ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 10720

B.D.Dorfel, D.I.Kazakov, D.V.Shirkov

THE UNIVERSALITY OF COUPLING CONSTANT SINGULARITY IN QFT

1977

E2 - 10720

B.D.Dorfel*, D.I.Kazakov, D.V.Shirkov

THE UNIVERSALITY OF COUPLING CONSTANT SINGULARITY IN QFT

Submitted to "Nuclear Physics"

* Permanent address: Gumboldt Universität, Berlin, DDR.

Дерфель Б.Д., Казаков Д.И., Ширков Д.В. E2 - 10720

Универсальность сингулярности по константе связи в квантовой теории поля

Исследуется структура сингулярности функции Грина по константе связи в нуле. Основной прием заключается в использовании представления функционального интеграла и приближенного вычисления последнего методом перевала в функциональном пространстве. Рассмотрен класс перенормируемых и суперперенормируемых скалярных моделей и установлено, что сингулярность имеет универсальный характер, независимо от наличия расходимостей. Полученные выражения представимы в виде спектрального интеграла по константе связи. Спектральная функция $\rho(g) \sim (-g)^{-\alpha} \exp(-A/g)$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1977

Dorfel B.D., Kazakov D.I., Shirkov D.V. E2 - 10720

The Universality of Coupling Constant Singularity in QFT

The structure of the Green function singularity near the origin of the coupling constant plane is studied. The method used exploits the functional integral representation and the procedure of the steepest descent method in the functional space. The class of renormalizable and superrenormalizable scalar field models is considered, and the universality of the singularity is established independently of the existence of divergences.

The obtained expressions are represented via the spectral integral over the coupling constant. The spectral function is of the form $\rho(g) \sim (-g)^{-\alpha} \exp(-A/g)$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1977

1. Introduction

Recent study of solutions of classical Euclidean field equations shed new light on the structure of the quantum-field amplitudes near the point $g=0$, g being the coupling constant. The method used exploits the functional integral representation and the procedure of the steepest descent method in the functional space. The saddle-point corresponds to classical solutions of the "instanton" type and the integration reduces to the Gaussian quadrature in the vicinity of instanton solution.

Using this way L.N.Lipatov /1/, starting from the functional integral representation for k-th order perturbation term of the expansion

$$G(\dots,g)=\sum_k g^k G_k(\dots),$$

has been able to evaluate the asymptotic (for k large) expression of $G_k(\dots)$ coefficients. The calculations were performed for n-prong vertex in the scalar model

$$\mathcal{L}=-\frac{g}{n!}\varphi^n_{(D)} \tag{1}$$

considered in the (Euclidean) space of $D=2n/(n-2)$ dimensions. His result is of the form

$$G(g)=a\sum_k(-g)^k b^k k^{k(n/2-1)+(n+D)/2} \tag{2}$$

a, b being some numbers depending on n.

The use of the steepest descent method for approximate calculation of functional integrals was proposed a decade ago by Langer [2]. In this paper the structure of singularity at the origin ($g=0$) was studied in a number of quantum statistical problems. Here, the functional integral was considered as a whole without its series expansion.

The expressions obtained there obey a singularity at the origin of complex g -plane and a cut along the real negative semiaxis. Due to this they can be expressed via the spectral Cauchy integral. It is just the jump across the cut that can be evaluated by the steepest descent method.

Following the Longer procedure E.B.Bogomolny [3] succeeded in finding the expression for $G(g)$ as a whole without using the power series over the coupling constant. The final expression

$$G(g)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{dz\,\Delta(z)}{z-g}\,,\quad \Delta(z)\simeq(-z)^{-\alpha}\exp(c/z) \qquad (3)$$

has the following properties:

(1) Its formal expansion gives the asymptotic series (2);

(2) Is real for real positive g ;

(3) For real negative g it has a cut with imaginary jump

$$G(g\pm i\epsilon)=R(g)\pm i\pi\Delta(g)\,,\quad g<0\,.$$

The function $\Delta(g)$, cannot be expanded in powers of g and "leaves no footprints" in the usual perturbation series;

(4) Corresponds [4] to the formal Borel summation of series (2).

