

The formulation of the two-body problem in the relativistic configurational space is based on geometrical properties of the field-theoretical QPE[/4/] and, especially,[/5/] and possesses many typical properties of quantum mechanics, that makes it possible to apply here a number of its methods. For instanse, in ref.[/6/] (see also App. II,III in the book[/7/]) a variant of the relativistic phase method was developed, based on the finite-difference Schrödinger equation[/2,3/]. The equations, deduced in[/6/], made it possible to introduce the relativistic notions of the effective radius and the scattering length, to obtain some qualitative estimates of the scattering parameters, and to elaborate a method of calculating the relativistic corrections. However, the finite-difference character of the phase equations required

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1) A more complete list of refs. on quasipotential approach is presented in[/1,3/].

the additional speculations to solve them unambiguously. From this point of view the relativistic Schrödinger equation, deduced in[1], is preferable.

The integral form of this equation is:

$$\Psi_q(\vec{r}) = \xi(\vec{q},\vec{r}) + \int g_q(\vec{r},\vec{r}\,')V(\vec{r}\,')\Psi_q(\vec{r}\,')d\vec{r}\,', \qquad (1.1)$$

where $2E_q = 2ch\chi_q = 2\sqrt{1+\vec{q}^{\,2}}$ is the total energy eigenvalue[2)], $\vec{q}$ is the relative momentum in the c.m. system, $\xi(\vec{q},\vec{r})$ is the kernel of relativistic Fourier transformation[2,3]:

$$\xi(\vec{q},\vec{r}) = (q_0 - \vec{q}\,\vec{n})^{-1-ir}, \qquad q_0 = E_q, \qquad (1.2)$$

$\vec{r} = r\vec{n}$ is the relativistic analogue of the relative radius-vector ($0 < r < \infty$). The nonrelativistic limit ($c \to \infty$) of the quantity ξ is the usual plane wave:

$$\xi(\vec{q},\vec{r}) \longrightarrow e^{i\vec{q}\vec{r}}$$

The relativistic plane waves (1.2) satisfy the completeness and orthogonality conditions:

$$\frac{1}{(2\pi)^3}\int \xi(\vec{q},\vec{r})\,\xi^*(\vec{q}\,',\vec{r})\,d\vec{r} = q_0\,\delta(\vec{q}-\vec{q}\,'),$$

$$\frac{1}{(2\pi)^3}\int \xi(\vec{p},\vec{r})\,\xi^*(\vec{p},\vec{r}\,')\,d\Omega_p = \delta(\vec{r}-\vec{r}\,'),$$

$$d\Omega_p = d\vec{p}\,/\sqrt{1+\vec{p}^{\,2}}.$$

The partial expansions of the plane waves

$$\xi(\vec{q},\vec{r}) = \frac{1}{r\,sh\,\chi_q}\sum_{\ell=0}^{\infty}(2\ell+1)\,i^\ell\, s_\ell(r,\chi_q)\,P_\ell\!\left(\frac{\vec{q}\,\vec{r}}{q\,r}\right), \qquad (1.3)$$

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2) We use the system of units, in which $\hbar = c = m = 1$, m is the mass of one of the interacting particles. The case of equal masses is considered.

and the expansion of the Green function $g_q(\vec{r},\vec{r}\,')$ [1]:

$$g_q(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\frac{\chi_q}{sh\,\chi_q}\int \frac{\xi(\vec{p},\vec{r})\,\xi^*(\vec{p},\vec{r}\,')}{\chi_q^2 - \chi_p^2 + i\varepsilon}\,d\Omega_p = \qquad (1.4)$$

$$= \frac{1}{4\pi\, r r'}\sum_{\ell=0}^{\infty}(2\ell+1)\,g_{q\ell}(r,r')\,P_\ell(\vec{n}\,\vec{n}\,'),$$

where

$$g_{q\ell}(r,r') = -\frac{1}{sh\,\chi_q}\{\theta(r-r')\,e_\ell^{(1)}(r,\chi_q)\,s_\ell^*(r',\chi_q) + \qquad (1.5)$$

$$+\,\theta(r'-r)\,e_\ell^{(2)}(r',\chi_q)\,s_\ell(r,\chi_q)\},$$

are defined now using the relativistic analogues of spherical Bessel-functions $s_\ell(r,\chi_q)$, $c_\ell(r,\chi_q)$ and spherical Hankel-functions $e_\ell^{(1,2)}(r,\chi_q)$ (see Appendix).

