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ON N-ORDERING OF EXPONENTIALS OF BILINEAR FORMS

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ON N-ORDERING OF EXPONENTIALS OF BILINEAR FORMS

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Об N -упорядочивании экспонент от квадратичных форм

Для получения матричных элементов от бозонного преобразования Боголюбова между состояниями Фока применено N-упорядочивание при помощи формулы Бекера-Кемпбелла-Хаусдорфа для операторов сдвига.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Polubarinov I.V. E2 - 10349

On N-Ordering of Exponentials of Bilinear Forms

Matrix elements of the boson Bogolubov transformation between Fock states are obtained using N-ordering with the help of Baker-Campbell-Hausdorff formula for shift operators.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

Normal ordering of exponential functions of bilinear forms of field operators is of interest from different aspects, in particular, in connection with the Bogolubov transformation [1], which in an operator form has such a construction. The Bogolubov transformation has a decisive role in the theoretical understanding of phenomena of superfluidity and superconductivity. It is applied now in many other fields [2-4] as well. Recently Tanabe [5] and Rashid [6] have considered a problem of obtaining general matrix elements of this transformation between Fock states, using different approaches.

In this work we treat this problem as a problem of N-ordering. For the boson Bogolubov transformation the last problem can be solved (Sec. 2) by means of the Baker-Campbell-Hausdorff formula for a particular case, when one operator is a shift operator for another. Such an approach was used by Sack [7] for calculating matrix elements between Fock states for the Gaussian potential. He used, in fact, an ordering, which differs somewhat from the N-ordering.

For ordering of more general expressions a more direct way is possible (Sec. 3). In literature we can find many different approaches to N-ordering in the general case (see, e.g., [8,9]).

The N-ordering is of interest also for calculation of matrix elements between coherent states (see Sec. 4), in particular, for operators e^{-iHt} in quantum mechanics or $e^{-H/kT}$ in statistical mechanics. This way permits us also to interpret Rashid's generating functions.

2. N-ORDERING OF THE BOGOLUBOV TRANSFORMATION WITH THE HELP OF THE BAKER-CAMPBELL-HAUSDORFF FORMULA FOR SHIFT OPERATORS

Let us begin with the list of necessary theorems. Note that the consideration of this Sec. is mainly based on particular case (3.a), (3.b) of the Baker-Campbell-Hausdorff formula /7/

If A is the shift operator for B, i.e.,

$$[A, B] = c_1 B , \tag{1}$$

then

$$e^{xA} e^{yB} = e^{y e^{c_1 x} B} e^{xA} , \tag{2.a}$$

$$e^{yB} e^{xA} = e^{xA} e^{y e^{-c_1 x} B} , \tag{2.b}$$

$$e^{\alpha(A+\lambda B)} = e^{\alpha A} e^{\frac{\lambda}{c_1} B(1-e^{-\alpha c_1})} = \tag{3.a}$$

$$= e^{-\frac{\lambda}{c_1} B(1-e^{\alpha c_1})} e^{\alpha A} . \tag{3.b}$$

If operators C and D are such that

$$[C, D] = c_2 (C+D) \tag{4.a}$$

(note, that from (4.a) there follow

$$[C, C+D] = c_2 (C+D) , \quad [D, C+D] = -c_2 (C+D)), \tag{4.b,c}$$

then

$$e^{\beta(C+D)} = e^{c_2^{-1} C \ln(1+\beta c_2)} e^{c_2^{-1} D \ln(1+\beta c_2)} = \tag{5.a}$$

$$= e^{-c_2^{-1} D \ln(1-\beta c_2)} e^{-c_2^{-1} C \ln(1-\beta c_2)} , \tag{5.b}$$

$$e^{\gamma(C-D)} = e^{c_2^{-1} C \ln \frac{1}{2}(1+e^{2\gamma c_2})} e^{c_2^{-1} D \ln \frac{1}{2}(1+e^{-2\gamma c_2})} = \tag{6.a}$$

$$= e^{-c_2^{-1} D \ln \frac{1}{2}(1+e^{2\gamma c_2})} e^{-c_2^{-1} C \ln \frac{1}{2}(1+e^{-2\gamma c_2})} , \tag{6.b}$$

$$e^{\gamma C+\delta D} = e^{c_2^{-1} C \ln[(\gamma+\delta)^{-1}(\gamma e^{(\gamma-\delta)c_2} - \delta)]} e^{c_2^{-1} D \ln[(\gamma-\delta)^{-1}(\gamma - \delta e^{(\delta-\gamma)c_2})]} = \tag{7.a}$$

$$= e^{-c_2^{-1} D \ln[(\gamma-\delta)^{-1}(\gamma - \delta e^{(\gamma-\delta)c_2})]} e^{-c_2^{-1} C \ln[(\gamma-\delta)^{-1}(\gamma e^{(\delta-\gamma)c_2} - \delta)]} . \tag{7.b}$$

Here the quantities $c_1, x, y, \alpha, \lambda, c_2, \beta, \gamma$ and δ are c-numbers.

Proofs of these relations are simple. However we show them for completeness. The proof of eq. (2.a) is

$$e^{xA} e^{yB} e^{-xA} = e^{y e^{xA} B e^{-xA}} = e^{y(B + x[A,B] + \frac{x^2}{2!}[A[A,B]] + \ldots)} = e^{y e^{c_1 x} B} , \tag{8}$$

and hence eq. (2.b) follows. For the proof of eq. (3.a) /7/ we represent $e^{\alpha(A+\lambda B)} = e^{\alpha A} X(\alpha, B)$, differentiate this equality with respect to α and make use of eq. ~~(2.a)~~ (2). This gives an equation for $X(\alpha, B)$, which can be elementary integrated and leads to eq. (3.a). Hence using eq. (2.a) one can obtain eq. (3.b).

