СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 10242

B.Timmermann, W.Timmermann

ON ULTRASTRONG AND ULTRAWEAK TOPOLOGIES ON ALGEBRAS OF UNBOUNDED OPERATORS

1976

E2 - 10242

B.Timmermann, W.Timmermann

ON ULTRASTRONG AND ULTRAWEAK TOPOLOGIES ON ALGEBRAS OF UNBOUNDED OPERATORS

Тиммерман Б., Тиммерман В. **E2 - 10242**

Ультрасильные и ультраслабые топологии в алгебрах неограниченных операторов

В алгебрах неограниченных операторов введены топологии, обобщающие известные ультрасильную и ультраслабую топологии в алгебрах ограниченных операторов. Исследование некоторых соответствий между структурой и непрерывностью линейных функционалов на таких алгебрах приводит к результатам подобным случаю ограниченных операторов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований

Дубна 1976

Timmermann B., Timmermann W. **E2 - 10242**

On Ultrastrong and Ultraweak Topologies on Algebras of Unbounded Operators

On algebras of unbounded operators there are introduced topologies which are generalizations of the well-known ultrastrong and ultraweak topologies on algebras of bounded operators. The investigation of some relationships between the structure and the continuity of linear functionals on such algebras leads to results which are quite similar to the bounded case.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research

Dubna 1976

0. INTRODUCTION

This paper deals with some investigations of the structure of linear functionals on algebras of unbounded operators on Hilbert space. Especially, there are regarded some relationships between normality of functionals and its continuity with respect to some topologies on such algebras.

These topologies will be defined in section 2, and they are generalizations of the well-known ultrastrong and ultraweak topologies on algebras of bounded operators.

In section 3 we will describe results which are quite analogous to the bounded case. For example, the class of ultrastrongly continuous functionals coincides with that of ultraweakly continuous functionals, a positive functional is normal if and only if it is ultrastrongly continuous. A result about dual pairs is also generalized to the unbounded case.

1. PRELIMINARIES

In this section we collect some definitions and notations used in the sequel. Details can be found for example in /3-5/ . Let $\mathcal{H}$ be a separable Hilbert space with the scalar product (,) and the norm $\|\cdot\|$. If $\mathfrak{D} \subset \mathcal{H}$ is a dense linear manifold, by $\mathcal{L}^+(\mathfrak{D})$ we denote the set of all linear operators A with $A\mathfrak{D} \subset \mathfrak{D}$, $\mathfrak{D} \subset \mathfrak{D}(A^*)$, $A^*\mathfrak{D} \subset \mathfrak{D}$. With the involution $A \longrightarrow A^+ = A^* \restriction \mathfrak{D}$ and the usual operations $\mathcal{L}^+(\mathfrak{D})$ becomes a $*$-algebra. A $*$-subalgebra $\mathcal{A} = \mathcal{A}(\mathfrak{D})$ of $\mathcal{L}^+(\mathfrak{D})$ containing the identity I will be called Op*-algebra. An Op*-algebra $\mathcal{A}(\mathfrak{D})$ induces a topology $t_{\mathcal{A}}$ on $\mathfrak{D}$ given by the seminorms $\varphi \longrightarrow \|A\varphi\|$ for any $A \in \mathcal{A}(\mathfrak{D})$.

Denote $t_{\mathcal{L}^+(\mathcal{D})}$ by t_+. An Op* -algebra $\mathcal{A}(\mathcal{D})$ is said to be closed (selfadjoint, resp.) if

$$\mathcal{D} = \bigcap_{A \in \mathcal{A}} \mathcal{D}(\bar{A}) \qquad (\mathcal{D} = \bigcap_{A \in \mathcal{A}} \mathcal{D}(A^*) \quad \text{, resp. })$$

By $S_1(\mathcal{A})$ we denote the set

$$S_1(\mathcal{A}) = \{ T \in \mathcal{L}^+(\mathcal{D}) : AT,\ AT^* \text{ nuclear for all } A \in \mathcal{A}(\mathcal{D}) \}$$