The result (3) obtained for logarithmic massless models (1) with $D = 2n/(n-2)$ is very close to the Langer formulas for the quantum-statistical models.

We show below that it is valid for models (1) with $D < 2n/(n-2)$ as well and discuss some consequences of such a universality.

2. Evaluation of Functional Integral by Saddle-Point Method

The class of models (1) with

$$D \leqslant 2n/(n-2)$$

for integers n and D consists of three models $\varphi^4_{(2,3,4)}$ and $\varphi^6_{(2,3)}$. We limit our analysis to even values $n = 2N$. Consider the functional integral

$$G_\nu(g) = \int \delta\varphi \exp\left\{ i \int \mathcal{L}(x)\,dx \right\} \underbrace{\cdots}_{\nu} ,$$

where

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{g}{(2N)!}\varphi^{2N}$$

and ... stands for the possible product of ν field functions. After Wick rotation, we get the Euclidean analog of the initial theory with

$$G_\nu(g) \to \int \delta\varphi \exp\left\{ -\int H(x)\,d^D x \right\} \cdots \qquad (4)$$

Here the Hamiltonian

$$H(x) = \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{g}{(2N)!}\varphi^{2N}$$

is a positively defined for $g > 0$ and has the only stationary point $\varphi = const = 0$. Hence, the use of the steepest descent method for positive g gives noting more that the usual perturbation expression.

Following Langer we shall consider the Eq.(4) as a basis for analytical continuation on complex g values till the negative real semiaxis. For real $g < 0$ the Hamiltonian H becomes unbounded from below and at $|g| \to 0$ the corresponding quantummechanical problem has only quasistationary levels (see a more detaild discussion of this matter on pages 116-117 of paper /2/ as well as in /3/). Due to this the analytical continuation of expression (4) can be presented as a function obeying a cut along the negative real semiaxis with pure imaginary jump.

Hence

$$G(g) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{dz\, \Delta(z)}{z-g}\,, \tag{5}$$

$$\Delta(g) = \frac{1}{2i}\left[G(g+i\epsilon) - G(g-i\epsilon)\right] = \mathrm{Im}\, G(g). \tag{6}$$

The functional integral

$$G(\gamma) = \int \delta\varphi \left\{ -\int\left[\frac{(\vec{\nabla}\varphi)^2}{2} + \frac{m^2\varphi^2}{2} - \frac{\gamma}{(2N)!}\varphi^{2N}\right] d^D x\right\}\ldots = G(-g) \tag{7}$$

$(\gamma = -g)$ should be evaluated by the saddle-point method. Changing the integration variable

$$\varphi(x) = \left[(2N-1)!/\gamma\right]^{\frac{1}{2(N-1)}}\psi(x)$$

we get

$$G(\gamma) = \gamma^{-\frac{\nu}{2(N-1)}}\int \delta\psi \exp\left[-A(\psi)/\gamma^{\frac{1}{N-1}}\right]\cdots, \tag{8}$$

where

$$A(\psi) = \frac{1}{2}\int \left[(\vec{\nabla}\psi)^2 + m^2\psi^2 - \frac{1}{N}\psi^{2N} \right] d^D x . \tag{9}$$

The saddle-point solution corresponding to $\delta A = 0$ is defined as a solution of the Euler equation

$$\Delta\tilde{\psi} - m^2\tilde{\psi} = -\tilde{\psi}^{2N-1} \tag{10}$$

with finite action (9).