The partial-wave equation of $\Psi_{q\ell}(r)$ has the form:

$$\Psi_{q\ell}(r) = s_\ell(r,\chi_q) + \int_0^\infty g_{q\ell}(r,r')\,V(r')\,\Psi_{q\ell}(r')\,dr'. \qquad (1.6)$$

2. <u>The Phase Type Equation for the Partial Scattering Amplitude</u>

Following the phase function method[7,8] let us consider the wave function $\Psi_{q\ell}(r)$ as a linear superposition of two free solutions $s_\ell(r,\chi_q)$ and $e_\ell^{(1)}(r,\chi_q)$ with coefficients, depending on r:

$$\Psi_{q\ell}(r) = B_\ell(r,\chi_q)\left[s_\ell(r,\chi_q) + A_\ell(r,\chi_q)\,e_\ell^{(1)}(r,\chi_q)\right]. \qquad (2.1)$$

Comparing (2.1) and (1.1), and taking into account (1.4), we come to the pair of first-order differential equations for $A_\ell(r,\chi_q)$ and $B_\ell(r,\chi_q)$:

$$\frac{d}{dr}\mathcal{A}_\ell(r,\chi_q) = -\frac{V(r)}{\hbar\chi_q}\left[s_\ell(r,\chi_q) + \mathcal{A}_\ell(r,\chi_q)e_\ell^{(1)}(r,\chi_q)\right] \times \left[s_\ell^*(r,\chi_q) + \mathcal{A}_\ell(r,\chi_q)e_\ell^{(2)*}(r,\chi_q)\right], \tag{2.2}$$

$$\mathcal{A}_\ell(0,\chi_q) = 0,$$

$$\frac{d}{dr}\mathcal{B}_\ell(r,\chi_q) = \frac{V(r)}{\hbar\chi_q}e_\ell^{(2)*}(r,\chi_q)\mathcal{B}_\ell(r,\chi_q) \times \left[s_\ell(r,\chi_q) + \mathcal{A}_\ell(r,\chi_q)e_\ell^{(1)}(r,\chi_q)\right], \tag{2.3}$$

$$\mathcal{B}_\ell(\infty,\chi_q) = 1.$$

If the solution of equation (2.2) is known, we find the solution of (2.3) in quadratures

$$\mathcal{B}_\ell(r,\chi_q) = \exp\left\{\frac{1}{\hbar\chi_q}\int_r^\infty V(r')e_\ell^{(2)*}(r',\chi_q)\left[s_\ell(r',\chi_q) + \mathcal{A}_\ell(r',\chi_q)e_\ell^{(1)}(r',\chi_q)\right]dr'\right\}.$$

By analogy with the nonrelativistic case, we may treat the quantities $\mathcal{A}_\ell(r,\chi_q)$ and $\mathcal{B}_\ell(r,\chi_q)$ as the scattering amplitude and the normalization constant of the wave function $\psi_{q\ell}(r')$, corresponding to a part of the potential $V(r';r)$ inside a sphere of the radius r :

$$V(r';r) = V(r')\theta(r-r'). \tag{2.4}$$

The equations for the S -matrix element $\hat{S}_\ell(r,\chi)$, phase function $\delta_\ell(r,\chi)$ and the tangent of the phase $t_\ell(r,\chi)$, corresponding to the scattering on the cut-off potential (2.4), may be obtained from (2.2) and the relations connecting these

quantities with the partial scattering amplitude:

$$\hat{S}_\ell(r,\chi) = e^{2i\delta_\ell(r,\chi)},$$

$$t_\ell(r,\chi) = \frac{\mathcal{A}_\ell(r,\chi)}{1+i\mathcal{A}_\ell(r,\chi)},$$

$$\mathcal{A}_\ell(r,\chi) = e^{i\delta_\ell(r,\chi)}\sin\delta_\ell(r,\chi).$$

These equations have the form:

$$\frac{d}{dr}\hat{S}_\ell(r,\chi) = i\frac{V(r)}{2\hbar\chi}\left[e_\ell^{(2)}(r,\chi) - \hat{S}_\ell(r,\chi)e_\ell^{(1)}(r,\chi)\right] \times \left[e_\ell^{(1)*}(r,\chi) - \hat{S}_\ell(r,\chi)e_\ell^{(2)*}(r,\chi)\right], \quad \hat{S}_\ell(0,\chi) = 1, \tag{2.5a}$$

$$\frac{d}{dr}t_\ell(r,\chi) = -\frac{V(r)}{\hbar\chi}\left[s_\ell(r,\chi) + t_\ell(r,\chi)c_\ell(r,\chi)\right] \times \left[s_\ell^*(r,\chi) + t_\ell(r,\chi)c_\ell^*(r,\chi)\right], \quad t_\ell(0,\chi) = 0, \tag{2.5b}$$

$$\frac{d}{dr}\delta_\ell(r,\chi) = -\frac{V(r)}{\hbar\chi}\left[\cos\delta_\ell(r,\chi)\,s_\ell(r,\chi) + \sin\delta_\ell(r,\chi)\,c_\ell(r,\chi)\right]\left[\cos\delta_\ell(r,\chi)\,s_\ell^*(r,\chi) + \sin\delta_\ell(r,\chi)\,c_\ell^*(r,\chi)\right], \quad \delta_\ell(0,\chi) = 0. \tag{2.5c}$$

We can come to (2.2) and (2.5), replacing in their nonrelativistic analogues /7,8/ the free solutions of the Schrödinger equation (Riccati-Bessel and Riccati-Hankel functions) by their relativistic analogues.

