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RELATIVISTIC STRING WITH MASSIVE ENDS

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Релятивистская струна с массами на концах

Рассматривается классическая и квантовая теория релятивистской струны с точечными массами на концах. Найдены решения динамических уравнений для определенного класса движений, когда параметр временной эволюции τ пропорционален собственному времени концов струны. Решение линейной краевой задачи в этом случае дается рядами Фурье. Ограничения на амплитуды Фурье, следующие из условий ортогональной калибровки, существенно отличаются от условий Виразоро для свободной струны. Найдены масса, импульс и угловой момент системы.

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Relativistic String with Massive Ends

The classical and quantum theory is discussed for the relativistic string with point masses at its ends. The dynamical equations are solved for the class of motions when the time evolution parameter τ is proportional to proper times of the string ends. The solution to the linear boundary value problem is given by the Fourier series. Constraints on the Fourier amplitudes resulting from the orthogonal gauge differ essentially from the Virasoro conditions for the free string. The string mass, momentum and angular momentum are found.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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Introduction

The study of the relativistic string is important for elementary particle physics for two reasons. First, quantization of the free string gives the mass spectrum of the reasonance states in dual models that allows this object to be considered as a dynamical basis of the dual-reasonance approach /1/. Second, the relativistic string can bind quarks inside hadrons /2,3/. For this purpose the relativistic string should be considered with masses at the ends. This essentially complicates the mathematical problem compared to the massless string due to the nonlinear boundary conditions. Only the simplest classical motions /4/ have been investigated for the massive string. In the case of two-dimensional space-time the most successful study has been made by authors of ref. /2/. A relativistic string with continuous mass distribution has been proposed in ref. /5/.

In this paper we consider the relativistic string with point masses attached to its ends in such a parametrization when the time evolution variable τ is proportional to the proper times of massive string ends. This gives a possibility to linearize the boundary conditions and the boundary value problem can be solved by using the Fourier series. The parametrization under consideration allows one to describe only a certain class of motions of this system . We find constraints on the Fourier amplitudes which follow from the orthogonal gauge. These constraints do not coincide with the well known

Virasoro conditions for the free string. Also the mass, momentum, and angular momentum of the system are obtained.

2. Lagrangian, Equations of Motion and Boundary Conditions

The action for the relativistic string with point masses attached to the ends is taken as follows

$$S=-\gamma\int_{\tau_1}^{\tau_2}d\tau\int_{\sigma_1(\tau)}^{\sigma_2(\tau)}d\sigma\sqrt{(\dot{x}x')^2-\dot{x}^2x'^2}-\sum_{i=1}^{2}m_i\int_{\tau_1}^{\tau_2}\sqrt{\left(\frac{dx_\nu(\sigma_i(\tau),\tau)}{d\tau}\right)^2}d\tau, \quad (1)$$

where γ is the constant of l^{-2} -dimensions, m_1, m_2 are the masses of endpoints, $\dot{x}_\mu=\partial x_\mu(\sigma,\tau)/\partial\tau$, $x'_\nu = \partial x_\nu(\sigma,\tau)/\partial\sigma$, $dx_\nu(\sigma_i(\tau),\tau)/d\tau=\dot{x}_\nu(\sigma_i(\tau),\tau)+x'_\nu(\sigma_i(\tau),\tau)\dot{\sigma}_i(\tau)$.
The functions $\sigma_i(\tau), (i=1,2)$ describe the motions of ends of the string in coordinates σ, τ.

