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QUASIPOTENTIAL EQUATION FOR TWO PARTICLES WITH SPINS 1/2 IN RELATIVISTIC CONFIGURATIONAL REPRESENTATION

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I. Introduction

In the present paper we develop the three-dimensional formalism [1,2] for the description of the interaction of two particles with spin 1/2 in the quasipotential approach [3,4]. This paper is a sequel to papers [1,2]. It was shown earlier [1] that the Feynman one-boson exchange matrix elements can be transformed to a form of the direct three-dimensional generalizarion of the corresponding nonrevalivistic one-boson exhcange potentials (OBEP). The transformation from the four-dimensional to the three-dimensional representation in terms of the Lobachevsky space may be treated as an alternative one to the Foldi-Wouthuysen transformation for two particles since it is exact and doesn not deal with the expansion of interaction terms in powers of v^2/c^2. In [2] the form was found for this transformed relativistic OBEP in the relativistic configurational representation (RCR) intriduced earlier for the spinless particles in [5].

In the second section the quasipotential equations for spin particles are transformed in the momentum space to a form of the direct geometrical generalization of the Lippmann-Schwinger and Schrodinger nonrelativistic equations. In the third section these equations are written in the RCR. In the fourth section we construct the local in RCR expressions for a relativistic spin-orbital and tensor forces.

II. The Local in the Lobachevsky Space Quasipotential Equations for Particles with Spins

Quasipotential equations for the relativistic scattering amplitude and thw wave function of two relativistic particles with spins 1/2 obtained in Kadyshevsky [6] approach in the c.m.s. have the form

$$T^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q}) = V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q};E_q) +$$

$$= \frac{1}{(2\pi)^3}\sum_{\sigma_1''\sigma_2''}\int d\Omega_k \frac{V^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)\,T^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{k},\vec{q})}{E_k(E_k - E_q - i\epsilon)} \qquad (1)$$

and

$$E_p(E_p - E_q)\Psi_q(\vec{p})_{\sigma_1\sigma_2} =$$

$$= \frac{1}{(2\pi)^3}\sum_{\sigma_1''\sigma_2''}\int d\Omega_k V^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{p},\vec{k},E_q)\Psi_q(\vec{k})_{\sigma_1''\sigma_2''}, \qquad (2)$$

where

$$E_p = \sqrt{M^2+\vec{p}^{\,2}},\ E_k = \sqrt{M^2+\vec{k}^{\,2}},\quad V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)$$

is the quasipotential dependent in general on the total energy of the system $s = 4E_q = 4\sqrt{M^2+\vec{q}^{\,2}}$. Since in the quasipotential equations of Kadyshevsky, like in the equation of Logunov and Tavkhelidze [3], all the momenta of particles belong to the mass shell

$$p_0^2 - \vec{p}^{\,2} = M^2 \qquad (3)$$

and the integration is performed with the volume element $d\Omega_k = \frac{d\vec{k}}{\sqrt{1+\vec{k}^{\,2}/M^2}}$ - the invariant measure on the hyperboloid (3), the momentum space in eqs. (1), (2) is the Lobachevsky space. We stress that in eqs. (1), (2) all the quantities are defined on the mass shell and off the energy shell $E_p \neq E_k \neq E_q$ like in the nonrelativistic equations. The amplitude $T^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q})$ on the energy shell obeys the relativistic condition of two-particle unitarity (cf. [3]):

$$\operatorname{Im} T^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q}) = \qquad (4)$$

$$= \frac{1}{(8\pi)^3}\sqrt{\frac{E_q^2 - M^2}{E_q^2}} \sum_{\sigma_1''\sigma_2''} \int d\omega_k \, T^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{p},\vec{k}) \overset{*}{T}{}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{k},\vec{q})$$

and relates to the elastic differential cross section as follows

$$\frac{d\sigma}{d\omega}(\sigma_1\sigma_2 \to \sigma_1'\sigma_2') = \sum_{\sigma_1''\sigma_2''} \frac{\overset{*}{T}{}^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{p},\vec{q})\, T^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(\vec{p},\vec{q})}{64\pi^2 s} \qquad (5)$$

The quasipotential $V(\vec{p},\vec{k};E_q)$ in eqs. (1), (2) is constructed of the matrix elements of the relativistic scattering amplitude as in refs. [3,4,5]. In the second order in coupling constant (see eq. (1)) it coincides with the Feynman matrix elements corresponding to the Born approximation of the scattering amplitude.