To prove eqs. (5.a) and (5.b) we make use of the fact that C and D are the shift operators for C + D (see eqs. (4b,c)). Then with the help of (3.a) and (3.b) one finds

$$e^{xD} = e^{-xC + x(C+D)} = e^{-xC} e^{-c_2^{-1}(C+D)(1-e^{xc_2})} = e^{c_2^{-1}(C+D)(1-e^{-xc_2})} e^{-xC} . \tag{9}$$

Hence the relations (5.a) and (5.b) are clear, and we must only find x in terms of β. For the first and second cases we obtain, respectively

$$\begin{aligned} \beta &= -c_2^{-1}(1-e^{xc_2}) \rightarrow x = c_2^{-1} \ln(1+\beta c_2) , \\ \beta &= c_2^{-1}(1-e^{-xc_2}) \rightarrow x = -c_2^{-1} \ln(1-\beta c_2) . \end{aligned} \tag{10}$$

Equations (6) are a particular case of eqs. (7). Therefore let us show the latter. To prove (7.a), first we make use of eq. (3.a), taking into account eq. (4.b), and then apply eq. (5.a)

$$\begin{aligned} e^{\gamma C + \delta D} &= e^{(\gamma-\delta)C + \delta(C+D)} = e^{(\gamma-\delta)C} e^{\frac{\delta}{(\gamma-\delta)c_2}(C+D)(1-e^{(\delta-\gamma)c_2})} = \\ &= e^{(\gamma-\delta)C} e^{c_2^{-1} C \ln[1 + c_2 \frac{\delta}{(\gamma-\delta)c_2}(1-e^{(\delta-\gamma)c_2})]} e^{c_2^{-1} D \ln[1 + c_2 \frac{\delta}{(\gamma-\delta)c_2}(1-e^{(\delta-\gamma)c_2})]} = \\ &= e^{c_2^{-1} C \ln[(\gamma-\delta)^{-1}(\gamma e^{(\gamma-\delta)c_2} - \delta)]} e^{c_2^{-1} D \ln[(\gamma-\delta)^{-1}(\gamma - \delta e^{(\delta-\gamma)c_2})]} . \end{aligned} \tag{11}$$

Eq. (7.b) is proved analogously.

Now, we turn to the N-ordering of the Bogolubov transformation, which has the forms

$$V = e^{-\frac{1}{2}x(a^{+2}-a^2)} \qquad \text{for} \qquad \vec{k}=0 \tag{12}$$

$$V = e^{-x_{\vec{k}}(a^+_{\vec{k}}a^+_{-\vec{k}} - a_{\vec{k}}a_{-\vec{k}})} \qquad \text{for} \qquad \vec{k}\neq 0 \tag{13}$$

where $a, a^+, a_{\vec{k}}$ and $a^+_{\vec{k}}$ are usual annihilation and creation operators $[a, a^+] = 1$, (14)

$$[a_{\vec{k}}, a^+_{\vec{k}'}] = \delta_{\vec{k}-\vec{k}'} \,. \tag{15}$$

In what follows we will use the well-known formula (without proof):

$$e^{xa^+a} = :e^{(e^x-1)a^+a}: , \tag{16}$$

where dots denote the N-ordering.

The case $\vec{k} = 0$. If we choose

$$C = a^{+2} + \frac{1}{2}(a^+a+aa^+) \,, \quad D = a^2 + \frac{1}{2}(a^+a+aa^+) \,, \quad [C,D] = -2(C+D) \,, \tag{17}$$

then according to eq. (6.a)

$$V = e^{\gamma(C-D)} = e^{-\frac{1}{2}C\ln\frac{1}{2}(1+e^{-4\gamma})}\, e^{-\frac{1}{2}D\ln\frac{1}{2}(1+e^{4\gamma})} \,. \quad (\gamma = -\tfrac{x}{2}) \tag{18}$$

Since the operator $\frac{1}{2}(a^+a + aa^+)$ is a shift operator for a^{+2} and a^2 , we put

$$A = \frac{1}{2}(a^+a+aa^+) \,, \quad B = a^{+2} \,, \qquad [A,B] = 2B \tag{19}$$

for the first factor of eq. (18), and

$$A = \frac{1}{2}(a^+a+aa^+) \,, \quad B = a^2 \,, \qquad [A,B] = -2B \tag{20}$$

for the second one. Then eq. (18) splits completely as follows:

$$V = e^{-\frac{1}{2}a^{+2}\left(1-e^{-\ln\frac{1}{2}(1+e^{-4\gamma})}\right)}\, e^{-\frac{1}{4}(a^+a+aa^+)\ln\frac{1}{2}(1+e^{-4\gamma})}$$
$$\cdot\, e^{-\frac{1}{4}(a^+a+aa^+)\ln\frac{1}{2}(1+e^{4\gamma})}\, e^{-\frac{1}{2}a^2\left(1-e^{-\ln\frac{1}{2}(1+e^{4\gamma})}\right)} =$$
$$= e^{\frac{1}{2}a^{+2}\,\mathrm{th}\,2\gamma}\, e^{-\frac{1}{2}(a^+a+aa^+)\ln \mathrm{ch}\,2\gamma}\, e^{-\frac{1}{2}a^2\,\mathrm{th}\,2\gamma} , \tag{21}$$

Finally we represent the Bogolubov transformation

$$V = e^{-\frac{x}{2}(a^{+2}-a^2)} = e^{-\frac{1}{2}a^{+2}\,\mathrm{th}\,x}\, e^{-\frac{1}{2}(a^+a+aa^+)\ln\mathrm{ch}\,x}\, e^{\frac{1}{2}a^2\,\mathrm{th}\,x} = \tag{22.a}$$

$$= (\mathrm{ch}\,x)^{-\frac{1}{2}}\, e^{-\frac{1}{2}a^{+2}\,\mathrm{th}\,x}\, e^{-a^+a\ln\mathrm{ch}\,x}\, e^{\frac{1}{2}a^2\,\mathrm{th}\,x} = \tag{22.b}$$

$$= (\mathrm{ch}\,x)^{-\frac{1}{2}} : e^{-\frac{1}{2}a^{+2}\,\mathrm{th}\,x}\, e^{(\mathrm{ch}^{-1}x-1)a^+a}\, e^{\frac{1}{2}a^2\,\mathrm{th}\,x} : = \tag{22.c}$$

