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Об инвариантной регуляризации

Определен набор требований, обеспечивающий инвариантность процедуры регуляризованного интегрирования по внутренним импульсам диаграмм относительно любых преобразований симметрии. Единственной регуляризационной схемой, удовлетворяющей этим требованиям, оказывается размерная регуляризация. Показано, что несмотря на инвариантность интегрирования по импульсам, при наличии аномалий вся регуляризационная схема как целое может быть неинвариантной.

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On the Invariant Regularization

A set of constraints is found which renders regularized momentum integration invariant under any symmetry transformations. Dimensional regularization proves to be the only scheme satisfying these constraints. It is shown, however, that even the invariant momentum integration in the presence of anomalies cannot ensure the invariance of the regularization scheme as a whole.

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1. Introduction

An invariant regularization respecting all symmetries of the Lagrangian is a usefull and highly desirable tool for treating theories based on various symmetry groups. For several most important groups the invariant regularization schemes have already been constructed. One of them, the dimensional regularization scheme /1/ proved to be suitable for an extremely wide class of field theory models. The gauge invariance of this scheme was shown in /2,3,4/ within different approaches.

In this paper the constraints imposed on the regularization scheme due to its invariance are formulated explicitly. Then the regularization procedure satisfying these constraints for the widest possible class of Lagrangians is constructed, the anomalous theories being the only exception. This procedure appears to be absolutely identical to the standard dimensional regularization which therefore may be considered as the regularization scheme invariant by construction.

On the basis of the invariant regularization, one can perform the renormalization in the invariant manner as well, for instance, with the help of the 't Hooft method /5/. The proof of the invariance of this approach becomes particularly easy in the framework of the background-field formalism /6/.

2. Invariant integration

Symmetry properties of the Lagrangian in quantum field theory result in some relations for the Green functions known as the Ward identities. In paper /7/ a convenient method to derive these identities using the generating functional technique was suggested. Ignoring the troubles with ultraviolet divergences and applying the recipes of /7/ formally, we arrive at the relations which make sense for the integrands of corresponding Green's functions only. To give a sense to the divergent integrals as well, we must regularize them.

The universally invariant regularization may be defined as the procedure that leaves all formally derived Ward identities valid also for regularized Green's functions. Consequently, the invariantly regularized integration must guarantee the correctness of all manipulations essential for the derivation of the Ward identities. Slavnov's papers /7,8/ contain an elaborate analysis of such manipulations with generating functional and allow us to list the required properties of the universally invariant regularized integration procedure in configuration space:

1) uniqueness (two different ways of evaluating an integral are to give the same result),

2) linearity,and

3) possibility to integrate by parts neglecting the surface terms.

The third condition is automatically satisfied if all X-space expressions are always interpreted as the Fourier-transforms of p-space ones. Therefore, it will be convenient to use below the p-representation only.

We shall treat the momentum integrands associated with the Feynman diagrams as formal expressions constructed of a set of symbols $(p_\mu, g_{\mu\nu}, \dots)$ possessing the Lorenz-covariant form and properties $(p_\mu g_{\mu\nu} = p_\nu,\ p_\mu p_\mu = p^2,\ g_{\mu\mu} = n, \dots)$. This algebra of symbols may be regarded as an extension of the usual 4-dimensional Lorenz algebra (in particular, n may be unequal to 4). The three dots refer to the possibility of using some other symbols, for instance, the generalizations of γ_μ, γ_5 and so on. Of course, the whole set of properties of such symbols must be self-consistent. An example of the appropriate system of symbols is presented in paper /4/. The functions with several numerical arguments replaced by the symbolic ones are treated as retaining all their usual properties with respect to these arguments, e.g. $e^{\alpha p^2} e^{\beta q^2} = e^{\alpha p^2 + \beta q^2}$ and so forth.

Now let us reformulate the constraints for the universally invariant regularized integration in terms of p-space.

$$\int dp\, f(p+k, \dots) = \int dp\, f(p, \dots), \tag{1}$$

$$\int dp\, f(-p, \dots) = \int dp\, f(p, \dots), \tag{2}$$

$$\int dp \int dq\, f(p, q, \dots) = \int dq \int dp\, f(p, q, \dots). \tag{3}$$

These relations prevent the dependence of the result of integration on the choice of the internal momentum variables.

$$\int dp \sum_i a_i f_i(p, \dots) = \sum_i a_i \int dp\, f_i(p, \dots). \tag{4a}$$

The sum in this equality (linearity condition) may be infinite. It allows us to transfer a part of the free Lagrangian to the interaction one and vice versa, that is necessary for the uniqueness of the path-integral representation of the Green functions. Note that infinite sums can emerge only due to the power series expansion. In such a case eq. (4a)means that the coefficients at equal powers of the expansion parameter are equal in both sides.