In [1] it is shown that after separating the Wigner rotation $D^{1/2}\{V^{-1}(\Lambda_p;k)\}$ originating from the relativistic spin kinematics, the Feynman matrix elements of the one-boson exchange in notation of [7] can be represented as follows

$$\frac{\bar{u}^{\sigma_1}(\vec{p}_1)\gamma_5 u^{\sigma_1'}(\vec{k}_1)\bar{u}^{\sigma_2}(\vec{p}_2)\gamma_5 u^{\sigma_2'}(\vec{k}_2)}{\mu^2-(p-k)^2} =$$

$$= \langle \vec{p}_1\sigma_1 ; \vec{p}_2\sigma_2 | T_{PS}^{(2)} | \vec{k}_1\sigma_1' ; \vec{k}_2\sigma_2' \rangle =$$

$$= \sum_{\sigma_{1p}\sigma_{2p}} \phi^{*\sigma_1}\phi^{*\sigma_2} T_{PS}^{(2)}(\vec{k}(-)\vec{p})\phi_{\sigma_{1p}}\phi_{\sigma_{2p}} \times$$

$$\times D^{1/2}\{V^{-1}(\Lambda_{p_1}, k_1)\} D^{1/2}\{V^{-1}(\Lambda_{p_2}, k_2)\}, \qquad (6)$$

where the amplitude

$$T_{PS}^{(2)}(\vec{k}(-)\vec{p}) = g^2 \frac{4(\vec{\sigma}_1\vec{\kappa}_1)(\vec{\sigma}_2\vec{\kappa}_2)}{\mu^2+4\vec{\kappa}^2} \qquad (7)$$

in forms does not differ from the nonrelativistic potential

$$V_{PS}(\vec{k}-\vec{p}) = \frac{\vec{\sigma}_1(\vec{k}_1-\vec{p}_1)\vec{\sigma}_2(\vec{k}_2-\vec{p}_2)}{\mu^2+(k-p)^2} =$$

$$= \frac{4(\vec{\sigma}_1\vec{\kappa}_1^e)(\vec{\sigma}_2\vec{\kappa}_2^e)}{\mu^2+4\vec{\kappa}_e^2} \qquad (8)$$

$$\vec{\kappa}^e = (\vec{k}-\vec{p})/2$$

widely used in mesin theory of nuclear forces (ϕ - are the two-component Pauli spinors normalized by the condition $\phi^{*\sigma_1}\phi_{\sigma_2} = \delta^{\sigma_1}_{\sigma_2}$).). The quantity $\vec{\kappa}$ defined in [1] is called the half-momentum transfer (an analog of the half-velocity of a particle from [8]) in the Lobachevsky space. It is related to the momentum transfer in this space

$$\vec{\Lambda}=\vec{k}(-)\vec{p}=\Lambda_p^{-1}\vec{k}=\vec{k}-\frac{\vec{p}}{M}\left(k_0-\frac{\vec{k}\cdot\vec{p}}{p_0+M}\right)=M\,\mathrm{sh}\chi_\Lambda\frac{\vec{\Lambda}}{|\vec{\Lambda}|},$$

$$\Lambda_0=\sqrt{M^2+\vec{\Lambda}^2}=(\Lambda^{-1}k)_0=(k_0\cdot p_0-\vec{k}\cdot\vec{p})/M=M\,\mathrm{ch}\chi_\Lambda \tag{9}$$

by the formula

$$\vec{\kappa}=\vec{\Lambda}\sqrt{\frac{M}{2(\Lambda_0+M)}}=M\,\mathrm{sh}\frac{\chi_\Lambda}{2}\cdot\frac{\vec{\Lambda}}{|\vec{\Lambda}|}. \tag{10}$$

In the nonrelativistic limit when the curvature of the Lobachevsky space tends to zero and it turns into the Euclidean three-dimensional space, $\vec{\kappa}\to\vec{\kappa}_0=(\vec{k}-\vec{p})/2$ and $\vec{\Lambda}\to\vec{k}-\vec{p}$.

Analogously, the amplitude of the vector meson exchange with the mass μ in the c.m.s. $(\vec{p}_1=-\vec{p}_2=\vec{p};\vec{k}_1=-\vec{k}_2=\vec{k})$ takes the form [1]

$$\langle\vec{p}\sigma_1;-\vec{p}\sigma_2|T_V^{(2)}|\vec{k}\sigma_1';-\vec{k}\sigma_2'\rangle=$$

$$=g_V^2\frac{\bar{u}^{\sigma_1}(\vec{p})\gamma_\mu u^{\sigma_1'}(\vec{k})\bar{u}^{\sigma_2}(-\vec{p})\gamma^\mu u^{\sigma_2'}(-\vec{k})}{\mu^2-(p-k)^2}=$$

$$=\sum_{\sigma_{1p}\sigma_{2p}}\overset{*}{\phi}{}^{\sigma_1}\overset{*}{\phi}{}^{\sigma_2}T_V^{(2)}(\vec{k}(-)\vec{p};\vec{p})\phi_{\sigma_{1p}}\phi_{\sigma_{2p}}\times$$

$$\times D^{1/2}_{\sigma_{1p}\sigma_1'}\{V^{-1}(\Lambda_p,\vec{k})\}D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_p,k)\}, \tag{11}$$