$$= (\mathrm{ch}\,x)^{-\frac{1}{2}}\, e^{-\frac{1}{2}a^+a\ln\mathrm{ch}\,x}\, e^{-\frac{1}{2}a^{+2}\,\mathrm{sh}\,x}\, e^{\frac{1}{2}a^2\,\mathrm{sh}\,x}\, e^{-\frac{1}{2}a^+a\ln\mathrm{ch}\,x} , \tag{22.d}$$

where eq. (22.c) is the N-ordered form, which follows from eq.(22.b), using eq. (16). Alternative form (22.d) is not N-ordered one, but it leads in the most simple way to matrix elements between Fock states. Between such states the first and last factors turn into c-numbers and there remains a simple N-ordered expression, which contains $\mathrm{sh}\,x$ only. Inserting a complete set of states between remaining exponentials, and expanding exponentials into series, one obtains

$$\langle 0|a^m e^{-\frac{x}{2}(a^{+2}-a^2)} a^{+n}|0\rangle = (\mathrm{ch}\,x)^{-\frac{m+n+1}{2}} \langle 0|a^m e^{-\frac{1}{2}a^{+2}\mathrm{sh}\,x}\, e^{\frac{1}{2}a^2\mathrm{sh}\,x}\, a^{+n}|0\rangle =$$
$$= (\mathrm{ch}\,x)^{-\frac{m+n+1}{2}} \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^p (\frac{1}{2}\mathrm{sh}\,x)^{p+r}}{l!\,p!\,r!} \langle 0|a^m a^{+2p} a^{+l}|0\rangle \langle 0|a^l a^{2r} a^{+n}|0\rangle =$$
$$= (\mathrm{ch}\,x)^{-\frac{m+n+1}{2}} \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^p (\frac{1}{2}\mathrm{sh}\,x)^{p+r}}{l!\,p!\,r!}\, m!\,n!\,\delta_{l+2p-m}\,\delta_{l+2r-n} =$$

$$= (\mathrm{ch}\, x)^{-\frac{m+n+1}{2}}\, m!\, n! \sum_{\ell=0}^{\infty} \frac{(-1)^{\frac{m-\ell}{2}} \left(\frac{1}{2}\mathrm{sh}\, x\right)^{\frac{m+n}{2}-\ell}}{\ell!\, \frac{m-\ell}{2}!\, \frac{n-\ell}{2}!} \cdot \begin{cases} 1 & \text{when } m-\ell \text{ and } n-\ell \text{ nonnegative integer} \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Therefore, the matrix element is non-zero only if 1) m and n are even, and ℓ runs over even numbers only 2) m and n are odd, and ℓ runs over odd numbers only.

The series cuts off at $\ell = \min(m,n)$. This coincides with the results of[5,6].

The N-ordered expression (22.b) has the same form as a generating function of Rashid[6], except for the factor e^{-a^+a}. This is the general fact, that not coefficients of the simple N-product decomposition

$$F = \sum_{k,\ell} c(k,\ell)\, a^{+k} a^{\ell} \quad , \qquad \langle 0| a^m F\, a^{+n} |0\rangle \neq c(m,n), \tag{24}$$

but coefficients of the modified one

$$F = \sum_{k,\ell} c(k,\ell) : a^{+k} a^{\ell} e^{-N} : , \qquad \langle 0| a^m F\, a^{+n} |0\rangle = m!\, n!\, c(m,n) \tag{25}$$

are equal to matrix elements between Fock states. The operator $\mathcal{N}$ is always the particle number operator, here $\mathcal{N} = a^+ a$.

This fact seems well-known[8–10]. However we show it in Appendix I because relations which prove it are of interest by themselves.

The case $\vec{k} \neq 0$. Choosing

$$C = a^+_{\vec k} a^+_{-\vec k} + \frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right), \quad D = a_{\vec k} a_{-\vec k} + \frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right),$$

$$[C, D] = -(C + D) \tag{26}$$

and using eq. (6.a), we obtain

$$V = e^{\gamma(C-D)} = e^{-C \ln \frac{1}{2}(1+e^{-2\gamma})}\; e^{-D \ln \frac{1}{2}(1+e^{2\gamma})} \quad (\gamma = -x_{\vec k}). \tag{27}$$

Since operator $\frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right)$ is the shift operator for $a^+_{\vec k} a^+_{-\vec k}$ and $a_{\vec k} a_{-\vec k}$ we put

$$A = \frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right), \quad B = a^+_{\vec k} a^+_{-\vec k} \quad , \quad [A, B] = B \tag{28}$$

for the first factor of eq. (27), and

$$A = \frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right), \quad B = a_{\vec k} a_{-\vec k} \quad , \quad [A, B] = -B \tag{29}$$

for the second one. Then eq. (27) splits completely

$$V = e^{\gamma(C-D)} = e^{-a^+_{\vec k} a^+_{-\vec k}\left(1 - e^{-\ln\frac{1}{2}(1+e^{-2\gamma})}\right)}\; e^{-\frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right)\ln\frac{1}{2}(1+e^{-2\gamma})} \cdot$$

$$\cdot\, e^{-\frac{1}{2}\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right)\ln\frac{1}{2}(1+e^{2\gamma})}\; e^{-a_{\vec k} a_{-\vec k}\left(1 - e^{-\ln\frac{1}{2}(1+e^{2\gamma})}\right)} =$$

$$= e^{a^+_{\vec k} a^+_{-\vec k}\, \mathrm{th}\,\gamma}\; e^{-\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right)\ln \mathrm{ch}\,\gamma}\; e^{-a_{\vec k} a_{-\vec k}\, \mathrm{th}\,\gamma}. \tag{30}$$