The equations of motion and subsidiary conditions are the same as in the free case

$$\ddot{x}_\nu(\sigma,\tau)-x''_\nu(\sigma,\tau)=0, \quad (2)$$

$$\dot{x}^2+x'^2=0, \quad \dot{x}x'=0 \quad (3)$$

but the boundary conditions are nonlinear

$$\begin{aligned}&\mu_1\frac{d}{d\tau}\left(\frac{\dot{x}_\nu+x'_\nu\dot{\sigma}}{\sqrt{\dot{x}^2(1-\dot{\sigma}^2)}}\right)=x'_\nu+\dot{x}_\nu\dot{\sigma}, \quad \sigma=\sigma_1(\tau),\\ &\mu_2\frac{d}{d\tau}\left(\frac{\dot{x}_\nu+x'_\nu\dot{\sigma}}{\sqrt{\dot{x}^2(1-\dot{\sigma}^2)}}\right)=-x'_\nu-\dot{x}_\nu\dot{\sigma}, \quad \sigma=\sigma_2(\tau),\\ &\qquad \mu_i=m_i/\gamma, \ i=1,2\end{aligned} \quad (4)$$

and are the main problem to be solved here.

The vector $x_\nu(\sigma,\tau)$ obeys the D'Alambert equation (2) with the general solution

$$x_\nu(\sigma,\tau)=\psi_{1\nu}(\sigma+\tau)+\psi_{2\nu}(\sigma-\tau)=\psi_{1\nu}(\alpha)+\psi_{2\nu}(\beta), \quad (5)$$

where $\alpha=\sigma+\tau$, $\beta=\sigma-\tau$. Substituting (5) into (3) and (4) gives the constraints on ψ_1 and ψ_2

$$\psi'^2_1(\alpha)=\psi'^2_2(\beta) \quad (6)$$

and the boundary conditions

$$\begin{aligned}&\mu_1\frac{d}{d\tau}\left(\frac{\psi'_{1\nu}(\alpha)\dot{\alpha}+\psi'_{2\nu}(\beta)\dot{\beta}}{\sqrt{2\psi'_{1\lambda}(\alpha)\psi'_{2\lambda}(\beta)\dot{\alpha}\dot{\beta}}}\right)=\psi'_{1\nu}(\alpha)\dot{\alpha}-\psi'_{2\nu}(\beta)\dot{\beta},\\ &\alpha=\alpha_1(\tau)=\sigma_1(\tau)+\tau, \quad \beta=\beta_1(\tau)=\sigma_1(\tau)-\tau,\\ &\mu_2\frac{d}{d\tau}\left(\frac{\psi'_{1\nu}(\alpha)\dot{\alpha}+\psi'_{2\nu}(\beta)\dot{\beta}}{\sqrt{2\psi'_{1\lambda}(\alpha)\psi'_{2\lambda}(\beta)\dot{\alpha}\dot{\beta}}}\right)=-\psi'_{1\nu}(\alpha)\dot{\alpha}+\psi'_{2\nu}(\beta)\dot{\beta},\\ &\alpha=\alpha_2(\tau)=\sigma_2(\tau)+\tau, \quad \beta=\beta_2(\tau)=\sigma_2(\tau)-\tau.\end{aligned} \quad (7)$$

The prime over ψ_i , $(i=1,2)$ denotes the differentiation with respect to argument.

From eqs. (5),(6) and (7) it follows that the parameters α and β can be changed by new variables such that

$$\tilde{\alpha}=f_1(\alpha), \quad \tilde{\beta}=f_2(\beta). \quad (8)$$

The boundary conditions (7) can be integrated over τ

$$\begin{aligned}&\mu_1\frac{\dot{x}_{1\nu}+x'_\nu\dot{\sigma}_1}{\sqrt{\dot{x}^2(1-\dot{\sigma}_1^2)}}=\psi_{1\nu}(\alpha_1)-\psi_{2\nu}(\beta_1)+C_{1\nu}\equiv\mathcal{E}_{1\nu}(\alpha_1,\beta_1),\\ &\mu_2\frac{\dot{x}_{1\nu}+x'_\nu\dot{\sigma}_2}{\sqrt{\dot{x}^2(1-\dot{\sigma}_2^2)}}=-\psi_{1\nu}(\alpha_2)+\psi_{2\nu}(\beta_2)-C_{2\nu}\equiv-\mathcal{E}_{2\nu}(\alpha_2,\beta_2).\end{aligned} \quad (9)$$