We will write $S_1(\mathcal{D})$ instead of $S_1(\mathcal{L}^+(\mathcal{D}))$. A functional f on $\mathcal{A}(\mathcal{D})$ is said to be normal if it has the representation

$$f(A) = \mathrm{Tr}\, AT \quad \text{with some } T \in S_1(\mathcal{A}).$$

In $S_1(\mathcal{A})$ we introduce the topology $\tau_1(\mathcal{A})$ given by the seminorms

$$T \longrightarrow \|AT\|_1 \quad \text{for any } A \in \mathcal{A}(\mathcal{D})$$

where $\|\cdot\|_1$ is the usual trace-norm in the space of nuclear operators.

Now there arise the problem to define topologies on Op* -algebras. There are many possibilities to introduce topologies, and it seems that the question whether or not a topology is a good one, in the case of non-normable algebras (of unbounded operators) much more depends on the problem which we will investigate then in the case of normed algebras. For details the reader is refered to the papers [3,4].

In the next section we give the definitions of the ultrastrong and ultraweak topologies which were not regarded up to now.

2. DEFINITION OF THE TOPOLOGIES

To make the formulation of the results more comprehensive we start with the following definition.

Definition 1

Let $\mathcal{A}(\mathcal{D})$ be an Op* -algebra. We introduce the following sets of (countable) sequences:

$$l^2(\|\ \|) = \{ (\varphi_i) : \varphi_i \in \mathcal{D},\ \sum \|\varphi_i\|^2 < \infty \}$$

$$l^2(t_{\mathcal{A}}) = \{ (\varphi_i) : \varphi_i \in \mathcal{D},\ \sum \|A\varphi_i\|^2 < \infty \text{ for all } A \in \mathcal{A}(\mathcal{D}) \}$$

Let us mention that these sets are linear spaces and can be equipped with a natural locally convex topology. We do not use the structure of these spaces in what follows; some information about spaces of this type can be found in [6].

Definition 2

On an Op^* -algebra $\mathfrak{A}(\mathfrak{D})$ we define

i) the ultrastrong topology τ_{us} given by the seminorms

$$A \longrightarrow \|A\|_{(\varphi_i)} = \left(\sum \|A\varphi_i\|^2 \right)^{1/2} \quad \text{for any } (\varphi_i) \in l^2(t_{\mathfrak{A}})$$

ii) the ultraweak topology τ_{uw} given by the seminorms

$$A \longrightarrow \|A\|_{(\varphi_i),(\psi_i)} = \left| \sum (\varphi_i, A\psi_i) \right| \quad \text{for any } (\varphi_i) \in l^2(\|\,\|), \; (\psi_i) \in l^2(t_{\mathfrak{A}})$$

iii) the proper ultraweak topology τ'_{uw} given by the seminorms

$$A \longrightarrow \|A\|'_{(\varphi_i),(\psi_i)} = \left| \sum (\varphi_i, A\psi_i) \right| \quad \text{for any } (\varphi_i),(\psi_i) \in l^2(t_{\mathfrak{A}})$$

Remarks

i) In the case of bounded operators the ultraweak and the proper ultraweak topology coincide, and if $\mathfrak{D} = \mathcal{H}$ we obtain the usual topologies /2/.

ii) Obviously the following relation $\tau'_{uw} \prec \tau_{uw} \prec \tau_{us}$ holds. **The set of seminorms for τ_{us} is directed.**

iii) There could be noted many properties of these topologies as for example the continuity or discontinuity of the operations in $\mathfrak{A}(\mathfrak{D})$. We do not need these for the following considerations and therefore this will be done elsewhere.

3. THE STRUCTURE OF CONTINUOUS FUNCTIONALS ON OP^*-ALGEBRAS

This section contains the main results on the connection between continuity of linear functionals with respect to one of the topologies described above and their simple structure. To see the almost complete anology to the bounded case, we repeat the results valid there in a very comprehensive form (for the exact formulation the reader can consult /2/ or /7/.).