Finally, we represent the Bogolubov transformation as follows

$$V = e^{-x_{\vec k}\left(a^+_{\vec k} a^+_{-\vec k} - a_{\vec k} a_{-\vec k}\right)} =$$

$$= e^{-a^+_{\vec k} a^+_{-\vec k}\, \mathrm{th}\, x_{\vec k}}\; e^{-\left(a^+_{\vec k} a_{\vec k} + a_{-\vec k} a^+_{-\vec k}\right)\ln \mathrm{ch}\, x_{\vec k}}\; e^{a_{\vec k} a_{-\vec k}\, \mathrm{th}\, x_{\vec k}} \underset{(31.a)}{=}$$

$$= (\mathrm{ch}\, x_{\vec k})^{-1}\, e^{-a^+_{\vec k} a^+_{-\vec k}\, \mathrm{th}\, x_{\vec k}}\; e^{-\left(a^+_{\vec k} a_{\vec k} + a^+_{-\vec k} a_{-\vec k}\right)\ln \mathrm{ch}\, x_{\vec k}}\; e^{a_{\vec k} a_{-\vec k}\, \mathrm{th}\, x_{\vec k}} \underset{(31.b)}{=}$$

$$= (\mathrm{ch}\, x_{\vec k})^{-1} : e^{-a^+_{\vec k} a^+_{-\vec k}\, \mathrm{th}\, x_{\vec k}}\; e^{\left(\mathrm{ch}^{-1} x_{\vec k} - 1\right)\left(a^+_{\vec k} a_{\vec k} + a^+_{-\vec k} a_{-\vec k}\right)}\; e^{a_{\vec k} a_{-\vec k}\, \mathrm{th}\, x_{\vec k}} : \underset{(31.c)}{=}$$

$$= e^{-\frac{1}{2}(a^+_{\vec k}a_{\vec k} + a_{-\vec k}a^+_{-\vec k})\ln ch x_{\vec k}}\; e^{-a^+_{\vec k}a^+_{-\vec k}\, sh x_{\vec k}}\; e^{a_{\vec k}a_{-\vec k}\, sh x_{\vec k}}\; e^{-\frac{1}{2}(a^+_{\vec k}a_{\vec k}+a_{-\vec k}a^+_{-\vec k})\ln ch x_{\vec k}} = \tag{31.d}$$

$$= e^{-a^+_{\vec k}a_{\vec k}\ln ch x_{\vec k}}\; e^{-a^+_{\vec k}a^+_{-\vec k}\, sh x_{\vec k}}\; e^{a_{\vec k}a_{-\vec k}\, sh x_{\vec k}}\; e^{-a_{-\vec k}a^+_{-\vec k}\ln ch x_{\vec k}}. \tag{31.e}$$

The N-ordered expression has form (31.c), and this construction again differs from a generating function of Rashid by the factor $e^{-N}=\exp[-(a^+_{\vec k}a_{\vec k}+a^+_{-\vec k}a_{-\vec k})]$. We stressed above, that it is natural. However, to get the desired matrix element, it is most convenient to use eq. (31.d) or (31.e):

$$\langle 0|a^p_{\vec k}a^q_{-\vec k}\; V\; a^{+r}_{\vec k}a^{+s}_{-\vec k}|0\rangle =$$

$$=(ch x_{\vec k})^{-p-s-1}\langle 0|a^p_{\vec k}a^q_{-\vec k}\, e^{-a^+_{\vec k}a^+_{-\vec k} sh x_{\vec k}}\, e^{a_{\vec k}a_{-\vec k} sh x_{\vec k}}\, a^{+r}_{\vec k}a^{+s}_{-\vec k}|0\rangle =$$

$$=(ch x_{\vec k})^{-p-s-1}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\ell!m!}\langle 0|a^p_{\vec k}a^q_{-\vec k}\, e^{-a^+_{\vec k}a^+_{-\vec k} sh x_{\vec k}}\, a^{+\ell}_{\vec k}a^{+m}_{-\vec k}|0\rangle$$

$$\langle 0|a^{\ell}_{\vec k}a^{m}_{-\vec k}\, e^{a_{\vec k}a_{-\vec k} sh x_{\vec k}}\, a^{+r}_{\vec k}a^{+s}_{-\vec k}|0\rangle =$$

$$=(ch x_{\vec k})^{-p-s-1}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(sh x_{\vec k})^{i+j}}{\ell!m!i!j!}$$

$$\langle 0|a^p_{\vec k}a^q_{-\vec k}(a^+_{\vec k}a^+_{-\vec k})^i a^{+\ell}_{\vec k}a^{+m}_{-\vec k}|0\rangle\langle 0|a^{\ell}_{\vec k}a^{m}_{-\vec k}(a_{\vec k}a_{-\vec k})^j a^{+r}_{\vec k}a^{+s}_{-\vec k}|0\rangle =$$

$$=(ch x_{\vec k})^{-p-s-1}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(sh x_{\vec k})^{i+j}}{\ell!m!i!j!}\;\cdot$$

$$\cdot\, p!q!r!s!\,\delta_{p-i-\ell}\,\delta_{q-i-m}\,\delta_{r-\ell-j}\,\delta_{s-m-j} =$$

$$=\delta_{p-q-r+s}\, p!q!r!s!(ch x_{\vec k})^{-p-s-1}\sum_{\ell=0}\frac{(-1)^{p-\ell}(sh x_{\vec k})^{p+r-2\ell}}{\ell!(q-p+\ell)!(p-\ell)!(r-\ell)!}. \tag{32}$$

The latter expression coincides with the Rashid's result/6/. Note, that like eq. (23), eq. (32) is a finite sum, since the series cuts off at $\ell=\min(p,r)$, and, when $q-p<0$, $p-q\le\ell\le\min(p,r)$.

We summarize also results of ordering, based on the Baker-Campbell-Hausdorff formula and corresponding matrix elements between Fock states for the Gaussian potential and other relative expressions

$$V=e^{-\varkappa q^2}=e^{-\frac{\varkappa}{2\omega}(C+D)}= \qquad \left(q=\frac{a+a^+}{\sqrt{2\omega}}\right)$$

$$=\sqrt{\frac{\omega}{\varkappa+\omega}}\; e^{-\frac{\varkappa}{2(\varkappa+\omega)}a^{+2}}\; e^{a^+a\ln\frac{\omega}{\varkappa+\omega}}\; e^{-\frac{\varkappa}{2(\varkappa+\omega)}a^2}= \tag{33.a}$$

$$=\sqrt{\frac{\lambda}{\varkappa}}\; :e^{-\lambda q^2}: = \qquad \left(\lambda=\frac{\varkappa\omega}{\varkappa+\omega}\right) \tag{33.b}$$