where the amplitude without the Wigner rotation

$$T_V^{(2)}(\vec{k}(-)\vec{p};\vec{p}) = -g_V^2 \frac{4M^2}{\mu^2+4\vec{\kappa}^2} -$$

$$-4g_V^2 \frac{(\vec{\sigma}_1\vec{\kappa})(\vec{\sigma}_2\vec{\kappa}) - (\vec{\sigma}_1\vec{\sigma}_2)\vec{\kappa}^2}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2 \frac{8p_0\kappa_0}{M^2} \frac{i(\vec{\sigma}_1+\vec{\sigma}_2)[\vec{p}\times\vec{\kappa}]}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2 \frac{8}{M^2} \frac{p_0^2\kappa_0^2 + 2p_0\kappa_0(\vec{p}\cdot\vec{\kappa}) - M^4}{\mu^2+4\vec{\kappa}^2} -$$

$$-g_V^2 \frac{8}{M^2} \frac{(\vec{\sigma}_1\vec{p})(\vec{\sigma}_1\vec{\kappa})(\vec{\sigma}_2\vec{p})(\vec{\sigma}_2\vec{\kappa})}{\mu^2+4\vec{\kappa}^2} \tag{12}$$

represents a direct geometrical generalization of the Breit potentials /9/ in the momentum space /1/. In /2/ the relativistic quasipotentials (11), (12) have been writtes in the RCR.

To facilitate fuether considerations we shall discuss the role of the Wigner rotation $D^{1/2}\{V^{-1}(\Lambda_p, k)\}$ in eqs. (6), (11). As is seen from the transformation law for the state vectors

$$U(\Lambda_p^{-1})|\vec{k},\sigma\rangle = \sum_{\sigma'} D^{1/2}_{\sigma\sigma'}\{V^{-1}(\Lambda_p, k)\}|\vec{k}(-)\vec{p},\sigma'\rangle \tag{13}$$

these matrices describe a spin rotation under the Lorentz transformations. Since the matrices $D^{1/2}\{V^{-1}(\Lambda_p\ k)\}$ depend on the momentum of the state, they are different for the spin indices in the left- and right-hadn sides of the matrix elements (6) and (11). By terminology of ref. /10/ spin indices are "sitting" each on its own momentum because by definition of the state vectors $|\vec{k},\sigma\rangle =$

$U(\Lambda_k)|\vec{0},\sigma\rangle$ the spin projections on the $|\vec{0},\sigma\rangle$-axis are given in the rest frames of particles $|\vec{0},\sigma\rangle$. And, in general, in each momentum there can be related its own coordinate system which axes may not coincide /11/.

This, the Wigner rotation in (6) and (11) perform a removal of all the spin indices onto one and the same momentum $\vec{p}$. As a result, with the Lorentz transformations, they begin to transform under the little group of the vector $\vec{p}$. In other words, the Wigner rotation superposes the axes in the rest frames of particles.

As is seen from eqs. (6), (7) and (11), (12), after separating the kinematical Wigner rotation the remaining part of the quasipotential $V^{\sigma_{1p}\sigma_{2p}}_{\sigma_1\sigma_2}(\vec{\Lambda};\vec{p})$ $T^{(2)\sigma_{1p}\sigma_{2p}}_{\sigma_1\sigma_2}(\vec{\Lambda};\vec{p})$ which spin indices are "sitting" on one momentum $\vec{p}$, is local in the Lobachevsky space as it depends on the difference of two vectors in that space $\vec{\Lambda}$ $\vec{k}(-)\vec{p}$. Note that in the l.h.s. of eq. (2) spin indices of the wave function $\Psi_q(\vec{p})_{\sigma_1\sigma_2}$, are "sitting" on the momentum $\vec{p}$ and those of $\Psi_q(\vec{k})_{\sigma_1'\sigma_2'}$ in the r.h.s. on the momentum $\vec{k}$. We pass now to such a form of eq. (2) where all the spin indices are sitting on one and the same momentum $\vec{p}$. To this end we "remove" all the spin indices of $\Psi_q(\vec{k})$ also onto the momentum $\vec{p}$. This transformation has the form

$$\Psi_q(\vec{k})_{\sigma_{1p}\sigma_{2p}} = \sum_{\sigma_1'}\sum_{\sigma_2'} D^{1/2}_{\sigma_{1p}\sigma_1'}\{V^{-1}(\Lambda_p,k)\} \times$$

$$\times D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_p,k)\}\Psi_q(\vec{k})_{\sigma_1'\sigma_2'} \quad . \tag{14}$$