Unlike the nonlinear finite-difference phase type equations, got in[6], equations (2.2) and (2.5) are the simplest ones (Riccati type equations). The natural boundary conditions choose their physical solutions unambiguously. Obviously, all approximate methods, developed in the nonrelativistic case (the perturbation theory, the linearization[7,8]), and the algorithms of numerical computation are directly applicable to (2.2) and (2.5).

3. The Phase Type Equation for the Total Scattering Amplitude

Let us now derive the phase type equation for the total amplitude without assuming the spherical symmetry of quasipotential. In quantum mechanics the same approach was developed in[9]; it allowed one to elaborate a new method for obtaining high-energy asymptotic representations of the scattering amplitude[10,11].

We begin with the more general case, than (2.4), where the potential variation is connected with the parametrical dependence on the cutoff radius.

Symbolically, the integral equations for the scattering amplitude and the wave function, and the connection between the wave function and the off-energy shell amplitude have the form:

$$\mathcal{A} = -\frac{1}{4\pi} V + V g_q \mathcal{A}, \tag{3.1}$$

$$\Psi_q = I + g_q V \Psi_q, \tag{3.2a}$$

$$\tilde{\Psi}_q = I + \tilde{\Psi}_q V g_q, \tag{3.2b}$$

$$\Psi_q = I - 2(2\pi)^4 g_q \mathcal{A}, \tag{3.3a}$$

$$\tilde{\Psi}_q = I - 2(2\pi)^4 \mathcal{A} g_q, \tag{3.3b}$$

where I is a kernel of the unity operator. Assuming now, that potential suffers a variation δV, let us calculate the amplitude variation $\delta\mathcal{A}$. So far as

$$\mathcal{A} = -\frac{1}{2(2\pi)^4} V \Psi_q,$$

we get for $\delta\mathcal{A}$:

$$\delta\mathcal{A} = -\frac{1}{2(2\pi)^4}\left[\delta V\cdot\Psi_q + V\cdot\delta\Psi_q\right].$$

Excluding $\delta\Psi_q$ with the help of (3.2a), we come to the relation:

$$\delta\mathcal{A} = -\frac{1}{2(2\pi)^7}\tilde{\Psi}_q\cdot\delta V\cdot\Psi_q. \tag{3.4}$$

Taking into account (3.3a,b), we obtain the equation for $\mathcal{A}$:

$$\delta\mathcal{A} = -\frac{1}{2(2\pi)^7}\left(I - 2(2\pi)^4\mathcal{A}g_q\right)\delta V\left(I - 2(2\pi)^4 g_q\mathcal{A}\right). \tag{3.5}$$

This equation is justified for any variation of potential, in particular, for cut-off potentials (2.4). Other surfaces of the potential cut-off are also possible (comp.[11]).

Let us now concentrate on the parametrical dependence of potential (2.4) and investigate equation (3.5) in configurational representation.

In this representation the relation (3.4) for the scattering amplitude on the energy shell $\chi_p = \chi_q$ is given by expression:

$$\frac{d}{dr}\mathcal{A}(\chi_q, \vec{n}_p, r, \vec{n}_q) = -\frac{r^2}{4\pi}\int d\vec{n}_{r'}\, \times \tag{3.6}$$

$$\times\,\tilde{\Psi}_r(\chi_q, \vec{n}_p; r', \vec{n}_{r'})\, V(\vec{r}\,')\, \Psi_r(\chi_q, \vec{n}_q; r', \vec{n}_{r'})\Big|_{r'=r},$$

where the wave functions Ψ and $\widetilde{\Psi}$ satisfy the equations:

$$\Psi_r(\chi_q,\vec{n}_q;r',\vec{n}_{r'})=\xi(\chi_q,r';\vec{n}_q\vec{n}_{r'})+ \qquad (3.7a)$$

$$+\int g_q(\vec{r}',\vec{r}'')V(\vec{r}'')\theta(r'-r'')\Psi_r(\chi_q,\vec{n}_q;r'',\vec{n}_{r''})d\vec{r}'',$$

$$\widetilde{\Psi}_r(\chi_q,\vec{n}_q;r',\vec{n}_{r'})=\xi^*(\chi_q,r';\vec{n}_q\vec{n}_{r'})+ \qquad (3.7b)$$

$$+\int\widetilde{\Psi}_r(\chi_q,\vec{n}_q;r'',\vec{n}_{r''})g_q(\vec{r}'',\vec{r}')V(\vec{r}'')\theta(r'-r'')d\vec{r}''.$$

For a more clear representation we denote the dependence of all quantities on the lengths of three-dimensional vectors and their directions, separately.

