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M.Bednář, P.Kolář

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M.Bednář,* P.Kolář

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*Institute of Physics of the Czechoslovak Academy of Sciences, Prague, Czechoslovakia.

Беднарж М., Коларж П. E2 - 10461

О проекционных операторах релятивистского спина

В формализме Рарита-Швингера при помощи вектора Паули-Лубанского рассматриваются проекционные операторы релятивистского спина. Показано, что такой подход эквивалентен стандартному подходу и позволяет получить рекуррентные формулы для проекционных операторов, полезные в практических применениях.

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On Relativistic Spin Projection Operators

In the Rarita-Schwinger formalism, relativistic spin projection operators are discussed by means of the Pauli-Lubanski four-vector. It is shown that this approach is equivalent to the conventional one, but moreover, it enables one to derive recurrence relations for the spin projection operators. These relations are useful for practical applications.

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1. INTRODUCTION

There are many possibilities for describing free relativistic particles of higher spins (Bargmann and Wigner [1], Rarita and Schwinger [5], Weinberg [6]).

The Rarita-Schwinger formalism is most frequently used in various applications. In this formalism a relativistic particle of integer or half-integer spin j is described by a tensor function of rank j or $j-1/2$, respectively. (Note that the components of the latter tensor are bispinors.)

The method of spin projection operators (SPO) yields a very useful tool for constructing higher spin wave functions. In the Rarita-Schwinger formalism the explicit expressions for these operators were obtained previously using the symmetry properties of the corresponding spin wave functions (Behrends and Fronsdal [2], Fronsdal [3]). The SPO for integer spins j, $P^{(j)}$, are defined in this approach by means of the following properties:

$$P^{(j)}_{\alpha_1\ldots\alpha_n\ldots\alpha_m\ldots\alpha_j;\beta_1\ldots\beta_j}=P^{(j)}_{\alpha_1\ldots\alpha_m\ldots\alpha_n\ldots\alpha_j;\beta_1\ldots\beta_j},$$
$$p^{\mu}P^{(j)}_{\mu\alpha_2\ldots;\beta_1\ldots}=0,$$
$$g^{\mu\nu}P^{(j)}_{\mu\nu\ldots;\ldots}=0,\qquad (1)$$
$$P^{(j)}\cdot P^{(j)}=P^{(j)},$$

where $g^{\mu\nu}=\mathrm{diag}\ (1,-1,-1,-1)$ is the metric tensor and $\mathbf{p}$ denotes the four-momentum of a free particle with spin j. For half-integer spin $s=j+1/2$ there is the additional condition

$$\gamma^{\mu}P^{(s)}_{\mu\alpha_2\ldots\alpha_j;\beta_1\ldots\beta_j}=0, \qquad (2)$$

where γ^{μ} are the Dirac matrices. It was shown by Fronsdal /3/ that these conditions determine uniquely the corresponding SPO.

In this paper we present another method for the explicit construction of SPO. In our approach the SPO are defined in terms of the Pauli-Lubanski spin operator (see, e.g., Gasiorowicz /4/, p. 69). It enables us to obtain useful relations for SPO.

2. PROJECTION OPERATORS FOR INTEGER SPINS

The Pauli-Lubanski four-vector operator $W^{\rho}_{(s)}$, representing the relativistic spin operator of a particle with spin s, four-momentum p and mass m, is defined by

$$W^{\rho}_{(s)}(p) = -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho} \quad M_{(s)\mu\nu} \; p_{\sigma} \;, \qquad (3)$$

where $M_{(s)\mu\nu}$ are the generators of the homogeneous Lorentz group acting in the space of spin states.

Let V_p be the space of spin-1 vector wave functions $e(p) = \{e_{\mu}(p)\}$ satisfying the subsidiary condition

$$p^{\mu} e_{\mu}(p) = 0 \,.$$

The generators of the Lorentz group are represented on V_p by the matrices $N_{\mu\nu}$ with the elements

$$(N_{\mu\nu})_{\alpha\beta} = -i(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \,.$$

Then, in the space

$$\underbrace{V_p \otimes V_p \otimes \ldots \otimes V_p}_{j \text{ times}} \qquad (4)$$

the matrix elements $(M_{(j)\mu\nu})_{\alpha_1 \ldots \alpha_j ; \beta_1 \ldots \beta_j}$ of the Lorentz group generators $M_{(j)\mu\nu}$ are given by

$$\begin{aligned}(M_{(j)\mu\nu})_{\alpha_1 \ldots \alpha_j ; \beta_1 \ldots \beta_j} = {} & (N_{\mu\nu})_{\alpha_1\beta_1} g_{\alpha_2\beta_2} \cdots \; g_{\alpha_j\beta_j} + \\ & + g_{\alpha_1\beta_1} (N_{\mu\nu})_{\alpha_2\beta_2} \cdots \; g_{\alpha_j\beta_j} + \qquad (5) \\ & \quad \vdots \qquad \vdots \\ & + g_{\alpha_1\beta_1} \cdots \; \cdots g_{\alpha_{j-1}\beta_{j-1}} (N_{\mu\nu})_{\alpha_j\beta_j}\end{aligned}$$