Since the quasipotentials (6), (11) contain the required D-functions this removal is performal automatically. As a result, we arrive at the equation for the wave function with all the spin indices on one momentum $\vec{p}$

$$E_p(E_p - E_q)\Psi_q(\vec{p})_{\sigma_1\sigma_2} =$$

$$= \frac{1}{(4\pi)^3} \sum_{\sigma_{1p}\sigma_{2p}} \int d\Omega_k V_{\sigma_1 \sigma_2}^{\sigma_{1p}\sigma_{2p}} (\vec{k}(-)\vec{p}, \vec{p}; E_q) \Psi_q(\vec{k})_{\sigma_{1p}\sigma_{2p}} \quad . \tag{15}$$

With this form of the quasipotential equation the interaction is described by the local in the Lobachevsky space quasipotential $V(\vec{k}(-)\vec{p}, \vec{p}; E_q)$ of type (7) and (12), and the integral part (15) looks like a convolution in the Lobachevsky space. It can be seen that the form (15) is natural for eq. (2) when it is solved for thw wave function of scattering of two particles /2/

$$\Psi_q(\vec{p})_{\sigma_1 \sigma_2} = (2\pi)^3 \delta(\vec{p} - \vec{k}) \sqrt{1 + \vec{p}^{\,2}/M^2}\, \phi_{\sigma_1} \phi_{\sigma_2} + \tag{16}$$

$$+ \frac{1}{(4\pi)^3} \frac{1}{E_p (E_p - E_q - i\epsilon)} \sum_{\sigma_1' \sigma_2'} \int d\Omega_k V_{\sigma_1 \sigma_2}^{\sigma_1' \sigma_2'} (\vec{p}, \vec{k}; E_q) \Psi_q(\vec{k})_{\sigma_1' \sigma_2'} \quad .$$

Really, when substituting into the r.h.s. of (16) the expression $(2\pi)^3 \delta(\vec{p} - \vec{k}) \sqrt{1 + \vec{p}^{\,2}/M^2}\, \phi_{\sigma_1} \phi_{\sigma_2}$ taken as the first approximation to $\Psi_q(k)_{\sigma_1 \sigma_2}$ and describing the free motion it is necessary to remove all the spin indices $\sigma_1 \sigma_2$ from p to $\vec{k}$. At the same time eqs. of form (15) do not require such an additional operation as it was already performed by the transformation (13).

To complete the analogy with the nonrelativistic formalism it is convenient to pass to the Green function linear in E_p. We therefore intriduce the new scattering amplitude

$$A_{\sigma_1 \sigma_2}^{\sigma_1' \sigma_2'}(s, t) = T_{\sigma_1 \sigma_2}^{\sigma_1' \sigma_2'}(s,t) / 8\pi\sqrt{s} \tag{17}$$

with the nonrelativistic normalization to the cross section

$$\frac{d\sigma}{d\omega}(\sigma_1 \sigma_2 \to \sigma_1' \sigma_2') = \sum_{\sigma_1'' \sigma_2''} A^{*\,\sigma_1'' \sigma_2''}_{\sigma_1 \sigma_2}(s,t) A^{\sigma_1' \sigma_2'}_{\sigma_1'' \sigma_2''}(s, t). \tag{18}$$

If one defines the amplitude (177) off the energy shell as follows [5]

$$A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(s,t) = A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q})\Big|_{E_p=E_q} ;$$

$$A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q}) = T^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q})/8\pi\sqrt{4E_pE_q}\;\Big|_{E_p\neq E_q}$$

and takes the new quasipotential to be

$$\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q) = -V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)/2M\sqrt{4E_pE_k} \tag{19}$$

and the wave function

$$\Phi_q(\vec{k})_{\sigma_1\sigma_2} = \Psi_q(\vec{k})_{\sigma_1\sigma_2}/\sqrt{2E_k}$$

then eqs. (1) and (2) in terms of the new quantities take the form

$$A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q}) = -\frac{M}{4\pi}\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q};E_q) +$$

$$+\frac{1}{(2\pi)^3}\sum_{\sigma_1''\sigma_2''}\int \tilde{V}^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)\frac{d\Omega_k}{2E_q-2E_k+i\epsilon}A^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(\vec{k},\vec{q}), \tag{20}$$

$$(2E_p - 2E_p)\Phi_q(\vec{p})_{\sigma_1\sigma_2} =$$

$$= \frac{1}{(2\pi)^3} \sum_{\sigma_1'\sigma_2'} \int d\Omega_k \tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)\Phi_q(\vec{k})_{\sigma_1'\sigma_2'} \quad . \tag{21}$$

Equations (20) and (21) have a form of the direct geometrical generalization of the Lippmann-Schwinger and Schrodinger equation in the case of spin particles. In terms of the amplitude $A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}$ on the energy shell the two-particle unitarity condition (4) takes the nonrelativistic form

$$\operatorname{Im} A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{q}) = \frac{|\vec{q}|}{4\pi} \sum_{\sigma_1''\sigma_2''} \int d\omega_k \overset{*}{A}{}^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\vec{p},\vec{k}) A^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(\vec{k},\vec{q}). \tag{22}$$

The quasipotential $\tilde{V}(\vec{p},\vec{k};E_q)$ is defined through the quasipotnetial $V(\vec{p},\vec{k};E_q)$ by formula (9). To solve the problem on bound states, in papers /4-6,19/ the procedure has been suggested for constructing the quasipotential $V(\vec{p},\vec{k};E_q)$ from the matrix elements of the relativistic scattering amplitude $T(\vec{p},\vec{k})$ given by quantum field theory.

