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LIE ALGEBRAICAL ASPECTS OF QUANTUM STATISTICS. PARAFERMI STATISTICS

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Алгебраические аспекты квантовой статистики. Параферми-статистика.

Изучаются алгебраические свойства квантового условия для спинорных полей $[[a_i^+, a_i^-], a_j^\pm] = \pm 2\delta_{ij} a_j^\pm$.

Оказывается, параферми-статистика является одним из решений этого соотношения - это простая алгебра Ли минимального ранга, порожденная входящими в вышеописанное равенство операторами $a_i^\pm$. Мы указываем также на некоторые другие решения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Lie Algebraical Aspects of Quantum Statistics. Parafermi Statistics.

We study in detail the Lie algebraical properties of the quantization condition for spinor fields

$$[[a_i^+, a_i^-], a_j^\pm] = \pm 2\delta_{ij} a_j^\pm .$$

It turns out, that the parafermi statistics is one particular solution of this relation: it is the minimal rank simple Lie algebra generated by the operators $a_i^\pm$ entering into the above relation. We point out some other solutions.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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In 1953 Green[1] showed that the quantum statistics can be considerably generalized if one quantizes fields according to a weaker system of axioms, abandoning the usually accepted c-number postulate, i.e., the requirement for the commutator or the anticommutator of two fields to be a c-number. As a consequence the Fermi anticommutation relations

$$\{a_i^{\xi}, a_j^{\eta}\} = \frac{1}{4}(\xi-\eta)^2 \delta_{ij}, \quad \xi, \eta = \pm \quad \text{or} \pm 1 \tag{1}$$

were replaced by a weaker system of three linear commutation relations

$$[[a_i^{\xi}, a_j^{\eta}], a_k^{\epsilon}] = \frac{1}{2}(\eta-\epsilon)^2 \delta_{jk} a_i^{\xi} - \frac{1}{2}(\xi-\epsilon)^2 \delta_{ik} a_j^{\eta}. \tag{2}$$

It is important to point out that the parafermi relations (2) do not follow necessarily from the weakened quantization procedure and in fact the commutation relations, which are a unique consequence, read as

$$[[a_i^+, a_i^-], a_j^{\pm}] = \pm 2\delta_{ij} a_j^{\pm}. \tag{3}$$

On the basis of what we call here main quantization condition (3) Green has postulated the more rigid set of double commutation relations (2). They can be also derived from (3) if one imposes the invariance of these relations under unitary transformations[2].

The commutation relations (2) exhibit some remarkable Lie algebraical properties. It turns out that the parafermi creation and annihilation operators can be viewed as a part of the generators of the algebra of the orthogonal group, generating the whole algebra [3]. To make the statement more precise consider a finite number of operators $a_1^{\pm},\ldots,a_n^{\pm}$. Then the linear envelope over **C** of the operators

$$a_i^{\xi},\ [\,a_j^{\eta},a_k^{\epsilon}\,];\quad i,j,k=1,\ldots,n\,;\quad \xi,\eta,\epsilon=\pm \tag{4}$$

is isomorphic to the classical Lie algebra B_n [4], whereas as a real Lie algebra it is so $(n,n+1)$.

The main purpose of the present note is to study in more detail the Lie algebraical properties of the main quantization condition (3). In order to use Lie algebraical methods in a rigorous way we restrict ourselves to case of a finite number of operators. The generalization of the results to the infinite number of operators is straightforward. We introduce first one definition.

Definition 1. The operators $a_1^{\epsilon},\ldots,a_n^{\epsilon}$ are called creation $(\epsilon=+)$ and annihilation $(\epsilon=-)$ operators if they satisfy the main quantization condition (3).

Apart from (3) these operators are arbitrary and they can satisfy several other relations. In particular, like in the parafermi case, the creation and annihilation operators could close or generate the Lie algebra*. The main result we prove in this paper is contained in the following theorem.