In the space, which is the direct product of space (4) and the Dirac bispinor space, the Lorentz group generators $M_{(s)\mu\nu}$ are given by

$$(M_{(s)\mu\nu})_{\alpha_1\ldots\alpha_j a;\beta_1\ldots\beta_j b} = (M_{(j)\mu\nu})_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j}\delta_{ab} + g_{\alpha_1\beta_1} g_{\alpha_2\beta_2} \cdots g_{\alpha_j\beta_j} (\Sigma_{\mu\nu})_{ab} , \qquad (6)$$

where $s = j + 1/2$, and δ_{ab} is the Kronecker symbol. The Lorentz group generators $\Sigma_{\mu\nu}$ in the bispinor space are expressed in terms of Dirac's γ^{μ}-matrices as

$$\Sigma_{\mu\nu} = \frac{1}{4i}[\gamma_\mu , \gamma_\nu].$$

We shall discuss now the SPO for a particle of integer spin j, mass m and momentum p. We define

$$P^{(j)}(p) \equiv \prod_{\ell=0}^{j-1} \left[\frac{(-W^2_{(j)}(p)/m^2) - \ell(\ell + 1)}{j(j+1) - \ell(\ell + 1)} \right], \qquad (7)$$

where $W^2_{(j)} = W^{\rho}_{(j)} W_{(j)\rho}$. From (7) it follows immediately that $P^{(j)}(p)$ projects space (4) on the subspace characterized by the maximal spin j.

We now prove the equivalence of (7) and (1). For this purpose it is convenient to rewrite (7) in the equivalent form:

$$P^{(j)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = \Big[\prod_{\ell=j-2}^{j-1} \frac{(-W^2_{(j)}/m^2) - \ell(\ell+1)}{j(j+1)-\ell(\ell+1)}\Big]_{\alpha_1\ldots\alpha_j;\beta_1\gamma_2\ldots\gamma_j} \times$$

$$\times P^{(j-1)\gamma_2\ldots\gamma_j;}{}_{\beta_2\ldots\beta_j} \tag{8}$$

valid for $j>1$. To prove relations (1) we shall use induction on j. It is readily verified that relations (1) are valid for the operator

$$P^{(2)}_{\alpha\beta;\gamma\delta} = \frac{1}{2}Q_{\alpha\gamma}Q_{\beta\delta} + \frac{1}{2}Q_{\alpha\delta}Q_{\beta\gamma} - \frac{1}{3}Q_{\alpha\beta}Q_{\gamma\delta},$$

where $Q_{\alpha\beta} = g_{\alpha\beta} - \frac{p_\alpha p_\beta}{m^2}$.

We suppose the relations true for $j-1$ and prove them true for j. However, if relations (1) are valid for $j-1$, then it follows from (3) and (5) that (8) can be rewritten as

$$P^{(j)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = \frac{1}{j}\sum_{n=1}^{j} Q_{\alpha_n\beta_1} P^{(j-1)}_{\alpha_1\ldots\alpha_{n-1}\alpha_{n+1}\ldots\alpha_j;\beta_2\ldots\beta_j} +$$

$$+\frac{2}{j(1-2j)}\sum_{\substack{n=1\\n<m}}^{j}\sum_{m=1}^{j} Q_{\alpha_n\alpha_m} P^{(j-1)}_{\alpha_1\ldots\alpha_{n-1}\alpha_{n+1}\ldots\alpha_{m-1}\alpha_{m+1}\ldots\alpha_j\beta_1;\beta_2\ldots\beta_j} \tag{9}$$

for $j>1$, and $P^{(1)}_{\alpha;\beta} = Q_{\alpha\beta}$. Recurrence relation (9) readily implies properties (1) for j, and the equivalence of (1) and (7) is proved. Furthermore, from (9) we obtain the following properties of $P^{(j)}$:

$$P^{(j)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = P^{(j)}_{\beta_1\ldots\beta_j;\alpha_1\ldots\alpha_j} ,$$

$$g^{\alpha\beta} P^{(j)}_{\alpha\ldots;\beta\ldots} = \frac{2j+1}{2j-1} P^{(j-1)}_{\ldots;\ldots} ,$$