These equations being squared give

$$\mathcal{E}^2_i(\alpha_i,\beta_i)=\mu_i^2, \quad i=1,2. \quad (10)$$

Next, we prove that $\dot{\sigma}_i(\tau)=0$. For this purpose we project conditions (9) onto $x'_\nu(\sigma_i(\tau),\tau)$ and because of

$x'_\nu(\sigma_i(\tau),\tau)=\dot{\mathcal{E}}_{i\nu}(\alpha_i,\beta_i)$ we obtain

$$\mu_i\frac{x'^2\dot{\sigma}_i}{\sqrt{\dot{x}^2(1-\dot{\sigma}_i^2)}}=\dot{\mathcal{E}}_{i\nu}(\alpha_i,\beta_i)\cdot\mathcal{E}'^\nu_i(\alpha_i,\beta_i)=\frac{1}{2}\frac{\partial}{\partial\tau}\mathcal{E}_i^2(\alpha_i,\beta_i)=0,\quad i=1,2.$$

The latter equality follows from (10). Thus we arrive at the equality $\dot{\sigma}_i=0,\ (i=1,2)$ and the string ends can be defined by setting $\sigma_1(\tau)=0,\quad \sigma_2(\tau)=\ell$. The boundary conditions (4) are now simplified but still nonlinear in $\dot{x}_\mu$

$$\mu_1\frac{d}{d\tau}\left(\frac{\dot{x}_\nu}{\sqrt{\dot{x}^2}}\right)=x'_\nu,\quad \sigma=0,$$
$$\mu_2\frac{d}{d\tau}\left(\frac{\dot{x}_\nu}{\sqrt{\dot{x}^2}}\right)=-x'_\nu,\quad \sigma=\ell. \tag{11}$$

Next, let us change the variables (8) so that in the new variables $\tilde{\sigma}$, $\tilde{\tau}$ the string ends be again described by equations $\tilde{\sigma}_1=0$ and $\tilde{\sigma}_2=\ell$. This condition is fulfilled by the following change

$$\tilde{\sigma}=\frac{1}{2}[f(\sigma+\tau)-f(\tau-\sigma)],\ \tilde{\tau}=\frac{1}{2}[f(\sigma+\tau)+f(\tau-\sigma)],$$

where $f(\tau+\ell)-f(\tau-\ell)=2\ell$ and, consequently, $f'(\tau)$ is a periodic function with period 2ℓ $f'(\tau)=f'(\tau+2\ell)$.

If one introduces the new Lorentz vector $\tilde{x}_\nu(\tilde{\sigma},\tilde{\tau})\equiv x_\nu(\sigma(\tilde{\sigma}),\tau(\tilde{\tau}))$ then $\dot{x}_\nu^2(\sigma_i,\tau),\ (i=1,2)$ may be expressed in terms of $\tilde{x}_\nu$ as follows:

$$\dot{x}_\nu^2(0,\tau)=\left[\frac{\partial}{\partial\tilde{\tau}}\tilde{x}_\nu(0,\tilde{\tau})\right]^2 f'^2(\tau),$$
$$\dot{x}_\nu^2(\ell,\tau)=\left[\frac{\partial}{\partial\tilde{\tau}}\tilde{x}_\nu(\ell,\tilde{\tau})\right]^2 f'^2(\ell+\tau). \tag{12}$$

The function $f(\tau)$ is to be chosen so that the new parameter $\tilde{\tau}$ be proportional to the proper times of the string ends

$$\left[\frac{\partial}{\partial\tilde{\tau}}\tilde{x}_\nu(\tilde{\sigma}_i,\tilde{\tau})\right]^2=\frac{C_i^2}{m_i^2},\ i=1,2. \tag{13}$$