Let us write the Green function (1.4) in the form:

$$g_q(\vec{r},\vec{r}')=-\frac{sh\,\chi_q}{(4\pi)^2}\int d\vec{n}_3\{\theta(r-r')\xi^*(\chi_q,r;\vec{n}_3\vec{n}') \qquad (3.8)$$

$$\times E^{(1)}(\chi_q,r;\vec{n}\vec{n}_3)+\theta(r'-r)\xi(\chi_q,r;\vec{n}_3\vec{n})E^{(2)*}(\chi_q,r';\vec{n}'\vec{n}_3)\},$$

where the generalized functions $E^{(1,2)}(\chi_q,r;\vec{n}\vec{n}')$ are formally given by partial expansions:

$$E^{(1,2)}(\chi_q,r;\vec{n}_q\vec{n}_r)=\frac{1}{r\,sh\,\chi_q}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}e_{\ell}^{(1,2)}(r,\chi_q)P_\ell(\cos\theta). \qquad (3.9)$$

For the quantities $E^{(1,2)}(\chi_q,r;\vec{n}_q\vec{n}_r)$ the relations

$$\int\xi^*(\chi_q,r;\vec{n}_q\vec{n}_r)E^{(1,2)}(\chi_p,r;\vec{n}_p\vec{n}_r)d\vec{n}_r= \qquad (3.10a)$$

$$=4\pi\,\frac{e^{\pm ir\chi_{pq}}}{r\,sh\,\chi_{pq}},$$

where

$$ch\,\chi_{pq}=ch\,\chi_p\,ch\,\chi_q-sh\,\chi_p\,sh\,\chi_q(\vec{n}_p\vec{n}_q),\quad \chi_p<\chi_q,$$

$$\int E^{(1,2)}(\chi_q,r;\vec{n}_q\vec{n}_r)d\vec{n}_r=\frac{4\pi}{r\,sh\,\chi_q}\times e^{\pm ir\chi_q}, \qquad (3.10b)$$

$$E^{(1,2)}(\chi_q,r;\vec{n}_q\vec{n}_r)\xrightarrow[r\chi_q\to\infty]{}\frac{4\pi}{r\,sh\,\chi_q}e^{\pm ir\chi_q}\delta(1\mp\vec{n}_q\vec{n}_r) \qquad (3.10c)$$

are fulfilled.

Taking into account (3.7a,b)-(3.10) and the expression for the amplitude depending on the cut-off radius r :

$$\mathcal{A}(\chi_q,\vec{n}_p;r,\vec{n}_q)=-\frac{1}{4\pi}\int\xi^*(\chi_q,r';\vec{n}_p\vec{n}_{r'})\times \qquad (3.11)$$

$$\times V(\vec{r}')\theta(r-r')\Psi_r(\chi_q,\vec{n}_q;r',\vec{n}_{r'})d\vec{r}'=-\frac{1}{4\pi}\times$$

$$\times\int\widetilde{\Psi}_r(\chi_q,\vec{n}_q;r',\vec{n}_{r'})V(\vec{r}')\theta(r-r')\times$$

$$\times\,\xi(\chi_q,r';\vec{n}_{r'},\vec{n}_3)d\vec{r}',$$

we obtain from (3.6) the sought - for equation:

$$\frac{d}{dr}\mathcal{A}(\chi_q,\vec{n}_p;r,\vec{n}_q)=-\frac{r^2}{4\pi}\int d\vec{n}_r\,V(\vec{r})\times \qquad (3.12)$$

$$\times\{[\xi^*(\chi_q,r;\vec{n}_p\vec{n}_r)+\frac{sh\,\chi_q}{4\pi}\int\mathcal{A}(\chi_q,\vec{n}_p;r,\vec{n}_1)\times$$

$$\times E^{(2)*}(\chi_q,r;\vec{n}_r\vec{n}_1)d\vec{n}_1][\xi(\chi_q,r;\vec{n}_q\vec{n}_r)+\frac{sh\,\chi_q}{4\pi}\times$$

$$\times\int\mathcal{A}(\chi_q,\vec{n}_q;r,\vec{n}_2)E^{(1)}(\chi_q,r;\vec{n}_r\vec{n}_2)d\vec{n}_2]\}$$

with the boundary condition $\mathcal{A}(\chi_q,\vec{n}_p,0,\vec{n}_q)=0$,

following from (3.11).