Theorem. The semisimple Lie Algebra $\mathfrak{A}$ of rank n is generated by n pairs of creation and annihilation operators if and only is it is a direct sum of classical Lie algebras $B_{m_1},\ldots,B_{m_k}$,

$$\mathfrak{A}=B_{m_1}+\ldots+B_{m_k}, \tag{5}$$

where

$$m_1+\ldots+m_k=n.$$

First we prove some preliminary results.

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* By the Lie algebra everywhere in the paper we meain the Lie algebra of finite dimension.

Lemma 1. Let $\tilde{\mathfrak{A}}$ be a semisimple Lie algebra generated by n pairs $a_1^{\pm},\dots,a_n^{\pm}$ of creation and annihilation operators. The elements

$$h_i=\frac{1}{2}[a_i^-,a_i^+],\ i=1,\dots,n \tag{6}$$

are contained in the Cartan subalgebra $\tilde{\mathcal{H}}$ of $\tilde{\mathfrak{A}}$. The rank of $\tilde{\mathfrak{A}}$ is equal or larger than n.

Proof.

We observe that the operators

$$a_i^+,\ h_i,\ a_i^- \tag{7}$$

constitute a basis of a subslgebra $A_1^i\subset\tilde{\mathfrak{A}}$, isomorphic to the algebra $A_1=s\ell(2,C)$. The operator h_i belongs to the Cartan subalgebra of A_1^i.

Consider the adjoint representation as $\tilde{\mathfrak{A}}$ of $\tilde{\mathfrak{A}}$. The latter restricted to A_1^i gives in general a reducible representation of A_1^i in the space $\tilde{\mathfrak{A}}$. As is known, the basis in any representation space of A_1^i and in particular in $\tilde{\mathfrak{A}}$ can always be chosen such that the Cartan element $\mathrm{ad}h_i$ will be diagonal. By definition such elements are called semisimple elements of $\tilde{\mathfrak{A}}$. From the main quantization condition (3) we have

$$[h_i,h_j]=0;\quad i,j=1,\dots,n. \tag{8}$$

Hence all operators $\mathrm{ad}\,h_1,\dots,\mathrm{ad}\,h_n$ can be simultaneously diagonalized in $\tilde{\mathfrak{A}}$ and therefore the linear envelope of $h_1,\dots,h_n$ is a semisimple subspace of $\tilde{\mathfrak{A}}$. The proof of the Lemma now follows from the observation that any semisimple subspace $\mathcal{H}$ of $\tilde{\mathfrak{A}}$ can be enlarged to a maximal semisimple subspace $\tilde{\mathcal{H}}\subset\tilde{\mathfrak{A}}$ and $\tilde{\mathcal{H}}$ is a Cartan subalgebra of $\tilde{\mathfrak{A}}$ /5/.

The elements $h_1,\dots,h_n$ are linearly independent. Indeed suppose

$$\sum_{i=1}^{n}\xi^i h_i=\frac{1}{2}\sum_{i=1}^{n}\xi^i[a_i^-,a_i^+]=0.$$

From (3) we obtain

$$[\sum_{i=1}^{n}\xi^i h_i,\ a_j^+]=-\xi^j a_j^+=0$$

and since $a_j^+ \neq 0$, we conclude that $\xi^j = 0$ for all $j = 1,\dots,n$ Therefore

$$\text{rank } \tilde{\mathcal{G}} = \dim \tilde{\mathcal{H}} \geq \dim \mathcal{H} = n , \tag{9}$$

which proves the Lemma.

Suppose now we are looking for a semisimple algebra $\mathcal{G}$ of rank n. From (9) we conclude that $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}$. Denote by $\overset{*}{\mathcal{H}}$ the space dual to $\mathcal{H}$. As an ordered basis in $\mathcal{H}$ we choose the vectors (6),

$$h_1, h_2, \dots, h_n \tag{10}$$

whereas for a basis in $\overset{*}{\mathcal{H}}$ we take the dual to (10) basis

$$\overset{*}{h}{}^1, \overset{*}{h}{}^2, \dots, \overset{*}{h}{}^n , \tag{11}$$

where by definition

$$\overset{*}{h}{}^i (h_j) = \delta^i_j . \tag{12}$$

Let h be an arbitrary element from the Cartan subalgebra. From (3) and (6) we have

$$[h, a_i^{\pm}] = \mp \overset{*}{h}{}^i (h) a_i^{\pm} \tag{13}$$

Therefore the operators $a_1^{\pm},\dots, a_n^{\pm}$ are root vectors of the algebra $\mathcal{G}_*$ we are looking for. The corresponding to $a_i^{\pm}$ root is $\mp \overset{*}{h}{}^i$.