The masses are introduced here to make arbitrary constants C_i dimensionless. Equations (12) and (13) give

$$f'^2(\tau)=\frac{m_1^2}{C_1^2}\dot{x}_\nu^2(0,\tau),\quad f'^2(\ell+\tau)=\frac{m_2^2}{C_2^2}\dot{x}_\nu^2(\ell,\tau). \tag{14}$$

Equations (14) define function $f(\tau)$ in a consistent way provided that the following conditions

$$\left(\frac{m_1}{C_1}\right)^2\dot{x}_\nu^2(0,\tau+\ell)=\left(\frac{m_2}{C_2}\right)^2\dot{x}_\nu^2(\ell,\tau),$$
$$\dot{x}_\nu^2(\sigma_i,\tau)=\dot{x}_\nu^2(\sigma_i,\tau+2\ell),\quad \sigma_i=0,\ell$$

hold for $\dot{x}_\nu^2(\sigma,\tau)$ at the string ends.

Thus, only this class of motions of the massive string enables one to introduce the time evolution parameter $\tilde{\tau}$ proportional to the proper times of the string ends. In this case, Eqs. (11) under the condition (13) become linear

$$\ddot{x}_\nu(0,\tau)=q_1 x'_\nu(0,\tau),$$
$$\ddot{x}_\nu(\ell,\tau)=-q_2 x'_\nu(\ell,\tau), \tag{15}$$

where

$$q_i=\frac{C_i}{m_i\mu_i}=\frac{C_i\gamma}{m_i^2},\ i=1,2.$$

Note that the linear boundary conditions (15) can be derived directly if the action for massive end points is taken in the form proposed by Fock[6]

$$S_m=-\frac{m^2}{2}\int_{\tau_1}^{\tau_2}\dot{x}^2(\tau)d\tau. \tag{16}$$

For the action (16) to be equivalent to the conventional one $S_m=-m\int_{\tau_1}^{\tau_2}d\tau\sqrt{\dot{x}^2}$ it is necessary to impose conditions (13) on $\dot{x}^2$.

3. Solution to the Boundary Value Problem

To find solutions to eqs.(2) obeying the boundary conditions (15) we apply the separation of variables

$$x_\mu(\sigma,\tau) = \varphi_\mu(\tau)\, u(\sigma).$$

Substituting this x_μ into (2) and (15) gives the following boundary value problem for function $u(\sigma)$

$$\begin{aligned} &u''(\sigma) + \omega^2 u(\sigma) = 0, \\ &\omega^2 u(0) = -q_1 u'(0), \\ &\omega^2 u(l) = q_2 u'(l). \end{aligned} \tag{17}$$

Further we put $m_1 = m_2 = m$ and $C_1 = C_2 = C$, hence $q_1 = q_2 = q$, as well.

Solutions to the problem (17) are of the form

$$u_n(\sigma) = N_n \left[\cos(\omega_n \sigma) + \frac{\omega_n}{q} \sin(\omega_n \sigma) \right], \tag{18}$$

where N_n are the normalization constants, ω_n are the roots of the transcendental equation

$$tg(\omega_n l) = \frac{2\omega_n q}{\omega_n^2 - q^2}. \tag{19}$$

ω_n are symmetrical with respect to zero therefore these may be numbered so that $\omega_0 = 0$, $\omega_{-n} = -\omega_n$, $n = \pm 1, \pm 2, \ldots$. The functions (18) then obey the conditions $u_n(\sigma) = u_{-n}(\sigma)$.

Equation (19) is equivalent to the two following ones

$$tg\left(\frac{\omega_n l}{2}\right) = -\frac{\omega_n}{q}, \quad \text{(n even)};$$

$$ctg\left(\frac{\omega_n l}{2}\right) = \frac{\omega_n}{q}, \quad \text{(n odd)}.$$

Similar equations but different in sign have been derived in paper[5] for the massive relativistic string in the framework of another approach.