Let $\mathfrak{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$, $S_1(\|\,\|_1)$ the set of all nuclear operators with the trace-norm. Then

i) $S_1(\|\,\|_1)^* = \mathfrak{B}(\mathcal{H})$

ii) f normal on $\mathfrak{B}(\mathcal{H})$ implies f ultraweakly continuous.

iii) f ultrastrongly continuous on $\mathfrak{B}(\mathcal{H})$ implies

$$f(A) = \sum (\varphi_i, A\psi_i) \quad \text{with } (\varphi_i),(\psi_i) \in l^2(\| \|).$$

iv) f positive and ultrastrongly continuous on $\mathfrak{B}(\mathcal{H})$ implies the normality of f.

v) $(S_1, \mathfrak{B}(\mathcal{H}))$ is a dual pair and the topology $\sigma(\mathfrak{B}(\mathcal{H}), S_1)$ is the ultraweak topology on $\mathfrak{B}(\mathcal{H})$.

Now we go on to the unbounded case and start with a technical lemma.

Lemma 1

Let $\mathfrak{F}(\mathfrak{D})$ be the set of all finite dimensional operators of $\mathcal{L}^+(\mathfrak{D})$. Then

i) $\mathfrak{F}(\mathfrak{D})$ is dense in $\mathcal{L}^+(\mathfrak{D})$ with respect to τ_{us} and consequently with respect to τ_{uw} and τ'_{uw}, too.

ii) $\mathfrak{F}(\mathfrak{D})$ is $\tau_1(\mathfrak{A})$-dense in $S_1(\mathfrak{A})$.

Proof:

i) Let $A \in \mathcal{L}^+(\mathfrak{D})$ be an arbitrary operator and $\mathcal{U}$ a τ_{us}-neighbourhood of A:

$$\mathcal{U} = \{ B \in \mathcal{L}^+(\mathfrak{D}) : \| A-B \|^2_{(\varphi_i)} = \sum \|(A-B)\varphi_i\|^2 < \varepsilon \}$$

It must be shown that there is an $F \in \mathfrak{F}(\mathfrak{D})$ with $F \in \mathcal{U}$. Because $\sum \| A\varphi_i \|^2 < \infty$ there is a natural n such that $\sum_{i>n} \| A\varphi_i \|^2 < \varepsilon$. Let P_n be the projection on the finite dimensional space spanned by $(A\varphi_1, \ldots, A\varphi_n)$ and put $F = P_n A$. Then $\| A - F \|^2_{(\varphi_i)} = \sum_1^\infty \|(I-P_n)A\varphi_i\|^2 = \sum_{i>n} \|(I-P_n)A\varphi_i\|^2 \leq$ $\leq \sum_{i>n} \| A\varphi_i \|^2 < \varepsilon$. Therefore $F \in \mathcal{U}$.

ii) can be proved by using the facts that the finite dimensional operators are $\| \|_1$-dense in the set of nuclear operators and that the system of seminorms defining the topology $\tau_1(\mathfrak{A})$ is directed.

Q.E.D.

Proposition 1

The dual space $S_1(\mathfrak{D})[\tau_1]^*$ of $S_1(\mathfrak{D})[\tau_1]$ is algebraically isomorphic to the set $\mathcal{L}(\mathfrak{D}(t_+), \mathcal{H})$ of all continuous linear operators from $\mathfrak{D}(t_+)$ to $\mathcal{H}$ equipped with the usual norm $\|\cdot\|$.

Proof:

First, remark that, of course, $\mathcal{L}^+(\mathfrak{D}) \subset \mathcal{L}(\mathfrak{D}(t_+), \mathcal{H})$. The isomorphism will be established by mapping

$S_1(\mathfrak{D})[\tau_1]^* \ni f \longleftrightarrow A \in \mathcal{L}(\mathfrak{D}(t_+), \mathcal{H})$ with $f(T) = \operatorname{Tr} AT$.

i) $\mathcal{L}(\mathfrak{D}(t_+), \mathcal{H}) \subset S_1(\mathfrak{D})[\tau_1]^*$:

If $R \in \mathcal{L}(\mathfrak{D}(t_+), \mathcal{H})$ then $\|R\varphi\| \leq \|A\varphi\|$ for all $\varphi \in \mathfrak{D}$ and consequently for all $T \in S_1(\mathfrak{D})$ $\|RT\varphi\| \leq \|AT\varphi\|$ which says RT nuclear. From the min-max-principle it is easily to see that $\|RT\|_1 \leq \|AT\|_1$ which implies the continuity of the functional $f(T) = \operatorname{Tr} RT$ on $S_1(\mathfrak{D})[\tau_1]$.

ii) $S_1(\mathfrak{D})[\tau_1]^* \subset \mathcal{L}(\mathfrak{D}(t_+), \mathcal{H})$:

Let $f \in S_1(\mathfrak{D})[\tau_1]^*$, i.e.,

$$|f(T)| \leq \|AT\|_1 \quad \text{for some } A \in \mathcal{L}^+(\mathfrak{D}) \tag{1}$$

For the operator $(\varphi, .)\psi$, $\varphi, \psi \in \mathfrak{D}$

$|f((\varphi, .)\psi)| \leq \|A(\varphi, .)\psi\|_1 \leq \|\varphi\| \cdot \|A\psi\|$, i.e., $f((\varphi, .)\psi)$ is a $\|\cdot\|$-continuous linear functional on $\mathfrak{D}$ for each fixed $\psi \in \mathfrak{D}$. Therefore $f((\varphi, .)\psi) = (\varphi, \chi)$ for some $\chi \in \mathcal{H}$. Putting $R\psi = \chi$ we get

$$f((\varphi, .)\psi) = (\varphi, R\psi) = \operatorname{Tr} R(\varphi, .)\psi \tag{2}$$

or by linearity

$f(F) = \operatorname{Tr} RF$ for all $F \in \mathcal{F}(\mathfrak{D})$.

(1) and (2) give $|(\varphi, R\psi)| \leq \|\varphi\| \cdot \|A\psi\|$ which says $\|R\psi\| \leq \|A\psi\|$ for all $\psi \in \mathfrak{D}$ and therefore R belongs to $\mathcal{L}(\mathfrak{D}(t_+), \mathcal{H})$. Using the facts that $\mathcal{F}(\mathfrak{D})$ τ_1-dense in $S_1(\mathfrak{D})[\tau_1]$ (Lemma 1,ii) and $T \longrightarrow \operatorname{Tr} RT$ τ_1-continuous (cf.i)), we can conclude that

$$f(T) = \mathrm{Tr}\, RT \quad \text{for all } T \in S_1(\mathfrak{D}).$$

Q.E.D.

We mention that there are generalizations of this result to the case $S_1(\mathcal{A})[\tau_1(\mathcal{A})]^*$ in a simple manner. Notice, that for $\mathfrak{D} = \mathcal{H}$ clearly $\mathcal{L}(\mathfrak{D}(t_+), \mathcal{H}) = \mathcal{B}(\mathcal{H})$ and one obtains the result for the bounded case mentioned above.

Proposition 2

Any normal functional f on a selfadjoint Op^*-algebra $\mathcal{A}(\mathfrak{D})$ is τ'_{uw}-continuous and therefore τ_{uw}- and τ_{us}-continuous, too.

Proof:

Let $T = U|T|$ be the polar decomposition of $T \in S_1(\mathcal{A})$. Remark that from the selfadjointness of $\mathcal{A}(\mathfrak{D})$ it follows that $|T| = (T^*T)^{1/2}$ also $\in S_1(\mathcal{A})$ /5/. If $|T|\varphi_i = t_i \varphi_i$ then

$$|f(A)| = |\mathrm{Tr}\, AT| = |\sum (\varphi_i, AU|T|\varphi_i)| = |\sum (t_i^{1/2}\varphi_i, A(t_i^{1/2} U \varphi_i))| =$$
$$= \|A\|'_{(\psi_i),(\rho_i)} \quad \text{with} \quad \psi_i = t_i^{1/2}\varphi_i, \quad \rho_i = t_i^{1/2} U \varphi_i .$$

Using that for any $B \in \mathcal{A}(\mathfrak{D})$ the operators $B^+B|T|$ and $U^*B^+BU|T|$ are nuclear ones one easily deduces that (ψ_i) and (ρ_i) both belong to $l^2(t_{\mathcal{A}})$. This completes the proof.