Expanding the relationship

$$\int dp\, f(\alpha, p, \dots) = g(\alpha, \dots)$$

in a parameter α and using (4a), we obtain

$$\int dp\, \frac{\partial}{\partial \alpha} f(\alpha, p, \dots) = \frac{\partial}{\partial \alpha} \int dp\, f(\alpha, p, \dots), \tag{4b}$$

i.e., the momentum integration and the differentiation with respect to a numerical parameter commute. From this there immediately follows an analogous feature of the inverse operation, integration over parameter:

$$\int d\alpha \int dp\, f(\alpha, p, \dots) = \int dp \int d\alpha\, f(\alpha, p, \dots) + \text{quantity independent of } \alpha. \tag{4c}$$

Among other invariance properties of the regularized integration the Lorenz-invariance should be noted. In our symbolic language it means just retaining the tensor structure of the integrand

$$\int dp\, f_{\mu\nu\dots\rho}(p, \dots) = g_{\mu\nu\dots\rho}(\dots). \tag{5}$$

Finally, the uniqueness condition is to be formulated: an arbitrary identity transformation of the integrand does not change the result of integration.

Thus, a set of constraints on the regularized integration procedure is found which assures the uniqueness of the representation of Green's functions in a path-integral form and the validity of essential manipulations with generating functionals in deriving the Ward identities, proving the equivalence theorem, etc.

Whether the properties (1)-(5) are the necessary conditions for the universal invariance of integration procedure and for what class of integrand functions a scheme obeying these properties exists, these are the open questions. However, for the integrands associated with the Feynman diagrams of any local field theory model, these conditions prove to determine the regularization scheme uniquely. In the next section this invariant procedure of the regularized momentum integration will be constructed explicitly.

3. Dimensional regularization

By virtue of (4c) one can use the well-known parametric representation for propagators:

$$\frac{1}{(p^2-m^2+i\varepsilon)^\lambda} = \frac{i^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty d\alpha\, \alpha^{\lambda-1} e^{i\alpha(p^2-m^2+i\varepsilon)} .$$

The small imaginary quantity $i\varepsilon$ $(\varepsilon > 0)$ serves here, as usual, to cut off the integral on its upper limit because the symbols p^2 and m^2 may be regarded as real quantities.

Now we encounter the problem of evaluating the typical integrals

$$\int dp\, p_{\mu_1} \dots p_{\mu_r} e^{i(\alpha p^2 + 2\beta K p)} . \tag{6}$$

Let us begin with more simple auxiliary integral

$$I(\alpha, \beta K) = \int dp\, e^{i(\alpha p^2 + 2\beta K p)} , \tag{7}$$

with α and β being numerical parameters and K — external momentum. After the shift $p \to p - \frac{\beta}{\alpha} K$ in accordance with (1) we get

$$I(\alpha, \beta k) = e^{-i\frac{\beta^2}{\alpha}k^2} I(\alpha) \quad ; \quad I(\alpha) = \int dp\, e^{i\alpha p^2}.$$

To find $I(\alpha)$ we may use the relations (1) - (5) only, so it is not allowed to apply the dimensional analysis for its computation, as it was done by Wilson /9/.

Therefore, let us consider the integral

$$\int dp\, p_\mu p_\nu e^{i\alpha p^2}, \tag{8}$$

which must be equal to $A(\alpha) g_{\mu\nu}$ due to Lorenz-invariance of the integration procedure. We obtain $A(\alpha)$, multiplicating (8) by $g_{\mu\nu}$:

$$n A(\alpha) = \int dp\, p^2 e^{i\alpha p^2} = -i \frac{\partial}{\partial \alpha} I(\alpha).$$

Consequently

$$\int dp\, p_\mu p_\nu e^{i\alpha p^2} = -\frac{i}{n} g_{\mu\nu} \frac{\partial}{\partial \alpha} I(\alpha). \tag{9}$$

Using (9) and the properties (1) and (2) we find

$$\int dp\, kp\, p_\mu e^{i(\alpha p^2 + 2\beta kp)} = e^{-i\frac{\beta^2}{\alpha}k^2} k_\mu \left(\frac{\beta^2}{\alpha^2}k^2 - \frac{i}{n}\frac{\partial}{\partial\alpha}\right) I(\alpha). \tag{10}$$

However, this integral may be evaluated in another way, namely differentiating it with respect to β :

$$\int dp\, kp\, p_\mu e^{i(\alpha p^2 + 2\beta kp)} = -\frac{i}{2}\frac{\partial}{\partial\beta}\int dp\, p_\mu e^{i(\alpha p^2 + 2\beta kp)} =$$
$$= -\frac{i}{2}\frac{\partial}{\partial\beta}\left(-\frac{\beta}{\alpha}k_\mu I(\alpha) e^{-i\frac{\beta^2}{\alpha}k^2}\right) = e^{-i\frac{\beta^2}{\alpha}k^2} k_\mu \left(\frac{\beta^2}{\alpha^2}k^2 + \frac{i}{2\alpha}\right) I(\alpha). \tag{11}$$

Compairing (10) with (11) gives

$$\frac{\partial}{\partial\alpha} I(\alpha) = -\frac{n}{2\alpha} I(\alpha). \tag{12}$$

From this equation one can find $I(\alpha)$

$$I(\alpha)=N\alpha^{-\frac{n}{2}} \tag{13}$$

and as a consequence

$$\underline{I}(\alpha,\beta K)\equiv\int dp\, e^{i(\alpha p^2+2\beta Kp)}=N\alpha^{-\frac{n}{2}}e^{-i\frac{\beta^2}{\alpha}K^2}, \tag{14}$$

where N is an arbitrary constant. It can be shown /10/ that this arbitrariness of the regularization procedure is nonessential because it can be absorbed in the redefinition of renormalization constants.