The eigenfunctions (18) of the boundary value problem (17) obey the orthogonality condition with weight[7]

$$\int_0^l d\sigma\, u_n(\sigma) u_m(\sigma) + \frac{1}{q} u_n(0) u_m(0) + \frac{1}{q} u_n(l) u_m(l) = \delta_{nm}. \tag{20}$$

It is convenient to introduce the weight function

$$\xi(\sigma) = 1 + \frac{1}{q}[\delta(\sigma) + \delta(\sigma - l)],$$

in terms of which the orthogonality condition (20) is written as follows

$$\int_0^l \xi(\sigma) u_n(\sigma) u_m(\sigma) d\sigma = \delta_{nm}.$$

The normalization constants N_n are

$$N_n^2 = \left\{ \frac{l}{2}\left(1 + \frac{\omega_n^2}{q^2}\right) + \frac{1}{q} \right\}^{-1},$$

$$N_0^2 = \left(l + \frac{2}{q}\right)^{-1}.$$

Further we need the condition of completeness of the system $u_n(\sigma)$

$$\sum_{n=0}^{\infty} u_n(\sigma) u_n(\sigma') \xi(\sigma') = \delta(\sigma, \sigma'). \tag{21}$$

The function $\delta(\sigma, \sigma')$ is defined by the requirement

$$\int_0^l d\sigma' \delta(\sigma, \sigma') f(\sigma') = f(\sigma),$$

where $f(\sigma)$ is an arbitrary smooth function given in the interval $0 \le \sigma \le l$.

Because of $\varphi_{\mu n} = A_{\mu n} e^{i\omega_n \tau}$, $x_\mu(\sigma, \tau)$ is expanded as follows

$$x_\mu(\sigma,\tau) = Q_\mu + \frac{q}{2 + ql} \frac{P_\mu \tau}{\gamma} + \frac{i}{\sqrt{2\gamma}} \sum_{n \neq 0} \frac{e^{-i\omega_n \tau}}{\omega_n} \alpha_{n\mu} u_n(\sigma) =$$

$$= Q_\mu + \frac{q}{2 + ql} \frac{P_\mu \tau}{\gamma} + \frac{i}{\sqrt{\gamma}} \sum_{n>0} \frac{1}{\sqrt{2\omega_n}} \left(a_{n\mu} e^{-i\omega_n \tau} + a_{n\mu}^+ e^{+i\omega_n \tau} \right) u_n(\sigma), \tag{22}$$

where

$$\alpha_n = \sqrt{\omega_n}\, a_n, \quad \alpha_{-n} = \alpha_n^+ = \sqrt{\omega_n}\, a_n^+, \quad n > 0.$$

For $q \to \infty$, $\omega_n \to \frac{n\pi}{l}$, $u_n(\sigma) \to \sqrt{2/l}\, \cos(n\pi\sigma/l)$ and solution (22) turns into the solution to the free string.

The subsidiary conditions (3) result in the constraints on amplitudes α_n:

$$\alpha_n \alpha_m = 0 \qquad \text{at } n \neq -m;\ n,m=\pm 1, \pm 2, \ldots \tag{23}$$

$$\alpha_n P = 0,\ n \neq 0, \tag{24}$$

$$P^2 = -\frac{\gamma}{2}\left(\frac{2}{q}+\ell\right)^2 \sum_{n\neq 0} N_n^2 \left(1+\frac{\omega_n^2}{q^2}\right)\alpha_{-n}\alpha_n . \tag{25}$$

In addition it is necessary that at the string ends the conditions (13) hold which leads to the equality

$$\frac{1}{2\gamma^2}\sum_{n\neq 0} N_n^2 \frac{\omega_n^2}{q^2}\alpha_{-n}\alpha_n = -\left(\frac{C}{m}\right)^2 . \tag{26}$$

By using (26) constraint (25) may be simplified as follows

$$P^2 = -\frac{\gamma}{2}\left(\frac{2}{q}+\ell\right)^2 \sum_{n\neq 0} N_n^2 |\alpha_n|^2 + 4m^2\left(1+\frac{q\ell}{2}\right)^2 .$$