Corollary 1. If the semisimple Lie algebra $\mathcal{G}$ of rank n is generated by n pairs of creation and annihilation operators, then with respect to the basis (10) of the Cartan subalgebra the creation (annihilation) operators are negative (positive) root vectors. The correspondence with their roots is

$$a_i^{\pm} \leftrightarrow \mp \overset{*}{h}{}^i \tag{14}$$

Let Σ be the root system of the semisimple Lie Algebra $\mathfrak{G}$.

Definition 2. We call the subset $\phi = (\omega_1, \ldots, \omega_m) \subset \Sigma$ of roots complete if an arbitrary root of $\mathfrak{G}$ is a linear combination with integer coefficients of the vectors from ϕ.

Clearly the positive and negative root vectors $e_{\pm\omega_i}$, $\omega_i \in \phi$, generate the whole algebra $\mathfrak{G}$.

Lemma 2. The semisimple Lie algebra $\mathfrak{G}$ of rank n is generated by n pairs of creation and annihilation operators if and only if it contains a complete system ϕ of orthogonal with respect to Cartan-Killing form roots.

Proof. Let $\phi = (\omega_1, \ldots, \omega_m)$ be a system with the above properties. The Cartan Killing form

$$(x, y) = \mathrm{Tr}\, \mathrm{ad}\, x\, \mathrm{ad}\, y\,; \quad x, y \in \mathfrak{G} \tag{15}$$

defines a scalar product in the real linear envelope of all roots. Therefore $m \leq n$. On the other hand ϕ is a complete system of vectors and hence $m \geq n$. Thus $m = n$.

Introduce a Cartan-Weyl basis in $\mathfrak{G}$, that is choose the roots $\omega_i \in \Sigma$ and the root vectors e_{ω_i},

$$\omega_i \leftrightarrow e_{\omega_i}$$

in such a way that

$$\begin{aligned} &[e_{\omega_i}, e_{-\omega_i}] = \omega_i \\ &[h, e_{\omega_j}] = (h, \omega_j) e_{\omega_j}, \qquad \forall\, h \in \mathcal{H} \\ &[e_{\omega_i}, e_{\omega_j}] = N_{\omega_i \omega_j} e_{\omega_i + \omega_j}, \quad N_{\omega_i \omega_j} - \text{number.} \end{aligned} \tag{16}$$

Let $\omega_i, \omega_j \in \phi$. Taking into account that

$$(\omega_i, \omega_i) = \delta_{ij} (\omega_i, \omega_i) \tag{17}$$

from the first two commutators (16) we have

$$[[e_{\omega_i}, e_{\omega_i}], e_{\pm\omega_j}] = \pm \delta_{ij} (\omega_i, \omega_i) e_{\pm\omega_j}. \tag{18}$$

Define

$$a_i^{\pm} = \sqrt{\frac{2}{(\omega_i, \omega_i)}}\, e_{\pm\omega_i} \quad . \tag{19}$$

In terms of $a_i^{\pm}$ (18) can be written as

$$[[a_i^+, a_i^-], a_j^{\pm}] = \pm 2\delta_{ij} a_j^{\pm} \tag{20}$$

and therefore $a_1^{\pm}, \ldots, a_n^{\pm}$ are creation and annihilation operators. Since ϕ is a complete system, these operators generate the algebra $\mathfrak{A}$. On the contrary, let the semisimple Lie algebra of rank n be generated by n pairs $a_1^{\pm}, \ldots, a_n^{\pm}$ of creation and annihilation operators. As we know from the *Corollary 1,* a_i^+ and a_i^- are negative and positive roots vectors correspondingly. Therefore