We give also another derivation of equation (3.12). In the spirit of phase function approach[8] we consider the wave function as a linear superposition of the solutions $\xi(\chi_q,\tau;\vec{n}_q\vec{n}_\tau)$ and $E^{(1)}(\chi_q,\tau;\vec{n}_q\vec{n}_\tau)$ of the free Schrödinger equation

$$\Psi(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)=\int\mathcal{B}(\chi_q,\vec{n}_q;\tau,\vec{n}_1)\times \tag{3.13}$$

$$\times\Big[\xi(\chi_q,\tau;\vec{n}_q\vec{n}_\tau)+\frac{sh\chi_q}{4\pi}\int d\vec{n}_2\,\mathcal{A}(\chi_q,\vec{n}_2;\tau,\vec{n}_1)\times$$

$$\times E^{(1)}(\chi_q,\tau;\vec{n}_2\vec{n}_\tau)\Big]\,d\vec{n}_1\,,$$

where the amplitude function $\mathcal{B}(\chi_q,\vec{n}_q;\tau,\vec{n}_1)$ is an analogue of the normalization factor $\mathcal{B}_\ell$ (2.1). Comparing (3.13) and (3.7) and taking into account (3.8), we find:

$$\mathcal{B}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)=\delta(\vec{n}_q-\vec{n}_\tau)- \tag{3.14}$$

$$-\frac{sh\chi_q}{(4\pi)^2}\int d\vec{\tau}'\,\theta(\tau'-\tau)\,E^{(2)*}(\chi_q,\tau';\vec{n}_{\tau'}\vec{n}_\tau)\,V(\vec{\tau}')\times$$

$$\times\Psi(\chi_q,\vec{n}_q;\tau',\vec{n}_{\tau'})\,,$$

$$\int d\vec{n}_\tau\,\mathcal{B}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)\,\mathcal{A}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)= \tag{3.15}$$

$$=-\frac{1}{4\pi}\int d\vec{\tau}'\,\theta(\tau-\tau')\,\xi^*(\chi_q,\tau';\vec{n}_\tau\vec{n}_q)\,V(\vec{\tau}')\times$$

$$\times\Psi(\chi_q,\vec{n}_q;\tau',\vec{n}_{\tau'}).$$

Hence it follows that $\mathcal{B}(\chi_q,\vec{n}_q;\infty,\vec{n}_\tau)=\delta(\vec{n}_q-\vec{n}_\tau)$ and $\mathcal{A}(\chi_q,\vec{n}_q;\infty,\vec{n}_p)$ is the scattering amplitude on the potential $V(\vec{\tau})$.

Excluding $\mathcal{B}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)$ from the pair of relations (3.14) and (3.15), we come again to the equation (3.13). It is easy to see, that at any finite value of τ the quantity $\mathcal{A}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)$ equals to the total scattering amplitude on the cut-off potential (2.4), since in this case $\mathcal{A}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)=\mathcal{A}(\chi_q,\vec{n}_q;\infty,\vec{n}_\tau)$.

It can be easily verified, that the solution of equation (3.13) satisfies the reciprocity relation

$$\mathcal{A}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)=\mathcal{A}(\chi_q,-\vec{n}_\tau;\tau,-\vec{n}_q) \tag{3.16}$$

and, for real potentials, the unitarity condition:

$$\operatorname{Im}\mathcal{A}(\chi_q,\vec{n}_q;\tau,\vec{n}_\tau)=\frac{sh\chi_q}{4\pi}\int d\vec{n}\times$$

$$\times\mathcal{A}(\chi_q,\vec{n};\tau,\vec{n}_\tau)\,\mathcal{A}^*(\chi_q,\vec{n};\tau,\vec{n}_q). \tag{3.17}$$

We give also the treatment of nonlinear equation for the scattering amplitude in the momentum representation, which can be useful for detailed calculations:

$$\delta A_\ell(p,q) = -\frac{1}{\mathrm{sh}\chi_q}\int_0^\infty\!\!\int_0^\infty \tilde{\Psi}_{pq}(\chi_k)\,\delta V_\ell(\chi_k,\chi_k')\,\Psi_{q\ell}(\chi_k')\times d\chi_k\,d\chi_k', \tag{3.18}$$

where

$$\Psi_{q\ell}(\chi_k) = \delta(\chi_p-\chi_k) - \mathrm{sh}\chi_q\cdot g_q(k)\,A_\ell(k,q), \tag{3.19}$$

$$\tilde{\Psi}_{pq}(\chi_k) = \delta(\chi_q-\chi_k) - \mathrm{sh}\chi_q\cdot A_\ell(p,k)\,g_q(k), \tag{3.20}$$

$$V_\ell(\chi_p,\chi_k) = \int_0^\infty s_\ell^*(r,\chi_p)\,V(r)\,s_\ell(r,\chi_k)\,dr, \tag{3.21}$$

$$A_\ell(p,q) = -\frac{1}{\mathrm{sh}\chi_q}\int_0^\infty s_\ell^*(r,\chi_p)\,V(r)\,\Psi_{q\ell}(r)\,dr, \tag{3.22}$$

$$g_q(k) = \frac{2}{\pi}\cdot\frac{\chi_q}{\mathrm{sh}\chi_q}\cdot\frac{1}{\chi_q^2-\chi_k^2+i\varepsilon}. \tag{3.23}$$

4. The Effective Radius Approximation. Relativistic Corrections

In the nonrelativistic case one comes to the known effective radius approximation by expanding the tangent of scattering phase on the short-acting potential with radius R in powers of the dimensionless parameter

$$t_\ell(r,q) = \mathrm{tg}\,\delta_\ell(r,q) = -\frac{1}{(2\ell+1)!!\,(2\ell-1)!!}\sum_{n=0}^{\infty} a_{\ell n}(r)\,q^{2n+2\ell+1}$$

and keeping the first terms of this expansion.

