ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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ДУБНА

E2 - 10259

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FIXED ORDER MATRIX ELEMENTS OF THE PARAFERMI OPERATORS

1976

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Submitted to "Rep.Math.Phys."

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E2 - 10259

Матричные элементы парафермн-операторов заданного порядка парастатистики

Найдены явные формулы для матричных элементов парафермиоператоров в представлении с произвольным порядком парастатистики p.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований
Дубна 1976

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E2 - 10259

Fixed Order Matrix Elements of the Parafermi Operators

Explicit representation formulae are found for the parafermi operators of arbitrary order p.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research
Dubna 1976

In 1953 Green [1] pointed out that the quantization rules can be considerably generalized if one does not impose the postulate for the commutator or the anti-commutator of two fields to be a c-number. As a result the Bose and Fermi commutation relations were replaced by more general threelinear structure relations. For the parafermi operators $a_i^{\pm}$, $i = 1, 2, \ldots$, the double commutation relations can be written in a compact form as follows ($\xi, \eta, \epsilon = \pm$ or ± 1)

$$[[a_i^{\xi}, a_j^{\eta}], a_k^{\epsilon}] = \frac{1}{2}(\eta - \epsilon)^2 \delta_{jk} a_i^{\xi} - \frac{1}{2}(\xi - \epsilon)^2 \delta_{ik} a_j^{\eta} \quad . \qquad (1)$$

All single vacuum irreducible representations of the paraoperators were classified by Greenberg and Messiah [2]. They have shown that to every positive integer p, called the order of the parastatistics, there corresponds an irreducible representation. In principle, it is known how one can find the matrix elements of the creation and annihilation operators. For every fixed order of the statistics this could be done through the Green ansatz [1]. In a more general case one can use the interrelations between the parafermi operators and the algebra of the orthogonal group [3]. Nevertheless, general expressions for the representations of the paraoperators were not written. In the present note we write down explicit formulae for the matrix elements of the parafermi operators valid for arbitrary order of the parastatistics. Since the derivation is somewhat complicated whereas the results are comparatively simple we give only the final expressions.

Consider first a finite number $a_1^{\pm}, \dots, a_n^{\pm}$ of para-fermi creation and annihilation operators. With energy fixed positive integer p we associate a set $K = \{k\}$ of symmetric matrices

$$k = \begin{matrix} k_{11} & \dots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \dots & k_{nn} \end{matrix} \tag{2}$$

with integer nonnegative matrix elements k_{ij}, satisfying the inequalities

$$0 \le \mathrm{Tr}\, k \le p$$

$$0 \le \sum_{j=1}^{n} k_{ij} \le p, \qquad i = 1, \dots, n. \tag{3}$$

To every such matrix k we put in correspondence a symbol

$$|k_{11}, k_{22}, \dots, k_{nn}, k_{12}, k_{13}, \dots, k_{1n}, k_{23}, k_{24}, \dots, k_{2n}, k_{34}, \dots, k_{3n}, \dots, k_{n-1,n}>. \tag{4}$$

Denote by W_p the linear space which is a real linear envelope of all vectors (4), for which (3) holds. Let ϵ_{ij} be a squared n-dimensional antisymmetric matrix, $\epsilon_{ij} = 1$ for all $i < j = 1, \dots, n$. The representation of the para-fermi operators of order p is realized in W_p through the following relations,

$$a_i^+ |k_{11}, \dots, k_{n-1,n}> = |\dots, k_{ii} + 1, \dots> +$$
$$+ \sum_{j=1}^{n} \epsilon_{ij} k_{jj} |\dots, k_{jj} - 1, \dots, k_{ij} + 1, \dots>, \tag{5a}$$

$$a_i^- |k_{11}, \dots, k_{n-1,n}> =$$

$$= k_{ii} [p - Trk - \sum_{j=1}^{n} k_{ij} + k_{ii} + 1] | \ldots, k_{ii} - 1, \ldots > +$$

$$+ \sum_{j=1}^{n} \epsilon_{ij} k_{ij} | \ldots, k_{jj} + 1, \ldots, k_{ij} - 1, \ldots > +$$

$$+ \sum_{\substack{j,r=1 \\ r \neq i}}^{n} \epsilon_{rj} \epsilon_{ij} k_{rr} k_{ij} | \ldots, k_{rr} - 1, \ldots, k_{rj} + 1, \ldots, k_{ij} - 1, \ldots >. \qquad (5b)$$

In the left-hand side of the above relations $|k_{11},\ldots,k_{n-1,n}>$ is abbreviation for the vector (4) .In the vectors from the right-hand side we indicate only the labels which differ from the corresponding indices on the left. It is to be understood that the vector $|k_{11},\ldots,>$ vanishes if some of its labels k_{ij} are negative or do not fulfill the inequalities (3).

In our notations the vacuum corresponds to the vector $|k_{11},\ldots,k_{n-1,n}>$ with all labels $k_{ij} = 0$,

$$|0> = | 0,0,\ldots, 0> .$$

From (5) we have

$$a_i^- a_j^+ |0> = \delta_{ij} \; p |0> \qquad (6)$$

as it should be for the representation with order of parastatistics p.

As one may easily see from the relations (5), the parafermi operators can be represented by differential operators in the following way. Introduce the variables x_{ij}, where $i,j = 1,\ldots,n$ and for $i \neq j$ $x_{ij} = -x_{ji}$. Let

$\partial_{ij} = \dfrac{\partial}{\partial x_{ij}}$. Then

$$a_i^- = \partial_{ii} [p - \sum_{j=1}^{n} x_{jj} \partial_{jj} - \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} \partial_{ij} + 1] +$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n} x_{jj}\,\partial_{ij} + \sum_{\substack{j,k=1\\ j\neq k\neq i}}^{n} x_{kj}\,\partial_{kk}\partial_{ij}\ ,$$

$$a_i^{+} = x_{ii} + \sum_{\substack{j=1\\ j\neq i}} x_{ij}\,\partial_{jj}\ . \qquad (7)$$

In these notations the basis vectors (4) are

$$|k_{11},\ldots,k_{n-1,n}> = \prod_{1\leq i\leq j\leq n} (x_{ij})^{k_{ij}}\ : \qquad (8)$$

The representation formulae (5) can be immediately generalized to the case of infinite n. For this purpose all sums in (5) should be extended to infinity. Because of the inequalities (3) no more than a finite number of labels in every vector $|k_{n},\ldots,k_{n-1,n}>$ are different from zero. Therefore all infinite sums in (5) will be convergent.

Because of the interrelations between a given number of pairs of parafermi creation and annihilation oeprators and the algebra B_n of the orthogonal group in $2n+1$ dimensions /4/ the relations (5) generate irreducible representation of the algebra B_n. The generators of the algebra are given with the operators $a_i^{\xi},[a_j^{\eta}, a_k^{\delta}];\ \xi,\eta,\delta = \pm\ ;\ i,j,k = 1,\ldots,n$.

The vacuum is the highest weight of the representation. The orthogonal signature of this representation is $(\frac{p}{2},\ldots,\frac{p}{2})$ /5/. If one does not impose the inequalities (3) the formulae (5), nevertheless, do define representation of B_n, however it becomes infinite dimensional.

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Received by Publishing Department
on November 29, 1976.