$$[a_i^+, a_i^-] = c\omega_i \quad , \tag{21}$$

where c is a number and ω_i is the root of a_i^+. Since the root vectors $a_1^{\pm}, \ldots, a_n^{\pm}$ generate $\mathfrak{A}$ the system $\phi = (\omega_1, \ldots, \omega_n)$ is complete. Whereover from (21) and the main quantization condition (3) we have that for $i \neq j$

$$[\omega_i, a_j^{\pm}] = \pm (\omega_i, \omega_j)\, a_j^{\pm} = 0 \tag{22}$$

and therefore

$$(\omega_i, \omega_j) = 0 \quad i \neq j \, . \tag{23}$$

Thus, ϕ is a complete orthogonal system of roots.

Proof of the theorem

In view of *Lemma 2* we have to find all semisimple Lie algebras containing complete orthogonal root systems. The problem can be simplified further. Every semisimple Lie algebra is a direct sum

$$\mathfrak{A} = \mathfrak{A}_1(m_1) + \ldots + \mathfrak{A}_k(m_k) \tag{24}$$

of simple subalgebras $\mathcal{A}_i(m_i)$, $m_i = \text{rank}\,\mathcal{A}_i(m_i)$. The algebra $\mathcal{A}$ contains a complete orthogonal root system ϕ if and only if every component $\mathcal{A}_i$ contains such a system ϕ_i. This result follows from the fact that the Cartan subalgebras of different $\mathcal{A}_i$ are orthogonal to each other. In this case

$$\phi = \bigcup_{i=1}^{k} \phi_i . \qquad (25)$$

Thus, it is enough to determine the simple algebras containing systems ϕ with the necessary properties. Looking at the root systems of the simple Lie algebras, we can easily convince ourselves that only the algebras B_n, $n=1,2,\ldots$, contain such systems. In an orthogonal basis $h^1,\ldots,h^n$ the root system Σ of B_n is given with the vectors

$$\Sigma = \{\, \xi h^i \,,\ \eta h^j + \delta h^k \mid j<k\,;\ i,j,k=1,\ldots,n\,;\ \xi,\eta,\delta = \pm \} \qquad (26)$$

This proves the theorem

Corollary 2. The simple Lie algebra $\mathcal{A}$ of rank n is generated from n pairs of creation and annihilation operators if only it is isomorphoc to the classical algebra B_n.

In this case one can easily derive the three-linear commutation relation (2) using for instance the following matrix realization of the creation and annihilation operators

$$a_i^- = 2e_{i,0} - e_{0,-i}$$
$$a_i^+ = -2e_{-i,0} + e_{0,i} \,, \quad i=1,\ldots,n . \qquad (27)$$

Here e_{pq} is a $(2n+1)$-square matrix, $p,q=0,1,\ldots,n,-1,\ldots,-n$ with 1 on the intersection of the p-th row and q-th column and zero elsewhere.

So we see that the parafermi commutation relations (2) appear as one particular solution: the simple Lie algebraic solution of the main quantization condition (3). Another limiting possibility is

$$\mathfrak{A} = B_1 + B_1 + \ldots + B_1 \tag{28}$$

In case $n \to \infty$ and for instance, if $n = \vec{p}$ runs over the three-dimensional momentum space, the commutation relations between the operators

$$J_1^{\circ} = \frac{1}{2}[a^-(\vec{p}) + a^+(\vec{p})]$$

$$J_2^{\circ} = \frac{i}{2}[a^-(\vec{p}) - a^+(\vec{p})] \tag{29}$$

$$J_3^{\circ} = \frac{1}{2}[a^+(\vec{p}), a^-(\vec{p})]$$

resemble the current algebra commutation relations in x -space,

$$[J_{\alpha}^{\circ}(\vec{p}), J_{\beta}^{\circ}(\vec{q})] = i\epsilon_{\alpha\beta\gamma}\delta(\vec{p}-\vec{q})J_{\gamma}^{\circ}(\vec{p}) \tag{30}$$

One may consider several other intermediate cases, which in the language of the parafermi quark model will correspond to commuting parafermi fields.

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