In the relativistic case, as was shown in[6], one should use the expansion:

$$t_\ell(r,\chi_q) = -\frac{1}{(2\ell+1)!!\,(2\ell-1)!!}\sum_{n=0}^{\infty} a_{\ell n}(r)\left(2\,\mathrm{sh}\frac{\chi_q}{2}\right)^{2n+2\ell+1} \tag{4.1}$$

in powers of the quantity $2\,\mathrm{sh}\,\chi_q/2$, which is the relativistic generalization of the momentum of relative motion. The expansion (4.1) is valid, if the interaction radius R is smaller than the Compton wave length $\bar{\lambda}$ of interacting particle

$$R/\bar{\lambda} < 1.$$

This condition is fulfilled, if the interaction is realized by the exchange of quanta with the mass larger than the masses of interacting particles.

Substituting the expansion (4.1) into (2.5b) and using the expansions of free solutions in powers of $2\,\mathrm{sh}\,\chi_q/2$, we obtain the differential equations for the functions $a_{\ell n}(r)$.

At $n=0$ these equations have the form:

$$\frac{d}{dr}a_{\ell 0}(r) = \frac{V(r)(-1)^{\ell+1}}{2\ell+1}\cdot\frac{r^{(\ell+1)}}{(-r)^{(\ell+1)}}\left[(-r)^{(\ell+1)} + a_{\ell 0}(r)(-r)^{(-\ell)}\right]^2 \tag{4.2}$$

with the boundary condition $a_{\ell 0}(0)=0$. In particular, the equation for the scattering length $a(r)=a_{00}(r)$ does not differ in form from the nonrelativistic one:

$$\frac{d}{dr}a(r) = V(r)\left[r-a(r)\right]^2. \tag{4.3}$$

The equations for the functions $a_{\ell n}(r)$ are linear at $n\geq 1$. We give only the equation for $a_{01}(r)$, defining the effective radius $r_{eff} = \lim_{r\to\infty} 2a_{01}(r)/a_{00}^2(r)$:

$$\frac{d}{dr}a_{01}(r) = X(r)\,a_{01}(r) + Y(r),\quad a_{01}(0)=0, \tag{4.4}$$

where

$$\chi(r) = 2V(r)[a(r)-r],$$

$$Y(r) = \frac{1}{8}\frac{da(r)}{dr} - V(r)(r-a(r))\left(\frac{r^3}{3}-r^2a(r)+\frac{r}{12}\right).$$

The solution of equation (4.4)

$$a_{01}(\infty) = \int_0^\infty dr'\, Y(r')\, e^{\int_{r'}^{\infty}\chi(r'')dr''} \qquad (4.5)$$

connects the effective radius with the scattering length.

To find relativistic corrections to the scattering parameters, we use the expansions of free solutions in powers of $\lambda = \frac{\hbar}{mc}$, given in /6/. Substituting, for example, the expansion

$$t_\ell(r,\chi_q) = \sum_{k=0}^{\infty} t_\ell^{(k)}(r,q)\,\lambda^k \qquad (4.6)$$

into equation (2.5b), we obtain taking into account boundary conditions:

$$t_\ell^{(1)}(r,q)=0,\quad t_\ell^{(2)}(r,q) = -\int_0^r dr'\,\frac{V(r')}{2q}\varphi_\ell(r',q)\times \qquad (4.7)$$

$$\times\exp\left\{-\frac{2}{q}\int_{r'}^{r}V(r'')c_\ell(qr'')M_\ell(q,r'')dr''\right\},$$

where $\varphi_\ell(r,q) = M_\ell(q,r)\left\{\frac{\ell(\ell+1)}{2r^2}\left[\frac{\ell(\ell+1)}{2}-\frac{1}{3}\right]M_\ell(q,r) + \right.$

$$\left.+\frac{1}{6}\left[\frac{\ell(\ell+1)}{r}-q^2r\right]\left(\frac{d}{dr}s_\ell(qr) - t_\ell^{(0)}(r,q)\frac{d}{dr}c_\ell(qr)\right)\right\},$$

$$M_\ell(q,r) = s_\ell(qr) + t_\ell^{(0)}(r,q)\,c_\ell(qr).$$

The authors express their gratitude to A.A.Atanasov, V.G.Kadyshevsky, A.N.Kvinikhidze, S.Cht.Mavrodiev, M.D.Mateev, A.N.Sissakian, N.B.Skachkov and L.A.Slepchenko for useful discussions.