$$=\sqrt{\frac{\omega}{\varkappa+\omega}}\; e^{\frac{1}{2}a^+a\ln\frac{\omega}{\varkappa+\omega}}\; e^{-\frac{\varkappa}{2\omega}a^{+2}}\; e^{-\frac{\varkappa}{2\omega}a^2}\; e^{\frac{1}{2}a^+a\ln\frac{\omega}{\varkappa+\omega}}, \tag{33.c}$$

$$\langle 0|a^m e^{-\varkappa q^2}a^{+n}|0\rangle =$$

$$=\left(\frac{\omega}{\varkappa+\omega}\right)^{\frac{m+n+1}{2}} m!n!\sum_{\ell=0}\frac{\left(-\frac{\varkappa}{2\omega}\right)^{\frac{m+n}{2}-\ell}}{\ell!\,\frac{m-\ell}{2}!\,\frac{n-\ell}{2}!}\cdot\begin{cases}1 & \text{if } m-\ell \text{ and } n-\ell \text{ nonpositive integer}\\ 0 & \text{otherwise}\end{cases}, \tag{34}$$

$$V=e^{-2\varkappa q_{-\vec k}q_{\vec k}}=e^{-\frac{\varkappa}{\omega}(C+D)}= \qquad \left(q_{\vec k}=\frac{a_{\vec k}+a^+_{-\vec k}}{\sqrt{2\omega}}\right)$$

$$=\frac{\omega}{\varkappa+\omega}e^{-\frac{\varkappa}{\varkappa+\omega}a^+_{\vec k}a^+_{-\vec k}}\; e^{(a^+_{\vec k}a_{\vec k}+a^+_{-\vec k}a_{-\vec k})\ln\frac{\omega}{\varkappa+\omega}}\; e^{-\frac{\varkappa}{\varkappa+\omega}a_{\vec k}a_{-\vec k}} = \tag{35.a}$$

$$=\frac{\lambda}{\varkappa}\; :e^{-2\lambda q_{-\vec k}q_{\vec k}}: = \qquad \left(\lambda=\frac{\varkappa\omega}{\varkappa+\omega}\right) \tag{35.b}$$

$$=\frac{\omega}{\varkappa+\omega}e^{a^+_{\vec k}a_{\vec k}\ln\frac{\omega}{\varkappa+\omega}}\; e^{-\frac{\varkappa}{\omega}a^+_{\vec k}a^+_{-\vec k}}\; e^{-\frac{\varkappa}{\omega}a_{\vec k}a_{-\vec k}}\; e^{a^+_{-\vec k}a_{-\vec k}\ln\frac{\omega}{\varkappa+\omega}}, \tag{35.c}$$

$$\langle 0|a^p_{\vec k}a^q_{-\vec k}\, e^{-2\varkappa q_{-\vec k}q_{\vec k}}\, a^{+r}_{\vec k}a^{+s}_{-\vec k}|0\rangle =$$

$$=\delta_{p-q-r+s}\left(\frac{\omega}{\varkappa+\omega}\right)^{p+s+1} p!q!r!s!\sum_{\ell=0}\frac{\left(-\frac{\varkappa}{\omega}\right)^{p+r-2\ell}}{\ell!(q-p+\ell)!(p-\ell)!(r-\ell)!}, \tag{36}$$

$$V = e^{-\varkappa p^2} = e^{\frac{\varkappa\omega}{2}(C+D)} = \qquad \left(p = i\sqrt{\frac{\omega}{2}}\,(a^+ - a)\right)$$

$$= \frac{1}{\sqrt{1+\varkappa\omega}}\, e^{\frac{1}{2}\frac{\varkappa\omega}{1+\varkappa\omega}a^{+2}}\, e^{-a^+a\ln(1+\varkappa\omega)}\, e^{\frac{1}{2}\frac{\varkappa\omega}{1+\varkappa\omega}a^{2}} = \qquad (37.a)$$

$$= \sqrt{\frac{\lambda}{\varkappa}}\; :e^{-\lambda p^2}: \; = \qquad \left(\lambda = \frac{\varkappa}{1+\varkappa\omega}\right) \qquad (37.b)$$

$$= \frac{1}{\sqrt{1+\varkappa\omega}}\, e^{-\frac{1}{2}a^+a\ln(1+\varkappa\omega)}\, e^{\frac{\varkappa\omega}{2}a^{+2}}\, e^{\frac{\varkappa\omega}{2}a^{2}}\, e^{-\frac{1}{2}a^+a\ln(1+\varkappa\omega)}, \qquad (37.c)$$

$$\langle 0|a^m e^{-\varkappa p^2} a^{+n}|0\rangle =$$

$$= (1+\varkappa\omega)^{-\frac{m+n+1}{2}}\, m!\,n! \sum_{\ell=0} \frac{\left(\frac{\varkappa\omega}{2}\right)^{\frac{m+n}{2}-\ell}}{\ell!\,\frac{m-\ell}{2}!\,\frac{n-\ell}{2}!} \cdot \begin{cases} 1 & \text{if } m-\ell \text{ and } n-\ell \text{ nonpositive integer} \\ 0 & \text{otherwise} \end{cases}, \qquad (38)$$

$$V = e^{-2\varkappa p_{-\vec{k}} p_{\vec{k}}} = e^{\varkappa\omega(C+D)} =$$

$$= \frac{1}{1+\varkappa\omega}\, e^{\frac{\varkappa\omega}{1+\varkappa\omega}a^+_{\vec{k}}a^+_{-\vec{k}}}\, e^{-(a^+_{\vec{k}}a_{\vec{k}} + a^+_{-\vec{k}}a_{-\vec{k}})\ln(1+\varkappa\omega)}\, e^{\frac{\varkappa\omega}{1+\varkappa\omega}a_{\vec{k}}a_{-\vec{k}}} = \qquad (39.a)$$

$$= \frac{1}{1+\varkappa\omega} : e^{-2\lambda p_{-\vec{k}}p_{\vec{k}}} : = \qquad \left(\lambda = \frac{\varkappa}{1+\varkappa\omega}\right) \qquad (39.b)$$

$$= \frac{1}{1+\varkappa\omega}\, e^{-a^+_{\vec{k}}a_{\vec{k}}\ln(1+\varkappa\omega)}\, e^{\varkappa\omega a^+_{\vec{k}}a^+_{-\vec{k}}}\, e^{\varkappa\omega a_{\vec{k}}a_{-\vec{k}}}\, e^{-a^+_{-\vec{k}}a_{-\vec{k}}\ln(1+\varkappa\omega)}, \qquad (39.c)$$

$$\langle 0|a_{\vec{k}}^{\,p}\, a_{-\vec{k}}^{\,q}\, e^{-2\varkappa p_{-\vec{k}}p_{\vec{k}}}\, a^{+r}_{\vec{k}}\, a^{+s}_{-\vec{k}}|0\rangle =$$

$$= \delta_{p-q-r+s}\, p!\,q!\,r!\,s!\,(1+\varkappa\omega)^{-p-s-1} \sum_{\ell=0} \frac{(\varkappa\omega)^{p+r-2\ell}}{\ell!\,(q-p+\ell)!\,(p-\ell)!\,(r-\ell)!}\,. \qquad (40)$$