Now let us consider the integrals of the following type:

$$\int dp\, p_{\mu_1}\dots p_{\mu_r}\, e^{i\alpha p^2}. \tag{15}$$

They vanish for odd r according to (2). For even r such an integral is proportional to appropriate symmetric expressions composed of $g_{\mu_i\mu_j}$, the proportionality coefficient being found as before, in the case of $A(\alpha)$.

It is easy to see that our initial integral (6) can be reduced to the type (15) by the shift $p\to p-\frac{\beta}{\alpha}K$. Consequently we are able now to perform any relevant momentum integrations.

We proceed now to the integrals over the α-parameters. These integrals may diverge in a neighbourhood of $\alpha=0$ and are to be correctly defined in such cases. The following chain of manipulations may be performed:

$$\int dp\frac{p^2 f(p,\dots)}{(p^2+i\varepsilon)^{\lambda+1}}=\frac{i^{-\lambda-1}}{\Gamma(\lambda+1)}\int_0^\infty d\alpha\,\alpha^{\lambda}\int dp\, f(p,\dots)p^2 e^{i\alpha(p^2+i\varepsilon)}=$$

$$=\frac{i^{-\lambda-2}}{\Gamma(\lambda+1)}\int_0^\infty d\alpha\,\alpha^{\lambda}\frac{\partial}{\partial\alpha}\int dp\, f(p,\dots)e^{i\alpha(p^2+i\varepsilon)}=$$

$$\frac{i^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty d\alpha\,\alpha^{\lambda-1}\int dp\,f(p,\dots)e^{i\alpha(p^2+i\varepsilon)}+\frac{i^{-\lambda-2}}{\Gamma(\lambda+1)}\int dp\,f(p,\dots)e^{i\alpha(p^2+i\varepsilon)}\alpha^\lambda\Big/_0^\infty=$$

$$=\int\frac{dp\,f(p,\dots)}{(p^2+i\varepsilon)^\lambda}+\frac{i^{-\lambda-2}}{\Gamma(\lambda+1)}\int dp\,f(p,\dots)e^{i\alpha(p^2+i\varepsilon)}\alpha^\lambda\Big/_0^\infty.$$

To treat the momentum integration as well-defined unambiguous recipe, one must demand the surface term to be zero, in other words

$$0^\lambda=0 \qquad \text{for arbitrary } \lambda. \tag{16}$$

Now let us integrate by parts taking into account (16) the relation defining Euler's Γ-function.

$$\Gamma(z)\equiv\int_0^\infty dx\,x^{z-1}e^{-x}=\frac{1}{z}\int_0^\infty dx\,x^{z}e^{-x}+\frac{1}{z}x^{z}e^{-x}\Big/_0^\infty=\frac{\Gamma(z+1)}{z}.$$

We see that in this case (16) enables us to continue $\Gamma(z)$ analytically to negative z. However, the singular parts of the α-parameter integrals may be reduced to the similar form,

$$\int_0^\infty d\alpha\,\alpha^{\lambda-1}e^{i\alpha(a+i\varepsilon)},$$

by standard changes of variables. From this it follows that the condition (16) defines α-integrals by means of the analytic continuation in λ,

$$\int_0^\infty d\alpha\,\alpha^{\lambda-1}e^{i\alpha(a+i\varepsilon)}=i^\lambda a^{-\lambda}\Gamma(\lambda). \tag{17}$$

From (14) and (17) we conclude that our regularization procedure, reconstructed on the basis of the constraints (1)- (5) only, is precisely the dimensional regularization. The proof of self-consistency of this approach, i.e., check of the validity of underlying properties (1) - (5) in the dimensional regularization scheme, has been given by Collins /3/. Thus, the relations (1) -(5) may be regarded as the definition of the dimensional regularization, its momentum integration procedure being universally invariant by construction. However, we show below that neither the dimensional nor any other regularization as a whole cannot be universally invariant.

4. Anomalies

The usual four-dimensional integration, though being invariant, produces ultraviolet divergences. This forces us to use the invariant (dimensional) integration formulas with $n \neq 4$. Therefore, apart from the momentum integration the regularization procedure as a whole includes another important step, namely the transition from $n = 4$ of the initial theory to $n \neq 4$ in the regularized one. To ensure the invariance of regularization, all symmetry properties of the Lagrangian must be respected at this step. It is the case when the symmetry properties hold for arbitrary n and have no explicit dependence on n /11/ . In opposite (anomalous) case the initial symmetry relations do fail when the transformation of the Lagrangian from $n = 4$ to $n \neq 4$ is done, and certainly cannot be restored through the invariant integration.
Thus, the dimensional regularization scheme is invariant only if the anomalies are absent.

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