To maintain the locality of the quasipotential in the Lobachevsky momentum space, we change the procedure of cinstructing the quasipotential $\tilde{V}(\vec{p},\vec{k};E_q)$. To this end we utilize the fact of a nonuniqueness of the extension off the energy shell.

Next, consider eqs. (20), (21) to be the basic ones and the quasipotential $\tilde{V}(\vec{p},\vec{k};E_q)$ on the energy shell to be connected with $\tilde{V}(\vec{p};\vec{k};E_q)$ (i.e., the set of Feynman matrix elements) through the relation

$$\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)\Big|_{E_p=E_k=E_q} = -V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{k}(-)\vec{p},\vec{p};E_q)/4ME_q \; . \tag{23}$$

Our method of the extension of the quasipotential $\tilde{V}(\vec{p},\vec{k};E_q)$ off the energy shell consists in its definition off the energy shell by formula (19) instead of (23). It is clear that all the relations (17), (18), (4), (22) are then fulfilled. Equations (20) and (21) with the

quasipotentials $\tilde{V}^{\sigma'_1\sigma'_2}_{\sigma_1\sigma_2}(\vec{k}(-)\vec{p},\vec{p},E_q)$ (7), (12) local

in the Lobachevsky space compose the formalism which looks like the direct geometrical generalization of the corresponding integral equations in the nonrelativistic quantum mechanics.

Next we shall pass to a configurational representation adeuqate to the above 'presented formalism in the momentum space.

III. The Quasipotential Equation in the Relativistic Coordinate Space

The relativistic configurational representation (RCR) has been proposed in /5/. The difference of the RCR from the nonrelativistic configurational representation introduced through the Fourier transformation consists in the following: The Fourier transformation has the group-theoretical meaning of an expansion over the unitary irreducible representations (UIR) of the Euclidean group, i.e., over the functions $\exp(i\vec{q}\vec{r})$, while the RCR is introduced with the help of expansions over UIR of the Lorentlz group. It is quite natural to use such an expansion as the Lorentz group is the group of motion of the Lobachevsky space realized on the upper sheet of the hyperboloid (3). Therefore if we want to obtain the local expressions in a coordinate space starting from the local in the Lobachevsky space quasipotentials (7), (12), then we should make a transition to it by using expansions on the group of motion of the Lobachevsly space.

In the quasipotential equation (2) the transition to the configurational representation can be achieved through the use of two different complete sets of functions on hyperboloid (3). One of them is

$$\xi(\vec{p};\vec{n},r) = \left(\frac{p_0 - \vec{p}\cdot\vec{n}}{M}\right)^{-1-irM} \quad . \tag{24}$$

Transformations with (24) for the wave function of a spinless particle

$$\Psi_q(\vec{r})_{\sigma_1\sigma_2} = \frac{1}{(2\pi)^3} \int d\Omega_p \, \xi(\vec{p};\vec{n},r)\Psi_q(\vec{p})_{\sigma_1\sigma_2} \tag{25}$$

from the group-theoretical point of wiev is the expansion over the principle series of UIR of the Lorentz gorup /14/. The second complete set has been obtained in /15/ and contains spin-dependence (see also /16, 17/). The difference between these two sets is due to different transformation laws for spin and spinless wave functions.

However, for the quasipotential equation in form (15,21) it suffices to use the expansion over "plane waves" (24). Indeed, the Green function $(E_p - E_q - i\epsilon)^{-1}$ written in the c.m.s. of eqs. (15), and (21) like in the covariant formulation of the same eqs./4/

$$\sqrt{s_p}(\sqrt{s_p} - \sqrt{s_q})\Psi_q(\vec{p})_{\sigma_1\sigma_2} =$$

$$= \frac{4}{(4\pi)^3} \sum_{\sigma_1'\sigma_2'} \int d\Omega_k \, V_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\vec{k}(-)\vec{p},\vec{p};E_q)\Psi_q(\vec{k})_{\sigma_1'\sigma_2'} \tag{26}$$

is scalar in the spin space, and all the spin dependence contains in a quasipotential*. Consequently after a "removal" in (26), of all spin indices onto one and the same momentum $\vec{p}$, like in (15), they begin to transform under the small group of one and the same vector $\vec{p}$ (i.e., they undergo the same Wigner rotation) under the

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*Such a formalism is close to the nonrelativistic one (the Pauli equations) where only the interaction terms depend on spin.