In eqs. (33), (35), (37) and (39) C and D are (17), (26),

$$C = a^{+2} - \tfrac{1}{2}(a^+a + aa^+), \quad D = a^2 - \tfrac{1}{2}(a^+a + aa^+), \quad [C,D] = 2(C+D) \qquad (41)$$

and

$$C = a^+_{\vec{k}}a^+_{-\vec{k}} - \tfrac{1}{2}(a^+_{\vec{k}}a_{\vec{k}} + a_{-\vec{k}}a^+_{-\vec{k}}), \quad D = a_{\vec{k}}a_{-\vec{k}} - \tfrac{1}{2}(a^+_{\vec{k}}a_{\vec{k}} + a_{-\vec{k}}a^+_{-\vec{k}}), \quad [C,D] = C+D, \qquad (42)$$

respectively.

It was just the Baker-Campbell-Hausdorff **formula** for the shift operator, which Sack/7/ used to obtain his results for the Gaussian potential somewhat different from eqs. (33.c) and (34) because of his unusual normalization of a and a^+.

3. DIRECT APPROACH TO THE ORDERING

Now we represent a more general expression in the ordered form:

$$F(\lambda) = e^{\lambda[\alpha a^{+2} + \mu a^+ + \beta\frac{1}{2}(a^+a + aa^+) + \nu a + \gamma a^2]} =$$

$$= e^{xa^{+2} + ua^+}\, e^{y\frac{1}{2}(a^+a + aa^+)}\, e^{za^2 + va}\, e^{w}, \qquad (43)$$

where in the first line an initial expression is written (the parameter λ being introduced), $\alpha, \beta, \gamma, \mu, \nu$ are constants, and in the second line the desired final form is given, where x, y, z, u, v and w are functions of λ yet to be defined. After differentiation with respect to λ, we obtain

$$\frac{dF}{d\lambda} = \left(\alpha a^{+2} + \mu a^+ + \beta\tfrac{1}{2}(a^+a + aa^+) + \nu a + \gamma a^2\right)F =$$

$$= \Big[(x' - 2xy' + 4x^2z'e^{-2y})a^{+2} + (y' - 4xz'e^{-2y})\tfrac{1}{2}(a^+a + aa^+) + z'e^{-2y}a^2 +$$

$$+ (u' - y'u + 4xuz'e^{-2y} - 2xv'e^{-y})a^+ + (-2uz'e^{-2y} + v'e^{-y})a +$$

$$+ u^2z'e^{-2y} - uv'e^{-y} + w'\Big]F\,. \qquad (44)$$

Therefore we reduce the problem of ordering to the problem of solution of the following system of ordinary differential equations of the first order

$$\begin{aligned} x' - 2xy' + 4x^2z'e^{-2y} &= \alpha \\ y' - 4xz'e^{-2y} &= \beta \\ z'e^{-2y} &= \gamma \\ u' - (y' - 4xz'e^{-2y})u - 2xv'e^{-y} &= \mu \\ v'e^{-y} - 2uz'e^{-2y} &= \nu \\ w' - uv'e^{-y} + u^2z'e^{-2y} &= 0 \end{aligned} \qquad (45)$$

with the initial conditions

$$x(0)=0\,,\; y(0)=0\,,\; z(0)=0\,,\; u(0)=0\,,\; v(0)=0,\; w(0)=0\,. \qquad (46)$$

for $\lambda=0$. After evident simplifications we obtain the solution in the form

$$x(\lambda)=\frac{\alpha\,\mathrm{th}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)}{\sqrt{\beta^2-4\alpha\gamma}-\beta\,\mathrm{th}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)}\,, \qquad (47)$$

$$y(\lambda)=-\ln\left[\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-\frac{\beta}{\sqrt{\beta^2-4\alpha\gamma}}\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)\right]\,, \qquad (48)$$

$$z(\lambda)=\frac{\gamma\,\mathrm{th}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)}{\sqrt{\beta^2-4\alpha\gamma}-\beta\,\mathrm{th}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)}\,, \qquad (49)$$

$$u(\lambda)=\frac{\mu\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)+\frac{2\alpha\nu-\beta\mu}{\sqrt{\beta^2-4\alpha\gamma}}\left[\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-1\right]}{\sqrt{\beta^2-4\alpha\gamma}\,\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-\beta\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)}\,, \qquad (50)$$

$$v(\lambda)=\frac{\nu\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)+\frac{2\gamma\mu-\beta\nu}{\sqrt{\beta^2-4\alpha\gamma}}\left[\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-1\right]}{\sqrt{\beta^2-4\alpha\gamma}\,\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-\beta\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)}\,, \qquad (51)$$

$$w(\lambda)=\frac{\alpha\nu^2+\gamma\mu^2-\beta\mu\nu}{\beta^2-4\alpha\gamma}-\frac{(2\alpha\nu-\beta\mu)(2\gamma\mu-\beta\nu)\left[\sqrt{\beta^2-4\alpha\gamma}\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-\beta\,\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)\right]}{(\beta^2-4\alpha\gamma)\left[\sqrt{\beta^2-4\alpha\gamma}\,\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-\beta\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)\right]^2}-$$
$$-\frac{\alpha\nu^2+\gamma\mu^2-\beta\mu\nu+\frac{\beta(2\alpha\nu-\beta\mu)(2\gamma\mu-\beta\nu)}{\beta^2-4\alpha\gamma}}{\left[\sqrt{\beta^2-4\alpha\gamma}\,\mathrm{ch}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)-\beta\,\mathrm{sh}\left(\lambda\sqrt{\beta^2-4\alpha\gamma}\right)\right]^2}\,. \qquad (52)$$