The conserved total momentum of the string with massive ends is of the form

$$\Pi_\mu = \frac{\gamma}{q}[\dot{x}_\mu(0,\tau)+\dot{x}_\mu(\ell,\tau)] + \gamma\int_0^\ell \dot{x}_\mu(\sigma,\tau)d\sigma = \int_0^\ell \pi_\mu(\sigma,\tau)\,d\sigma , \tag{27}$$

where $\pi_\mu(\sigma,\tau) = \gamma\,\xi(\sigma)\,\dot{x}_\mu(\sigma,\tau)$ is the canonical momentum density. Inserting (22) into (27) gives

$$\Pi_\mu = P_\mu$$

For the squared mass of the string we have

$$M^2 = P^2 = -\frac{\gamma}{2}\left(\frac{2}{q}+\ell\right)^2 \sum_{n\neq 0} N_n^2 \alpha_{-n}\alpha_n + 4m^2\left(1+\frac{q\ell}{2}\right)^2 . \tag{28}$$

If the string does not vibrate then its squared mass differs, nevertheless, from $4m^2$:

$$M_0^2 = 4m^2\left(1+\frac{\ell q}{2}\right)^2,\ \alpha_n = 0,\ n\neq 0.$$

Proceeding from action (1) one may construct the conserved tensor of the angular momentum of the system under consideration

$$M_{\mu\nu} = \frac{\gamma}{q}[x_\mu(0,\tau)\dot{x}_\nu(0,\tau) - x_\nu(0,\tau)\dot{x}_\mu(0,\tau)] +$$
$$+ \frac{\gamma}{q}[x_\mu(\ell,\tau)\dot{x}_\nu(\ell,\tau) - x_\nu(\ell,\tau)\dot{x}_\mu(\ell,\tau)] +$$
$$+ \gamma\int_0^\ell d\sigma[x_\mu(\sigma,\tau)\dot{x}_\nu(\sigma,\tau) - x_\nu(\sigma,\tau)\dot{x}_\mu(\sigma,\tau)] =$$
$$= \int_0^\ell d\sigma[x_\mu(\sigma,\tau)\pi_\nu(\sigma,\tau) - x_\nu(\sigma,\tau)\pi_\mu(\sigma,\tau)].$$

On substituting expansion (22) $M_{\mu\nu}$ takes the form

$$M_{\mu\nu} = \frac{1}{\ell}(Q_\mu P_\nu - P_\mu Q_\nu) - \frac{i}{2}\sum_{n\neq 0}\frac{1}{\omega_n}(\alpha_{-n\mu}\alpha_{n\nu} - \alpha_{-n\nu}\alpha_{n\mu}).$$

4. Transition to Quantum Theory

The obtained classical solutions for the string with massive ends allow a direct quantization of this system as in the case of the free string /1/. Indeed, the quantities α_n^+ and α_n may be considered as usual creation and annihilation operators acting in the Fock space

$$[\alpha_n, \alpha_m] = \omega_n \delta_{n+m,0} .$$

Using the condition of completeness (21) this gives the following commutators

$$[x_\nu(\sigma,\tau), \pi_\mu(\sigma',\tau)] = [x_\nu(\sigma,\tau), \xi(\sigma')\dot{x}_\mu(\sigma',\tau)] = -i g_{\mu\nu}\delta(\sigma,\sigma'),$$

$$[x_\nu(\sigma,\tau), x_\mu(\sigma',\tau)] = [\pi_\nu(\sigma,\tau), \pi_\mu(\sigma',\tau)] = 0.$$

The Hamiltonian of the system which yields the correct equations of motion may be chosen in the form

$$H = \frac{P^2}{2}\cdot\frac{q}{\gamma(2+q\ell)} + \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\alpha_n .$$

In the quantum theory constraints (23) and (24) on the normal modes α_n should be imposed as conditions on the physical state vectors $|\Phi\rangle$

$$L_{nm}|\Phi\rangle \equiv \alpha_n \alpha_m |\Phi\rangle = 0, \quad n,m>0, \tag{29}$$

$$G_n|\Phi\rangle \equiv \alpha_n P|\Phi\rangle = 0, \quad n>0. \tag{30}$$

Contrary to classical theory, it is enough to require that these constraints are valid only for $n, m > 0$ [8]. Then, obviously, the commutators between constraints will be zero and moreover L_{nm} and G_n will commute weakly [9] with the Hamiltonian H.