Lorentz transformation. Due to the unitariry of the matrix $D^{1/2}\{V^{-1}(\Lambda_p,\vec{k})\}$ this Wigner rotation is factorized in the left- and right-hand sides of eqs. (15), (16) without changing the form of eqs. and potentials. Thus, for our aim it is sufficient to have a complete and orthogonal system of functions in the Lobachevsky space without account of spin dependence. We shall emphasize that spin indices σ_p like σ take numerical values $\pm 1/2$.

We shall work with the wave function equation in the fixed reference frame - c.m.s. After transformation (25) for (21) we have

$$(2E_q - 2\hat{H}_0)\Psi_q(\vec{r})_{\sigma_1\sigma_2} =$$

$$= \int d\Omega_p\, \xi(\vec{p};\vec{n},r) \sum_{\sigma_1'\sigma_2'} \int d\Omega_k V_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\vec{\Delta},\vec{p};E_q)\Psi_q(\vec{k})_{\sigma_1'\sigma_2'} \qquad (27)$$

Our purpose is to transform the containing the interaction r.h.s. of eq. (27) to the local form in a relativistic coordinate space.

To this end we pass, under the integral (27), to the wave function in the r -space

$$\Psi_q(\vec{k})_{\sigma_1\sigma_2} = \int d\vec{r}_1 \overset{*}{\xi}(\vec{k};\vec{n}_1,r_1)\Psi_q(\vec{r}_1)_{\sigma_1\sigma_2}$$

and apply the equality

$$\xi(\vec{k};\vec{n},r) = \xi(\vec{k}(-)\vec{p};\vec{n}_{\Lambda_p},r)\,\xi(\vec{p};\vec{n},r)$$

where the unit vector

$$\vec{n}_{\Lambda_p} = [M\vec{n} - \vec{p}(1 - \vec{p}\cdot\vec{n}/(p_0 + M))]/(p_0 - \vec{p}\cdot\vec{n}). \qquad (28)$$

Using the invariance of the volume element $d\Omega_k = d\Omega_{k(-)p}$ in (27) we pass to the potential in the RCR

$$V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(r,\vec{n};\vec{p};E_q)=\frac{1}{(2\pi)^3}\int d\Omega_\Delta \overset{*}{\xi}(\vec{\Delta};\vec{n},r)V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{\Delta},\vec{p};E_q). \tag{29}$$

By means of (29) in /2/ the transform of the potential (7) has been found is the relativistic analog of tensor forces and has the form

$$V(\vec{r})=V_S(r)(\vec{\sigma}_1\cdot\vec{\sigma}_2)+V_T(r)S_{1,2},$$
$$S_{1,2}=3(\vec{\sigma}_1\vec{n})(\vec{\sigma}_2\vec{n})-(\vec{\sigma}_1\vec{\sigma}_2), \tag{30}$$

where

$$V_S(r)=\frac{\mu^2}{3}V_{YUK}(r)-\frac{8\pi}{3}\delta(r^2+1/M^2)\delta(\vec{n})/r \tag{31}$$

and

$$V_T(r)=\frac{1}{3}\frac{r^2}{(r+i/M)(r+2i/M)}\times$$
$$\times[\mu^2+\frac{3\mu}{r}(1-\mu^2/2M^2)\mathrm{th}\,rMa/\sqrt{1-\mu^2/4M^2}+$$
$$+\frac{1}{r^2}\frac{3-2(\mu^2/M^2)\cdot(1-\mu^2/4M^2)-3/2\,\mathrm{ch}\,rMa}{1-\mu^2/4M^2}]V_{YUK}(r), \tag{32}$$

$$V_{YUK.}(r) = \begin{cases} \dfrac{1}{4\pi r}\,\dfrac{\mathrm{ch}\, rMa}{\mathrm{sh}\, rM\pi}, & \mu^2 < 4M^2, \quad a = \arccos\dfrac{\mu^2 - 2M^2}{2M^2} \\[2ex] \dfrac{1}{4\pi r}\,\dfrac{\cos rMb}{\mathrm{sh}\, rM\pi}, & \mu^2 > 4M^2, \quad b = \mathrm{Arch}\dfrac{\mu^2 - 2M^2}{2M^2} \end{cases} \qquad (33)$$