It can be shown using the method of paper /5/ that for logarithmic theories $\left(D = \frac{2N}{N-1} \right)$ such solutions exist only for $m = 0$. It was obtained in paper /1/ and has the form

$$\tilde{\psi}(x) = \lambda^{\frac{D-2}{2}} \psi_0(\lambda(x-a)), \quad \psi_0(x) = \left[\frac{2\sqrt{N}}{N-1} \cdot \frac{1}{1+x^2} \right]^{\frac{1}{N-1}} . \tag{11}$$

On the contrary in the superrenormalizable case $(D < 2N/_{N-1})$ such solutions are possible only for $m \neq 0$. They were obtained numerically in papers /6,7/.

At the same time in the nonrenormalizable case $(D > 2N/_{N-1})$ there are no solutions with finite A.

To calculate integral (8) by the saddle-point method first of all we have to find the action $A(\psi)$ corresponding to the classical solution $\tilde{\psi}$. Substituting $\psi = \tilde{\psi}$ in (9) and taking into account eq.(10), we have

$$A(\tilde{\psi}) = \frac{N-1}{2N}\int d^D x \left[(\vec{\nabla}\tilde{\psi})^2 + m^2\tilde{\psi}^2 \right] > 0 . \tag{12}$$

If equation (10) has several different solutions $\tilde{\psi}_i$ one has to sum the contributions from all of them. In the limit of small g we should be interested in the solution (or solutions) with minimal value of $A(\psi)$.

Consider now the integral over small fluctuations around the stationary solution. For this purpose put

$$\psi = \tilde{\psi} + \psi'$$

and expand the Hamiltonian H over ψ'. Keeping only quadratic terms we have

$$H(\psi) \simeq H(\tilde{\psi}) + \psi' \frac{H''(\tilde{\psi})}{2}\psi',$$

where

$$H''(\tilde{\psi}) = -\Delta_D + m^2 - (2N-1)\tilde{\psi}^{2N-2}. \tag{13}$$

For evaluation of the integral over ψ' it is necessary to solve the eigenvalue (e.v.) problem for the Hamiltonian (13) and to expand ψ' over the eigenfunctions

$$\psi' = \sum_n C_n \psi_n .$$

Then the problem reduces to the ordinary integrations over C_n /2/.

There exist (see /1,8,7/) only <u>one</u> negative e.v. $E_0 < 0$ and one or two zero e.v. The first leads to the rotation of the integration contour over C_0 on 90° in integral (8) which becomes pure imaginary. The appearance of zero e.v. is connected with additional invariance of action. In the $\varphi^4_{(D)}$ theory with $m \neq 0$ we have only translational invariance and consequently <u>one</u>(D-fold degenerated)zero e.v.and in the $\varphi^4_{(4)}$ and $\varphi^6_{(3)}$ theories with $m = 0$ there exists additional dilatational invariance and one more zero e.v.. This leads to the existence of free parameters in the solution of equation (10) (see, for example, λ and a in (11)). Integration over these parameters leads to the additional dependence on the coupling constant of the type

$$\left[\gamma^{\frac{1}{N-1}}\right]^{-D/2} \quad \text{and} \quad \left[\gamma^{\frac{1}{N-1}}\right]^{-1/2}$$

for translational and dilatational invariance, respectively.

These calculations are analogous to those in the original paper by Lipatov /1/ (For another version of this calculation see paper /8,9/). We write only the final expression (A reader interested in details is refered to papers /1,8/)

$$\mathrm{Im}\, G_\nu(\gamma) = c_\nu \exp\left[-A/\gamma^{\frac{1}{N-1}}\right]\cdot\left[\gamma^{\frac{1}{N-1}}\right]^{-\alpha/2}, \tag{14}$$

where $\alpha = D+\nu+1$ for logarithmic theories and $\alpha = D+\nu$ for superrenormalizable model, C_ν is the independent of γ function of other parameters.