4. COHERENT STATE EXPECTATION VALUES

If we know the N-ordered form of an operator

$$Q=:\mathcal{Q}(\hat{a}^+,\hat{a}):\,, \qquad (53)$$

then its expectation value in the coherent state $|\alpha\rangle=e^{(\alpha a^+-\alpha^*a)}|0\rangle$ is given

$$\langle\alpha|Q|\alpha\rangle=\mathcal{Q}(\alpha^*,\alpha) \qquad (54)$$

(the non-diagonal elements are $\langle\alpha_2|Q|\alpha_1\rangle=\mathcal{Q}(\alpha_2^*,\alpha_1)$)). These expectation values permit one to restore the initial operator, and, in particular,to find matrix elements of Q between Fock states. The generating function for the latter matrix elements is known to be:

$$F(\alpha^*,\alpha)=\frac{\langle\alpha|Q|\alpha\rangle}{|\langle\alpha|0\rangle|^2}=\mathcal{Q}(\alpha^*,\alpha)e^{\alpha^*\alpha}\,, \qquad (55)$$

$$\langle 0|a^m Q a^{+n}|0\rangle=m!n!c(m,n)=\left(\frac{d}{d\alpha^*}\right)^m\left(\frac{d}{d\alpha}\right)^n F(\alpha^*,\alpha)\Big|_{\alpha=\alpha^*=0}\,. \qquad (56)$$

This holds for any number of degrees of freedom ($\alpha=\{\alpha_1\cdots\alpha_k\}$, $\left(\frac{d}{d\alpha}\right)^n=\left(\frac{d}{d\alpha_1}\right)^{n_1}\cdots\left(\frac{d}{d\alpha_k}\right)^{n_k}$). It is clear, Rashid's generating functions coincide exactly with those we obtain in this way.

Further using eq. (37.b) we immediately obtain,in the coherent state representation, the representative of the operator of evolution for one-dimensional free motion*)

$$\langle\alpha|e^{-i\hbar^{-1}Ht}|\alpha\rangle=\langle\alpha|e^{-i\hbar^{-1}\frac{p^2}{2m}t}|\alpha\rangle=\sqrt{\frac{\lambda}{\varkappa}}\,e^{-\lambda p^2}\quad\left(\varkappa^{-1}=2im\hbar,\ p=i\sqrt{\frac{\omega}{2}}(\alpha^*-\alpha)\right). \qquad (57)$$

Equation (43) permits one to represent analogously evolution for any bilinear one-dimensional Hamiltonian H

$$\langle\alpha|e^{-i\hbar^{-1}Ht}|\alpha\rangle=e^{x\alpha^{*2}+u\alpha^*+(e^y-1)\alpha^*\alpha+z\alpha^2+v\alpha+w+\frac{y}{2}}\,. \qquad (58)$$

It is easy to extend this approach to many-dimensional case.

Note also, that the Gaussian potential in this representation is (according to eq. (33.b))

$$\langle\alpha|e^{-\varkappa\hat{q}^2}|\alpha\rangle=\sqrt{\frac{\lambda}{\varkappa}}\,e^{-\lambda q^2}\,. \qquad \left(q=\frac{\alpha^*+\alpha}{\sqrt{2\omega}}\right) \qquad (59)$$

*) Equation (38) gives its matrix elements between Fock states.

APPENDIX A

If $|n\rangle$ are n-quantum states

$$|n\rangle = a^{+n}|0\rangle , \tag{A.1}$$

N is the quantum number operator

$$N = a^+ a \tag{A.2}$$

and Q is any operator, constructed out of a and a^+ then we have the following relations involving N

$$
\begin{array}{lll}
:N:|n\rangle = n|n\rangle , & :QN:|n\rangle = n\,a^+:Q:|n-1\rangle , & \\
:N^2:|n\rangle = n(n-1)|n\rangle , & :QN^2:|n\rangle = n(n-1)a^{+2}:Q:|n-2\rangle , & \\
\cdots\cdots & \cdots\cdots & \\
:N^m:|n\rangle = n^{[m]}|n\rangle , & :QN^m:|n\rangle = n^{[m]}a^{+m}:Q:|n-m\rangle , & \text{for } m \leq n \\
:N^n:|n\rangle = n!\,|n\rangle , & :QN^n:|n\rangle = n!\,a^{+n}:Q:|0\rangle , & \\
:N^m:|n\rangle = 0 \quad , & :QN^m:|n\rangle = 0 & \text{for } m > n ,
\end{array} \tag{A.3}
$$

where $n^{[m]} = n(n-1)\cdots(n-m-1)$. The relations of the right column are clear from

$$:QN^m: = a^{+m}:Q:a^m . \tag{A.4}$$

Further

$$:e^{-N}:|n\rangle = \left(1 - :N: + \frac{1}{2!}:N^2: - \cdots + \frac{(-1)^n}{n!}:N^n:\right)|n\rangle = (1-1)^n|n\rangle =$$

$$= \begin{cases} |0\rangle & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \neq 0 \end{cases} , \tag{A.5}$$

$$:Qe^{-N}:|0\rangle = :Q:|0\rangle ,$$

$$:Qe^{-N}:|1\rangle = :Q(1-N):|1\rangle = [:Q:,a^+]|0\rangle ,$$

$$\cdots\cdots\cdots$$

$$:Qe^{-N}:|n\rangle = :Q\left(1 - N + \frac{1}{2!}N^2 - \cdots + \frac{(-1)^n}{n!}N^n\right):|n\rangle =$$

$$= \underbrace{[\cdots[[:Q:,a^+]a^+]\cdots]}_{\text{n-fold commutator}}|0\rangle \tag{A.6}$$

Hence

$$\langle 0|a^m : a^{+k} a^{\ell}\, e^{-a^+a} : a^{+n}|0\rangle = m!\,n!\,\delta_{km}\delta_{\ell n} \tag{A.7}$$

and now eq. (25) becomes clear.