$$(-W^2_{(j)}/m^2) P^{(j)} = j(j+1) P^{(j)} .$$

Moreover, from (9) one can find the Behrends-Fronsdal formula

$$P^{(j)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = \left(\frac{1}{j!}\right)^2 \sum_{\substack{p(\alpha)\\ p(\beta)}} \Big(\prod_{\ell=1}^{j} Q_{\alpha_\ell\beta_\ell} + a^{(j)}_1 Q_{\alpha_1\alpha_2} Q_{\beta_1\beta_2} \prod_{\ell=3}^{j} Q_{\alpha_\ell\beta_\ell} +$$

$$+\ldots+ \begin{cases} a^{(j)}_{j/2} Q_{\alpha_1\alpha_2} Q_{\beta_1\beta_2} \ldots Q_{\alpha_{j-1}\alpha_j} Q_{\beta_{j-1}\beta_j} \Big), & j \text{ even} \\ a^{(j)}_{(j-1)/2} Q_{\alpha_1\alpha_2} \ldots\ldots Q_{\alpha_{j-2}\alpha_{j-1}} Q_{\alpha_j\beta_j} \Big), & j \text{ odd} , \end{cases} \qquad (10)$$

where

$$a^{(j)}_\ell = (-1)^\ell \frac{\binom{j}{\ell}\binom{j}{2\ell}}{\binom{2j}{2\ell}} .$$

The sum in (10) is taken over all permutations of α and β (Behrends and Fronsdal [2], Fronsdal [3]).

3. PROJECTION OPERATORS FOR HALF-INTEGER SPINS

For a half-integer spin $s = j + 1/2$ we define the projection operator by

$$P^{(s)}(p) \equiv \frac{(-W^2_{(s)}(p)/m^2) - s(s-1)}{s(s+1) - s(s-1)} P^{(j)}(p)\,. \tag{11}$$

Repeating the considerations of Sec. 2 we obtain from (3) and (6) the following relation:

$$P^{(s)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = P^{(j)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} -$$

$$- \frac{1}{2j+1}\sum_{k=1}^{j} Q_{\alpha_k\delta}\,\gamma^{\delta}\gamma^{\rho} P^{(j)}_{\alpha_1\ldots\alpha_{k-1}\alpha_{k+1}\ldots\alpha_j\rho;\beta_1\ldots\beta_j} \tag{12}$$

for $s > \frac{3}{2}$, and

$$P^{(3/2)}_{\alpha;\beta} = g_{\alpha\beta} - \frac{1}{3}\gamma_\alpha\gamma_\beta - \frac{2}{3m^2}p_\alpha p_\beta - \frac{1}{3m^2}(p_\alpha\gamma_\beta - p_\beta\gamma_\alpha)(\gamma p).$$

Besides (1) and (2) we obtain from (12) the following relations:

$$P^{(s)}_{\ldots\alpha_n\ldots\alpha_m\ldots;\beta_1\ldots\beta_j} = P^{(s)}_{\ldots\alpha_m\ldots\alpha_n\ldots;\beta_1\ldots\beta_j}$$

$$P^{(s)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = P^{(s)}_{\beta_1\ldots\beta_j;\alpha_1\ldots\alpha_j},$$

$$[(\gamma\cdot p), P^{(s)}] = 0$$

$$g^{\alpha\beta} P^{(s)}_{\alpha\ldots;\beta\ldots} = \frac{j+1}{2j+1} \gamma^{\alpha} \gamma^{\beta} P^{(j)}_{\alpha\ldots;\beta\ldots.}$$

$$-(W^{2}_{(s)}/m^{2}) P^{(s)} = s(s+1) P^{(s)} .$$

Likewise the Behrends-Fronsdal relation for $P^{(s)}$, i.e.,

$$P^{(s)}_{\alpha_1\ldots\alpha_j;\beta_1\ldots\beta_j} = \frac{j+1}{2j+3} \gamma^{\alpha} \gamma^{\beta} P^{(j+1)}_{\alpha\alpha_1\ldots\alpha_j;\beta\beta_1\ldots\beta_j} \qquad (13)$$

can be obtained from (12).

4. CONCLUDING REMARKS

We now turn to a comparison between our approach and the one presented by Behrends and Fronsdal. While Behrends and Fronsdal obtained the explicit expressions for SPO as a consequence of properties (1) and (2), in our approach these properties follow from definitions (7) and (11). Although our relations (9) and (12) do not exhibit the symmetry properties so directly like the Behrends-Fronsdal formulae, they can be useful in practical applications. In particular, formula (12) expresses the SPO for half-integer spin $P^{(s)}$ in terms of $P^{(j)}$ for integer spin $j = s - 1/2$, in the most effective way. Indeed, if $P^{(j)}$ contains only independent terms, then we obtain immediately, from (12), the corresponding $P^{(s)}$ with the same property.

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