Substituting expession (14) into eq. (5), according to (6,7) we have

$$G_\nu(g) = \frac{C_\nu}{\pi}\int_{-\infty}^{0}\frac{dz\,\exp\left[-A/(-z)^{\frac{1}{N-1}}\right]}{(z-g)(-z)^{\frac{\alpha}{2(N-1)}}}. \tag{15}$$

3. Discussion

i) Formula (15) just represents the desired result. For $N=2$ it coincides with eq.(3). Being expanded in powers of g it leads to the asymptotic series

$$G(g) \sim \frac{-C_\nu(N-1)}{\pi A^{\alpha/2}}\sum_k (-g)^k A^{-k(N-1)}\,[k(N-1)]!\,[k(N-1)]^{\frac{\alpha}{2}-1} \tag{16}$$

obtained for the logarithmic scalar theories in papers /1,8/ and for $\varphi^4_{(3)}$ model in paper /8/.

ii) By changing the integration variable, formula (15) can be reduced to the form close to eq. (7) from paper /4/ obtained there by Borel summation of the series (16). Thus, treating of the functional integral as a whole validates the Borel summation of asymptotic series of perturbation theory. This gives additional arguments (to those of refs. /10, 11/) in favour of this summation. Of course, while examining formulas of the type (15), one should take into account that they are correct only for $g \ll 1$ and give mainly qualitative information about the structure of singularity at the origin. They cannot be source of the quantitative information about the Green functions, for $g \sim 1$. (see the discussion of this question in /4/).

iii) There is a view point (see, for example /12/ that a functional integral par definition is only a compact from of perturbation series. The Langer procedure of approximate evaluation of the functional integral by the saddle-point method, as we have verified, gives something more than the divergent perturbation series. It gives the sum of this series, i.e., contains the way of summation.

iv) A characteristic feature of formula (15) is its universality. The cut along the negative axis exists for all (renormalizable and superrenormalizable) nonlinear scalar interactions of the type (1) for $n = 2N$.

v) The physically important consequence of the mentioned universality consists in the independence of nonanalyticity at the origin of the character and the very fact of existence of UV divergences in perturbation theory.

This fact, in its turn, allows one to hope that for properly formulated nonrenormalizable models it is possbile to carry out the analysis by the functional saddle-point method, and to get the formula analogous to (15).

The authors express their gratitude to E.B.Bogomolny, A.A.Vladimirov, O.I.Zavjalov, L.N.Lipatov and V.G.Makhankov for useful discussions.

References:

1. L.N.Lipatov. Soviet JETP, 72, 411 (1977).
2. J.S.Langer. Annals of Physics 41, 108-157 (1967).
3. E.B.Bogomolny. Phys.Lett. 67B, 193 (1977).
4. D.V.Shirkov. JINR, E2-10217, Dubna, 1977, submitted to Lettere al Nuovo Cimento , 18, 452 (1977).
5. E.P.Zhidkov, V.P.Shirikov. Zh.Vytchisl. Matem. Matem.Fiz., 4, 804 (1964).
6. V.B.Glasko, Ph.Leruste, Ya.P.Terletsky, S.Ph.Shushurin. Soviet JETP, 35, 452 (1968).
7. D.L.T.Anderson, G.N.Derrick. J.Math.Phys. 11, 1336 (1970).
8. E.Brezin, J.C.Le Guillon, J.Zinn-Justin. Phys.Rev. D15, 1544 (1977).
9. G 't Hooft. Phys.Rev. D14, 3432 (1976).

10. J.Glimm, A.Jaffe, T.Spencer. Proceedings of the International School of Mathematical Physics "Ettore Majorana", Erice, Italy (1973). edited by G.Velo and A.Wightman (Springer, New York, 1973) Vol. 25, pp. 199-242.
11. J.P.Eckmann, J.Magnen, R.Sénéor . Comm.Math.Phys., 39, 251 (1975).
12. N.N.Bogoliubov, D.V.Shirkov. Third Ed.Moscow 1976. Interscience Pub. 1959. "Introduction to the Theory of Quantized Fields".

Received by Publishing Department
on June 7, 1977.