Analogously, the spin-orbital interaction term from (12)

$$V_{SL}(\vec{\Delta},\vec{p}) = -ig_V^2\,\frac{4p_0}{M}\,\frac{(\vec{S}\cdot[\vec{p}\times\Delta]}{\mu^2 + 4\vec{\kappa}^2}, \quad \vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2)/2 \qquad (34)$$

after transformation (29) in the RCR takes the form

$$V_{SL}(\vec{r},\vec{p}) = \frac{g_V^2}{(2\pi)^3}\,\frac{4p_0}{M}\,\frac{1}{r + i/M}\,\frac{1}{2i/M}\,\times$$

$$\times\,[1 - \exp(-2\frac{i}{M}\frac{\partial}{\partial r})](\vec{S}\cdot\vec{L})V_{YUK.}(r) \qquad (35)$$

where the orbital momentum operator L is expressed through the momentum operator $\hat{\vec{P}} = -i\vec{\nabla}_{fd}$ (see [2]) by formula

$$\vec{L} = [\vec{r}\times\hat{\vec{P}}]\exp(-\frac{i}{M}\frac{\partial}{\partial r}).$$

On substituting (29) into (27) the r.h.s. takes the form

$$\int d\vec{r}\,\sum_{\sigma_1'\sigma_2'}\int d\Omega_p\,\xi(\vec{p};\vec{n},r)\,\xi^*(\vec{p};\vec{n}_1,r_1)V_{\sigma\sigma}^{\sigma_1'\sigma_2'}(r_1,\vec{n}_1,\Lambda_p;\vec{p},E_q)\Psi_q(\vec{r}_1)_{\sigma_1'\sigma_2'} \quad (36)$$

It is clear from the preceding section that the dependence of the potential $V(r_1, \vec{n}_{1\Lambda_p}; \vec{p}; E_q)$ on the unit vector $\vec{n}_{1\Lambda_p}$ is concentrated only in the spin structures $S^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}$. As a result, the potential can be represented as follows

$$V^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}(r, \vec{n}_{\Lambda_p}; \vec{p}; E_q) = V(r; \vec{p}; E_q) S^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}(\vec{p}; \vec{n}_{\Lambda_p}). \tag{37}$$

The function $V(r; \vec{p}; E_q)$ dependent on the coordinate modulus only can be taken out of the integral over momentum provided the vector $\vec{p}$ is replaced by the operator $\hat{\vec{p}}$ (see Appendix A to [2]). As a result, (36) takes the form

$$\int d\vec{r}_1 V(r; \hat{\vec{P}}; E_q) \sum_{\sigma_1' \sigma_2'} Z^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}(\vec{r}, \vec{r}_1) \Psi_q(\vec{r}_1)_{\sigma_1' \sigma_2'}, \tag{38}$$

where the function $Z(r, r_1)$ is defined by the spin structures of the potential

$$Z^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}(\vec{r}, \vec{r}_1) = \int d\Omega_p \xi(\vec{p}; \vec{n}, r) S^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}(\vec{p}, \vec{n}_{\Lambda_p}) \xi^*(\vec{p}; \vec{n}_1, r_1), \tag{39}$$

It is clear from (39) that for the part of the potential independent of spin variables, i.e., $S^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2} \sim \delta^{\sigma_1'}_{\sigma_1} \cdot \delta^{\sigma_2'}_{\sigma_2}$, the interaction is described in the local way

$$(2E_q - 2\hat{H}_0) \Psi_q(\vec{r})_{\sigma_1 \sigma_2} = \sum_{\sigma_1' \sigma_2'} V^{\sigma_1' \sigma_2'}_{\sigma_1 \sigma_2}(r; \hat{\vec{P}}; E_q) \Psi_q(\vec{r})_{\sigma_1' \sigma_2'} \tag{40}$$

since the function $Z(\vec{r}, \vec{r}_1)$ turns into the δ-function.

Next we shall show that for other forms of interaction an analogous localization can be achieved too. Thus, for the spin-orbital interaction the Z-function appears to be proportional to the δ-function, and for the tensor forces the part of the function $Z(\vec{r},\vec{r}_1)$ proportional to the δ-function can be separated. The remaining integral term is a higher-order relativistic correction to the obtained tensor potential.

IV. Equations for Spin-Orbital and Tensot Forces

In accordance with (12) and (34) for the spin-orbital interaction the structure of the potential is

$$S(\vec{p},\vec{n}_{\Lambda_p}) = (\vec{S}\cdot[\vec{n}_{\Lambda_p}\times\vec{p}]) = (\vec{S}\cdot[\vec{n}\times\vec{p}])\frac{M}{\vec{p}_0 - \vec{p}\cdot\vec{n}} .$$