Further note that

$$:e^{-N}::e^{-N}: = :e^{-N}: \tag{A.8}$$

Equations (A.5) and (A.8) are clear, due to the well-known relation

$$:e^{-N}: = |0\rangle\langle 0| , \tag{A.9}$$

the simplest proof of which may be given in terms of coherent states,

Note also that in these terms the completeness relation, i.e., decomposition of unity into 0-, 1-, 2-... - quantum projectors, is written

$$1 = :e^{N}e^{-N}: = :\left(1 + N + \frac{1}{2!}N^2 + \cdots\right)e^{-N}: =$$

$$= :e^{-N}: + :Ne^{-N}: + \frac{1}{2!}:N^2e^{-N}: + \cdots = \Lambda_0 + \Lambda_1 + \Lambda_2 + \cdots \tag{A.10}$$

$$\Lambda_0 = :e^{-N}: = |0\rangle\langle 0| , \quad \Lambda_1 = :Ne^{-N}: = a^+|0\rangle\langle 0|a ,$$

$$\Lambda_2 = \frac{1}{2!}:N^2e^{-N} = \frac{1}{2!}a^{+2}|0\rangle\langle 0|a^2 , \quad \cdots \tag{A.11}$$

All the above relations are general. They hold for any number of degrees of freedom, when annihilation and creation operators are labelled by some quantum numbers (a_i and a_i^+) in the Bose and Fermi cases.

The operator N is always the operator of total number of particles, $N = \sum_i a_i^+ a_i$. By $|n\rangle$ we now understand n-quantum states

$$|i_1 i_2 \cdots i_n\rangle = a^+_{i_1}a^+_{i_2}\cdots a^+_{i_n}|0\rangle$$

*) Let us illustrate the appearance of these commutators:

$$:Qe^{-N}:a^+|0\rangle = (:Q: - :QN:)a^+|0\rangle = (:Q: - a^+:Q:a)a^+|0\rangle = [:Q:,a^+]|0\rangle$$

$$:Qe^{-N}:a^{+2}|0\rangle = \left(:Q: - a^+:Q:a + \frac{1}{2!}a^{+2}:Q:a^2\right)a^{+2}|0\rangle =$$

$$= (:Q:a^{+2} - 2a^+:Q:a^+ + a^{+2}:Q:)|0\rangle = [[:Q:,a^+]a^+]|0\rangle$$

Then relations of the right column of (A.3) change in the following manner

$$:QN:|i_1 i_2 \cdots i_n\rangle = a^+_{i_1}:Q:|i_2 \cdots i_n\rangle + \ldots + a^+_{i_n}:Q:|i_1 \cdots i_{n-1}\rangle, \quad (A.12)$$

etc., and relations (A.6) as follows

$$:Qe^{-N}:|i_1 i_2 \cdots i_n\rangle = [\ldots[[:Q:, a^+_{i_1}]a^+_{i_2}]\ldots a^+_{i_n}]|0\rangle . \qquad (A.13)$$

Of course, now for projectors we have

$$\Lambda_1 = :Ne^{-N}: = \sum_i a^+_i|0\rangle\langle 0|a_i, \ \Lambda_2 = \frac{1}{2!}:N^2 e^{-N}: = \frac{1}{2!}\sum_{i,j} a^+_i a^+_j|0\rangle\langle 0|a_j a_i . \ (A.14)$$

For Fermi statistics in relations like (A.13) commutators and anti-commutators are alternating. If Q is an even function of a and a^+, the first operation is commutator $[:Q:, a^+_{i_1}]$, and for odd Q we begin with anticommutator $\{:Q:, a^+_{i_1}\}$.

APPENDIX B

Any function of one operator may be easily N-ordered, if we use a suitable integral representation. For example, for the Gaussian potential we have

$$e^{-\varkappa \hat{q}^2} = \frac{1}{\sqrt{4\pi\varkappa}}\int d\xi\, e^{i\hat{q}\xi}\, e^{-\frac{\xi^2}{4\varkappa}} = \frac{1}{\sqrt{4\pi\varkappa}}\int d\xi\, e^{\frac{i}{\sqrt{2\omega}}(a+a^+)\xi}\, e^{-\frac{\xi^2}{4\varkappa}} =$$

$$= \frac{1}{\sqrt{4\pi\varkappa}}\int d\xi\, e^{\frac{i}{\sqrt{2\omega}}a^+\xi}\, e^{\frac{i}{\sqrt{2\omega}}a\xi}\, e^{-\frac{\xi^2}{4\omega}-\frac{\xi^2}{4\varkappa}} = \frac{1}{\sqrt{4\pi\varkappa}}:\int d\xi\, e^{i\hat{q}\xi}\, e^{-\frac{\xi^2}{4\lambda}}: ,$$

where we use the simplest case of the Baker-Campbell-Hausdorff formula $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$. One can integrate, in usual sense, under the N-product sign what again leads us to eq. (33.b).

REFERENCES

I. N.N.Bogolubov. Journ.Phys.of USSR, 11, 23, 1947;
ЖЭТФ, 34, 58, 1958;
Н.Н.Боголюбов. Избранные труды. Тт 2 и 3. "Наукова думка", Киев, 1970, 1971.

2. Е.П.Солодовникова, А.Н.Тавхелидзе, О.А.Хрусталев. ТМФ, 10, 162, 1972.

3. Е.П.Солодовникова, А.Н.Тавхелидзе. ТМФ, 21, 13, 1976.

4. А.А.Гриб, В.М.Мостепаненко, В.М.Фролов. ТМФ, 26, 221, 1976.

5. K.Tanabe. Journ. Math.Phys., 14, 618, 1973.

6. M.A.Rashid. Journ. Math.Phys., 16, 378, 1975.

7. R.A.Sack. Phys.Mag., 3, 497, 1958.

8. P.Ziesche. Commun. Math.Phys., 5, 191, 301, 1967.

9. G.S.Agarwal, E.Wolf. Phys.Rev., D2, 2161, 2187, 2206, 1970.

10. Ф.А.Березин. Метод вторичного квантования. "Наука", Москва, 1965.

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