Q.E.D.

Remark:

From the proof above the following result follows:

Let f be a normal functional on a selfadjoint Op^*-algebra $\mathcal{A}(\mathfrak{D})$, $f(A) = \mathrm{Tr}\, AT$, $T = U|T|$ the polar decomposition. Then both, $B|T|^{1/2}$ and $B|T|^{1/2}U$ are Hilbert-Schmidt-operators for all $B \in \mathcal{A}(\mathfrak{D})$.

Our next result gives the generalization of the third fact valid in algebras of bounded operators. The proof is, with the necessary modifications the same as in /2/.

Proposition 3

Let f be a τ_{us} -continuous linear functional on $\mathcal{A}(\mathcal{D})$. Then, f has the form

$$f(A) = \sum (\varphi_i, A\psi_i) \quad \text{with } (\varphi_i) \in l^2(\|\cdot\|),\ (\psi_i) \in l^2(t_{\mathcal{A}}),$$

i.e., the set of τ_{us} - continuous functionals coincides with the set of τ_{uw}-continuous functionals on $\mathcal{A}(\mathcal{D})$.

Proof:

f τ_{us}-continuous means

$$|f(A)| \le \|A\|_{(\psi_i)} = (\sum \|A\psi_i\|^2)^{1/2} \quad \text{for all } A \in \mathcal{A}(\mathcal{D}) \qquad (3)$$

Let $\hat{\mathcal{H}} = \sum_i \oplus \mathcal{H}_i$, $\mathcal{H}_i \equiv \mathcal{H}$ for all i. In $\hat{\mathcal{H}}$ let

$$\mathcal{D}_o = \{\Phi = (\varphi_i): \ \varphi_i = A\psi_i \quad \text{for an } A \in \mathcal{A}(\mathcal{D})\}$$

$\mathcal{D}_o$ is a (not necessary dense) linear manifold in $\hat{\mathcal{H}}$ and $\|\Phi\|^2 = \sum \|\varphi_i\|^2 = \sum \|A\psi_i\|^2 = \|A\|^2_{(\psi_i)}$. The functional f induces a linear functional f_o on $\mathcal{D}_o$ by

$f_o(\Phi) = f_o((A\psi_i)) := f(A)$. From (3) it can be seen that the value $f_o(\Phi)$ does not depend on the representation of Φ ,i.e., f_o is correctly defined. Moreover:

$|f_o(\Phi)|^2 = |f(A)|^2 \le \sum \|A\psi_i\|^2 = \|\Phi\|^2$, this means that f_o can be extended to a continuous linear functional on $\overline{\mathcal{D}_o}$ (closure in $\hat{\mathcal{H}}$) which will be denoted also by f_o. Therefore there is a $\Phi \in \overline{\mathcal{D}_o}$ such that $f_o(\Psi) = (\Phi, \Psi)$ for all $\Psi \in \overline{\mathcal{D}_o}$. Especially for $\Psi \in \mathcal{D}_o$, $\Psi = (A\psi_i)$

$$f(A) = f_o(\Psi) = (\Phi, \Psi) = \sum (\varphi_i, A\psi_i)$$

The properties of (φ_i),(ψ_i) mentioned in the Proposition are clear.

Q.E.D.

Our next result makes use of a result of Uhlmann[8] which we state here in an appropriate form.

Proposition

Let f be a positive functional on $\mathcal{L}^+(\mathfrak{D})$. Then $f(F) = \operatorname{Tr} FT$ for all $F \in \mathfrak{F}(\mathfrak{D})$ with an operator $T \in S_1(\mathfrak{D})$, $T \geq 0$.