Appendix

The relativistic analogues of the Bessel function $s_\ell(r,\chi_q)$, Neuman function $c_\ell(r,\chi_q)$ and Hankel functions $e_\ell^{(1,2)}(r,\chi_q)$ are expressed through Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ in the following manner:

$$s_\ell(r,\chi_q) = \sqrt{\frac{\pi\,\mathrm{sh}\,\chi_q}{2}}(-1)^{\ell+1}(-r)^{(\ell+1)}P^{-\ell-1/2}_{ir-1/2}(\mathrm{ch}\,\chi_q), \qquad (A.1)$$

$$c_\ell(r,\chi_q) = (-1)^\ell s_{-\ell-1}(r,\chi_q) = \sqrt{\frac{\pi\,\mathrm{sh}\,\chi_q}{2}}(-r)^{(-\ell)}P^{\ell+1/2}_{ir-1/2}(\mathrm{ch}\,\chi_q), \qquad (A.2)$$

$$e_\ell^{(1,2)}(r,\chi_q) = c_\ell(r,\chi_q) \pm i\,s_\ell(r,\chi_q) = \qquad (A.3)$$

$$= \frac{1}{i}\sqrt{\frac{2\,\mathrm{sh}\,\chi_q}{\pi}}(-1)^{\ell+1}(-r)^{(\ell+1)}Q^{-\ell-1/2}_{\mp ir-1/2}(\mathrm{ch}\,\chi_q),$$

where

$$r^{(\lambda)} = i^\lambda\,\frac{\Gamma(-ir+\lambda)}{\Gamma(-ir)} \qquad (A.4)$$

is the so-called generalized power.

The quantities $s_\ell(r,\chi_q)$, $c_\ell(r,\chi_q)$ and $e_\ell^{(1,2)}(r,\chi_q)$ have the following nonrelativistic limit:

$$s_\ell(r,\chi_q) \longrightarrow s_\ell(rq) = \sqrt{\frac{\pi rq}{2}}\,J_{\ell+1/2}(rq), \qquad (A.5)$$

$$c_\ell(r,\chi_q) \longrightarrow c_\ell(rq) = -\sqrt{\frac{\pi rq}{2}}\,Y_{\ell+1/2}(rq), \qquad (A.6)$$

$$e_\ell^{(1,2)}(r,\chi_q) \longrightarrow e_\ell^{(1,2)}(rq) = \pm i\sqrt{\frac{\pi rq}{2}}\,H^{(1,2)}_{\ell+1/2}(rq). \qquad (A.7)$$

For the functions $S_\ell(\tau,\chi_q)$ the following addition theorem /2,12/ is fulfilled:

$$\frac{S_0(\tau,\chi_{pq})}{\operatorname{sh}\chi_{pq}} = \frac{1}{\tau\,\operatorname{sh}\chi_p\,\operatorname{sh}\chi_q}\sum_{\ell=0}^{\infty}(2\ell+1)\,S_\ell^*(\tau,\chi_p)\,\times \tag{A.8}$$

$$\times\, S_\ell(\tau,\chi_q)\,P_\ell(\cos\theta),$$

$$\operatorname{ch}\chi_{pq} = \operatorname{ch}\chi_p\cdot\operatorname{ch}\chi_q - \operatorname{sh}\chi_p\cdot\operatorname{sh}\chi_q\cdot\cos\theta,$$

or, in the integral form:

$$\int \xi^*(\vec p,\vec r)\,\xi(\vec q,\vec r)\,d\vec n_r = \frac{4\pi\, S_0(\tau,\chi_{pq})}{\tau\,\operatorname{sh}\chi_{pq}} = \tag{A.9}$$

$$= 4\pi\,\frac{\sin\tau\chi_{pq}}{\tau\,\operatorname{sh}\chi_{pq}}.$$

Let us prove an analogous theorem for the functions $e_\ell^{(1,2)}$ and C_ℓ. First of all, we note, that the addition theorem (A.8) is justified in a more wide region, than the "physical" one (turning into the halfline $0 \le \chi \le \infty$ or $1 \le \operatorname{ch}\chi = z < \infty$), namely, the relation

$$\frac{1}{(z^2-1)^{1/2}}\,P^{-1/2}_{i\tau-1/2}(z) = \frac{\sqrt{\pi/2}}{(z_1^2-1)^{1/2}\,(z_2^2-1)^{1/2}}\,\times \tag{A.10}$$

$$\times\sum_{\ell=0}^{\infty}(2\ell+1)\left|\frac{\Gamma(i\tau+\ell+1)}{\Gamma(i\tau+1)}\right|^2 P^{-\ell-1/2}_{i\tau-1/2}(z_1)\,P^{-\ell-1/2}_{i\tau-1/2}(z_2)\,P_\ell(\cos\theta)$$

is fulfilled.

Here

$$z = z_1 z_2 - (z_1^2-1)^{1/2}(z_2^2-1)^{1/2}\cos\theta, \tag{A.11}$$

θ is considered real everywhere, $0 < \theta < \pi$.