Following ref. [8], condition (30) may be used to construct the physical space of state vectors with positive definite norm in the center-of-momentum frame of the string where $\vec{P} = 0$. In this frame

$$\alpha_n P|\Phi\rangle = \alpha_n^0 P^0 |\Phi\rangle = 0, \quad n>0.$$

Assuming that $P^0|\Phi\rangle \neq 0$ we obtain

$$\alpha_n^0 |\Phi\rangle = 0, \quad n>0,$$

i.e. the time components of operators α_n which just produce the negative norms ("ghost" states) in fact are zero. Vectors of the Fock space in this system are constructed by the action on vacuum only of spatial components of the creation operators $\vec{a}_n^{+}$. In particular, formula (28) for the squared-mass operator of the system now takes the form

$$M^2 = \gamma\left(\frac{2}{q}+\ell\right)^2 \sum_{n=1}^{\infty} N_n^2 \omega_n \vec{a}_n^{+}\vec{a}_n + m_0^2,$$

where m_0^2 contains in addition to the classical term $M_0^2 = 4m^2\left(1+\frac{\ell q}{2}\right)^2$ the arbitrary constant arising in the transition to the normal product of $\vec{a}_n^{+}\vec{a}_n$. It is clear that for $n \to \infty$ and $\omega_n \to \frac{(n-1)\pi}{\ell}$ the spectrum of operator M^2 becomes equidistant.

The dynamical variables M^2, Π_μ, $M_{\mu\nu}$ in the case under consideration are expressed in the same form as in the theory of the free massless string, but the frequences ω_n do not equal $n\pi/\ell$ and the subsidiary conditions on the physical state vectors (29) essentially differ from the known Virasoro conditions [1].

Conclusion

Let us discuss in short the geometrical approach to the theory of a string with massive ends. The action of the free massless string is proportional to the area of its world sheet [1,10] so that it is invariant under the transformation of parameters of this sheet. Other invariants of the string world sheet may be considered, as well, e.g. its integral Gaussian curvature, and the string mass may be introduced into its action by means of this invariant

$$S_m = \frac{m^2}{\gamma} \int_{\tau_1}^{\tau_2}\int_{\sigma_1}^{\sigma_2} d\Sigma \, R(\dot{x}, x'), \tag{31}$$

where $R(\dot{x}, x')$ is the Gaussian curvature. The conclusion in paper [12], that a Lagrangian of this type will produce the equations of motion of higher than second order is not correct. In fact, this Lagrangian changes only the boundary conditions because action (31) can be transformed, by the Gauss-Bonnet theorem [11], to the contour integral

$$S_m = \frac{m^2}{\gamma}\int_{\tau_1}^{\tau_2} ds\, k_g(\sigma_1,\tau) - \frac{m^2}{\gamma}\int_{\tau_1}^{\tau_2} ds\, k_g(\sigma_2,\tau), \tag{32}$$

where k_g is the geodesic curvature of the world lines of the string ends. Mathematically, the boundary conditions following from (32) are more complicated than those considered above. However, physically action (32) is interesting since its variation minimizes not the length of world line of massive ends of the string but their geodesic curvature. For a free point these two requirements give the same equations of motion.

After completing this work it was pointed out to us that a similar model was considered in paper /13/. But these authors do not use the reparametrization invariant Lagrangian and as a consequence have not obtained the conditions (29) that in our case replace the Virasoro gauge conditions.

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