The essential point is the use of positivity to show $T \in S_1(\mathfrak{D})$. We use this idea to prove the following result.

Proposition 4

Let $\mathcal{L}^+(\mathfrak{D})$ be selfadjoint. Then all positive τ_{us}-continuous linear functionals on $\mathcal{L}^+(\mathfrak{D})$ are normal, i.e.,

$f(A) = \operatorname{Tr} AT$ with $T = T^* \geq 0$, $T \in S_1(\mathfrak{D})$.

Proof:

The fact that $f(F) = \operatorname{Tr} FT$ for all $F \in \mathfrak{F}(\mathfrak{D})$ with $T = T^* \geq 0$ can be established similarly to Proposition 1 or according to the Proposition above. By standard considerations [5] the selfadjointness of $\mathcal{L}^+(\mathfrak{D})$ gives $T \in \mathcal{L}^+(\mathfrak{D})$ and AT (and consequently TA) bounded for all $A \in \mathcal{L}^+(\mathfrak{D})$. We prove $T \in S_1(\mathfrak{D})$ which is also the proof of the second part of the Proposition of Uhlmann.
Because f is positive, for any finite dimensional projection $P \in \mathfrak{F}(\mathfrak{D})$: $f(APA^+) \leq f(AA^+)$ and $\operatorname{Tr} T(APA^+) = \operatorname{Tr} T(AP)(AP)^+ = \operatorname{Tr}(AP)^+T(AP)$. Let (ψ_i) be an orthonormal basis of $\mathcal{H}$ contained in $\mathfrak{D}$ and P_n the projection on the space spanned by $(\psi_1, \ldots, \psi_n)$.
Then: $f(APA^+) = \operatorname{Tr} T(APA^+) = \operatorname{Tr} (AP)^+T(AP) = \sum_{i=1}^{\infty}(AP\psi_i, TAP\psi_i) = \sum_{i=1}^{n}(A\psi_i, TA\psi_i) \leq f(AA^+)$.
That means $\sum \| T^{1/2}A\psi_i\|^2$ converges and therefore $T^{1/2}A$ is a Hilbert-Schmidt operator for all $A \in \mathcal{L}^+(\mathfrak{D})$. Especially, $T^{1/2}$ is a Hilbert-Schmidt operator, i.e., TA is nuclear for all $A \in \mathcal{L}^+(\mathfrak{D})$. Because $\mathcal{L}^+(\mathfrak{D})$ is selfadjoint, this means $T \in S_1(\mathfrak{D})$. This gives together with Lemma 1 and Proposition 2 the desired result $f(A) = \operatorname{Tr} AT$.

Q.E.D.

Clearly, Proposition 4 is a consequence of the Proposition of Uhlmann and our foregoing results. We have repeated the proof only because of the interesting part $T \in S_1(\mathfrak{D})$. A more general result can be formulated as follows:

Proposition 4'

Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint and τ a topology on $\mathcal{L}^+(\mathcal{D})$ with

i) $\mathcal{F}(\mathcal{D})$ is τ-dense in $\mathcal{L}^+(\mathcal{D})$

ii) any normal (positive) functional on $\mathcal{L}^+(\mathcal{D})$ is τ-continuous.

Then any τ-continuous positive functional on $\mathcal{L}^+(\mathcal{D})$ is normal.

Proposition 5

Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint. $(\mathcal{L}^+(\mathcal{D}), S_1(\mathcal{D}))$ is a dual pair with respect to the bilinear form $(A,T) \longrightarrow \mathrm{Tr}\, AT$. The weak topology $\sigma(\mathcal{L}^+(\mathcal{D}), S_1(\mathcal{D}))$ is the proper ultraweak topology τ'_{uw} on $\mathcal{L}^+(\mathcal{D})$.