The addition theorem (A.10) is justified for z_1, z_2 satisfying condition:

$$\left|\frac{(z_1+1)(z_2+1)}{(z_1-1)(z_2-1)}\right| > 1. \tag{A.12}$$

This condition follows from the asymptotic expansion of the expression, entering into the sum (A.10), at $\ell\to\infty$.

Let us now replace in (A.10) $z_2 \to -z_2$. Taking into account the relations

$$-z-1 = e^{\mp i\pi}(z+1),\quad -z+1 = e^{\mp i\pi}(z-1), \tag{A.13}$$

where the upper or lower signs are taken in accordance with the sign of $\operatorname{Im} z$ /13/, we pass to the expression:

$$\frac{1}{(z^2-1)^{1/2}}\,P^{-1/2}_{i\tau-1/2}(-z) = \frac{\sqrt{\pi/2}}{(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}}\,\times \tag{A.14}$$

$$\times\sum_{\ell=0}^{\infty}(2\ell+1)\left|\frac{\Gamma(i\tau+\ell+1)}{\Gamma(i\tau+1)}\right|^2 P^{-\ell-1/2}_{i\tau-1/2}(z_1)\,P^{-\ell-1/2}_{i\tau-1/2}(-z_2)\,P_\ell(\cos\theta),$$

which is fulfilled in the region:

$$\left|\frac{(z_1+1)(z_2-1)}{(z_1-1)(z_2+1)}\right| > 1. \tag{A.15}$$

Finally, we use the relation

$$\operatorname{sh}\pi\tau\cdot Q^{-\ell-1/2}_{-i\tau-1/2}(z) = \frac{\pi}{2}\left[\mp i\,e^{\mp\pi\tau}P^{-\ell-1/2}_{i\tau-1/2}(z) + P^{-\ell-1/2}_{i\tau-1/2}(-z)\right] \tag{A.16}$$

which gives the sought-for addition theorem

$$\frac{1}{(z^2-1)^{1/2}}\,Q^{-1/2}_{-i\tau-1/2}(z) = \frac{\sqrt{\pi/2}}{(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}}\,\times \tag{A.17}$$

$$\times\sum_{\ell=0}^{\infty}(2\ell+1)\left|\frac{\Gamma(i\tau+\ell+1)}{\Gamma(i\tau+1)}\right|^2 P^{-\ell-1/2}_{i\tau-1/2}(z_1)\,Q^{-\ell-1/2}_{-i\tau-1/2}(z_2)\,P_\ell(\cos\theta),$$

fulfilled in the region

$$\left| \frac{(z_1+1)(z_2 \pm 1)}{(z_1-1)(z_2 \mp 1)} \right| > 1. \tag{A.18}$$

Using the relation (A.3) we obtain in the "physical" region of arguments $z_1 = ch\chi_p$, $z_2 = ch\chi_q$:

$$\frac{\mathcal{E}_0^{(1)}(r, \chi_{pq})}{sh\,\chi_{pq}} = \frac{1}{r\,sh\chi_p\,sh\chi_q} \sum_{\ell=0}^{\infty} (2\ell+1) s_\ell^*(r,\chi_p)\,\mathcal{E}_\ell^{(1)}(r,\chi_q) \times P_\ell(\cos\theta), \tag{A.19}$$

and, analogously

$$\frac{\mathcal{E}_\ell^{(2)}(r, \chi_{pq})}{sh\,\chi_{pq}} = \frac{1}{r\,sh\chi_p\,sh\chi_q} \sum_{\ell=0}^{\infty} (2\ell+1) s_\ell^*(r,\chi_p)\,\mathcal{E}_\ell^{(2)}(r,\chi_q) \times P_\ell(\cos\theta), \tag{A.20}$$

$$\frac{\mathcal{C}_0(r, \chi_{pq})}{sh\,\chi_{pq}} = \frac{1}{r\,sh\chi_p\,sh\chi_q} \sum_{\ell=0}^{\infty} (2\ell+1) s_\ell^*(r,\chi_p)\,\mathcal{C}_\ell(r,\chi_q) \times P_\ell(\cos\theta). \tag{A.21}$$

The condition of fulfillment of addition theorems (A.19)-(A.21), in terms of the rapidities has the following form:

$$\chi_p < \chi_q .$$

The relations (A.19) and (A.20) are reduced in the integral form to (3.10a).

Passing in (A.19) and (A.20) to the limit $\chi_q \to \infty$, we obtain the partial expansion of the plane wave (1.3), like the expansion of the usual plane wave $e^{ipr\cos\theta}$ in Legendre polynoms is obtained from the addition theorem for $\mathcal{E}_0^{(1,2)}(r|\vec{p}-\vec{q}|)$. Let us emphasize that in the nonrelativistic limit formulae (A.9), (A.19)-(A.21) turn into addition theorems for the corresponding cylindrical functions.

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Received by Publishing Department on June 13, 1977.