Then, by definition (39)

$$Z(\vec{r},\vec{r}_1) = \exp(-\frac{i}{M}\frac{\partial}{\partial r})\int d\Omega_p(\vec{S}\cdot[\vec{n}\times\vec{p}])\xi(\vec{p};\vec{n},r)\xi^*(\vec{p};\vec{n}_1,r_1) =$$

$$= \frac{1}{r}(\vec{S}\cdot\vec{L})\delta(\vec{r}-\vec{r}_1). \qquad (41)$$

Therefore the integral in (38) is omitted and the equation reads

$$(2E_q - 2\hat{H}_0)\Psi_q(\vec{r}) = \hat{V}_{SL}(\vec{r})\Psi_q(\vec{r}),$$

where

$$\hat{V}_{SL}(\vec{r}) = -\frac{g_V^2}{(2\pi)^3}\frac{r+i/M}{r-i/M}\frac{1}{r}\times$$

$$\times\frac{1}{2i/M}[\exp(\frac{i}{M}\frac{\partial}{\partial r}) - \exp(-\frac{i}{M}\frac{\partial}{\partial r})]V_{YUK.}\frac{\hat{H}_0}{M}(\vec{S}\cdot\vec{L}).$$

The angular dependence of the tensor interaction is contained in the operator

$$(\vec{\sigma}_1 \vec{n}_{\Lambda_p})(\vec{\sigma}_2 \vec{n}_{\Lambda_p}), \tag{42}$$

where the unit vector n_{Λ_p} is given by (28). For (42) the function $Z(\vec{r},\vec{r}_1)$ is splitted into two parts

$$Z(\vec{r},\vec{r}_1) = \exp(-2\frac{i}{M}\frac{\partial}{\partial r})(\vec{\sigma}_1 \vec{n})(\vec{\sigma}_2 \vec{n})\delta(\vec{r}-\vec{r}_1) + \tilde{Z}(\vec{r},\vec{r}_1). \tag{43}$$

The first term obviously gives the local relativistic tensor potential, the second term as compared with the first one is a relativistic correction of the order $1/M$:

$$\tilde{Z}(\vec{r},\vec{r}_1) = \frac{1}{M}\exp(-2\frac{i}{M}\frac{\partial}{\partial r})\int d\Omega_p\, \xi(\vec{p};\vec{n},r)\times$$

$$\times\{-[(\vec{\sigma}_1\vec{n})(\vec{\sigma}_2\vec{p})+(\vec{\sigma}_1\vec{p})(\vec{\sigma}_2\vec{n})](1-\frac{\vec{p}\cdot\vec{n}}{p_0+M})+$$

$$+\frac{1}{M}(\sigma_1 p)(\sigma_2 p)(1-\frac{\vec{p}\cdot\vec{n}}{p_0+M})^2\}\xi^*(\vec{p};\vec{n}_1,r_1). \tag{44}$$

Hence, the r.h.s. of the quasipotential equation takes the form

$$-g_V^2[V_S(r)(\vec{\sigma}_1\cdot\vec{\sigma}_2) + V_T^*(r-2i/M)S_{1,2}]\Psi_q(\vec{r}) -$$

$$-g_V^2\int d\vec{r}_1\, V_T^*(r_1)\tilde{Z}(\vec{r},\vec{r}_1)\Psi_q(\vec{r}_1). \tag{45}$$

Thus we see that the first term in the r.h.s. of (45) is the local relativistic tensor interaction. Its difference from the nonrelativistic interaction is contained in new expressions for radial functions (31) and (32).

Note, that if one desired, in the relativistic integral term (45) the integral in any order of p/M can be taken. To this end, using (30) and performing standard algebraic transformations in the function $\tilde{Z}(\vec{r},\vec{r}_1)$ one can easily obtain terms proportional to the δ-function up to the third order in p/M. This procedure may be extended further to any order in p/M provided the remaining integrand in (14) is expanded in power series of p/M and the orthogonality condition of "plane waves" is used. However, at a phenomenological description one may not consider the correction terms to the local part of the tensor interaction as it itself is completely relativistic. In this case the equation with the relativistic tensor potential is written in complete analogy with the Schrodinger equation

$$(2E_q - 2\hat{H}_0)\Psi_q(\vec{r})_{\sigma_1\sigma_2} = \\ = -g_V^2 \sum_{\sigma_1'\sigma_2'} [V_S(r)(\vec{\sigma}_1\vec{\sigma}_2) + V_T^*(r-2i/M)S_{1,2}]^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2} \Psi_q(\vec{r})_{\sigma_1'\sigma_2'} . \tag{46}$$

Conclusion

Thus, we have constructed the three-dimensional formalism for description of the interaction of two relativistic spin particles in the coordinate space. The transition to the relativistic coordinate space has allowed us to obtain from the local in the Lobachevsky space quasipotential the local potentials in the relativistic coordinate space.

Note here that while the transformation from the four-dimensional form of the Feynmam matrix elements to the three-dimensional representation in the Lobachavsky space (see (7) and (12)) plays the same role as passing to the Foldy-Wouthuysen representation, the new relativistic generalization of the relative coordinate proposed in ref. /5/ is the most convenient tool for formulating the theory in the coordinate space.

Further our aim will be to apply the developed here formalism to the relativistic description to composite particles. In a sunsequent paper we shall introduce a system of partial equations for the system of two particles with spins 1/2.

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