Proof:

The first part is a simple consequence of the fact that $\mathcal{F}(\mathcal{D}) \subset \mathcal{L}^+(\mathcal{D})$, $\mathcal{F}(\mathcal{D}) \subset S_1(\mathcal{D})$. The weak topology σ is given by the family of seminorms

$$A \longrightarrow |\mathrm{Tr}\, AT| \quad \text{for all } T \in S_1(\mathcal{D}).$$

Hence Proposition 2 gives $\sigma \prec \tau'_{uw}$. Let $\|\cdot\|'_{(\varphi_i),(\psi_i)}$ be a seminorm defining the topology τ'_{uw}. If we can find an operator $T \in S_1(\mathcal{D})$ with $|\mathrm{Tr}\, AT| \geq \|A\|'_{(\varphi_i),(\psi_i)}$ for all $A \in \mathcal{L}^+(\mathcal{D})$, then $\tau'_{uw} \prec \sigma$ and the proof is completed. We show that $T = \sum_i (\varphi_i, \cdot\,)\psi_i$ is such an operator.

$AT = \sum_i (\varphi_i, \cdot\,)A\psi_i$ and $AT^* = \sum_i (\psi_i, \cdot\,)A\varphi_i$ are bounded operators because $(\varphi_i), (\psi_i) \in l^2(t_+)$. Let (χ_i) be an arbitrary orthonormal basis of $\mathcal{H}$, then

$$\mathrm{Tr}\, AT = \sum_j \sum_i (\varphi_i, \chi_j)(\chi_j, A\psi_i) = \sum_i \sum_j (\varphi_i, \chi_j)(\chi_j, A\psi_i) = \sum_i (\varphi_i, A\psi_i)\,. \tag{4}$$

The last series is absolutly convergent, i.e., AT is a nuclear operator for all $A \in \mathcal{L}^+(\mathcal{D})$. Analogously one sees that AT^* is a nuclear operator, hence $T \in S_1(\mathcal{D})$. Moreover (4) means that $|\mathrm{Tr}\, AT| = \|A\|'_{(\varphi_i),(\psi_i)}$.
To complete the proof let us remark that the interchanging of the sums in (4) is justified because both series are absolutly convergent for fixed second index.

Q.E.D.

4. CONCLUDING REMARKS

To investigate relationships between continuity and normality of positive linear functionals, one must seek in some sense "optimal" topologies. Proposition 4' says us how to understand "optimal". On the one hand, the topology must not be too strong otherwise $\mathcal{F}(\mathcal{D})$ fails to be dense in $\mathcal{L}^+(\mathcal{D})$. On the other hand, the topology must be strong enough to guarantee the continuity of the normal functionals. Proposition 4 and Proposition 2 say that the ultrastrong topology is such a good one.

Another topology on Op^*-algebras is the so called uniform topology $\tau_{\mathcal{D}}$ /3,4/. In /5/ it was pointed out that this topology under some restrictions on the domain $\mathcal{D}$ is also such an "optimal" topology. We conclude this section with the following remark. Instead of $l^2(t_{\mathcal{A}})$ we could use also $l^1(t_{\mathcal{A}})$ or general $l^p(t_{\mathcal{A}})$, $p \geq 1$ and then investigate the same questions. Note that there are results also along this line.

Acknowledgement

We wish to thank Professors G.Lassner and A.Uhlmann for stimulating discussions.

References

1. N.Bourbaki. Espaces vectoriels topologiques, Paris, 1953/55.
2. J.Dixmier. Les algebres d'operateurs dans l'espace Hilbertien, Paris, 1957.
3. G.Lassner. Topological Algebras of Operators, JINR Preprint E5-4606, Dubna, 1969; Rep.Math.Phys. 3, 279 (1972).
4. G.Lassner. Mathematische Beschreibung von Observablen-Zustandssystemen, Wiss.Zeitschrift, KMU, Leipzig, 22,103 (1973).
5. G.Lassner,W.Timmermann. Normal States on Algebras of Unbounded Operators, Preprint, Kiew, 1971; Rep.Math.Phys.3,295 (1972).
6. A.Pietsch. Nukleare lokalkonvexe Räume, Berlin, 1965.
7. S.Sakai. C^*-Algebras and W^*-Algebras, Berlin, 1971.
8. A.Uhlmann. unpublished results.

Received by Publishing Department on November 23, 1976.