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D.V.Shirkov

NONANALYTICITY IN COUPLING CONSTANT AND TROUBLES OF ULTRAVIOLET ANALYSIS

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Ширков Д.В.

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Неаналитичность по константе связи и трудности ультрафиолетового анализа

Обсуждается структура существенной особенности по константе связи. Показано, что особенность составляется из слагаемых двух типов: разлагаемых в степенной ряд и неразложимых. Первые приводят к асимптотическим разложениям. Формальные суммы асимптотических рядов обладают рядом свойств, которые затрудняют получение физических результатов.

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Shirkov D.V.

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Nonanalyticity in Coupling Constant and Troubles of Ultraviolet Analysis

The structure of essential singularity in the coupling constant is discussed. It is shown that the singularity consists of two terms: one that can be expanded in perturbation series and another that cannot. The first term yields the asymptotic expansion. The formal sums of asymptotic series obey specific properties which prevent obtaining physical results.

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The problem of analyticity with respect to the coupling constant g in the quantum field theory (QFT) models has a rather long history, started by the well-known Dyson paper [1], published in 1952 and containing a qualitative discussion of the subject based on general physical arguments, and by a set of investigations of the general structure of perturbation series, performed at the beginning of the fifties.

In recent years this problem attracts attention due to the study of ultraviolet asymptotic (UVA) behaviour based on the information obtained from perturbation theory and analysed with the help of the renormalization group (RG) formalism.

Within the RG method it is possible to show [2] that in the logarithmic QFT models the properties of renormalizability and causality (taken in the form of Källen-Lehmann spectral representation) are not consistent with the property of analyticity at the point $g=0$ and yield the conclusion about existence of an essential singularity of the form

$$\exp(-1/g). \tag{1}$$

In quite recent papers [3,4] Lipatov has attacked the problem of explicit determination of an arbitrary coefficient of the power expansion

$$\psi(g) = \sum_k (-g)^k \psi_k \qquad (2)$$

of the so-called Gell-Mann-Low function ψ, entering into the r.h.s. of the well-known differential Lie RG equation for the invariant (effective) coupling constant $\bar{g}$.

This problem was formulated for the class of renormalizable (logarithmic) nonlinear scalar models $\mathcal{L} \sim -g\varphi^n$ in the space-time of $\mathcal{D} = 2n/(n-2)$ dimensions. In the first paper[3] the limit $n \to \infty$ ($\mathcal{D} \to 2$) was studied, while in the second[4] the problem was considered with the help of functional integral representation. Here the procedure of steepest descent method around the stationary classical Euclidean soliton-type solutions is employed and yields the explicit expressions for the $\psi_k(n)$ in the large k limit

$$\psi_k(n) \simeq \tilde{\psi}_k(n)\left[1 + O(1/k)\right], \quad k \to \infty. \qquad (3)$$

The general result is of the form

$$\tilde{\psi}_k(n) = a(n)\, b^k(n)\, k^{k(n/2-1)+(n+2)/2}.$$

For $n = \mathcal{D} = 4$ one gets

$$\tilde{\psi}_k = \tilde{\psi}_k(4) = \alpha\left(\frac{2k}{e}\right)^k k^4 \qquad (4)$$

with

$$\mathcal{L} = -\frac{4\pi^2}{3} g\varphi^4 \quad \text{and} \quad \alpha = \frac{1.37}{16\pi^2} \simeq 0.87 \times 10^{-2}.$$

Let us put the question: What physical information can be extracted from the results (4), (3), i.e., from the sum

$$\tilde{\psi}(g) \sim \sum (-g)^k \tilde{\psi}_k. \qquad (5)$$

In other words: What can be learned about the properties of the sum of series (2) from the study of the series (5)?

Due to its asymptotic nature ($\tilde{\psi}_k \sim k!$) the series in the r.h.s. of Eq.(5) does not determine unambiguously the function $\tilde{\psi}$. To get the unique answer one has to make some assumptions about analytic properties of the "sum" in the complex g-plane.

One of the possible ways is the Borel summation which is equivalent to the summation under the sign of the Laplace integral

$$\tilde{\psi}_B(g) = \int_0^\infty e^{-x/g}\Psi(x)\,dx/g, \quad \Psi(x) = \sum_k (-x)^k \tilde{\psi}_k / k! \qquad (6)$$

In favour of such a procedure one can use the structure of exact solutions of some two-dimensional QFT models (see the discussion of this subject in paper[5]).

We take it for granted having in mind that (a) the differences between various procedures do not change seriously the relation between (2) and (5) if the same procedure is used in both the cases, and that (b) different summations yield the same type of singularity at the origin $g = 0$, which, as will be shown, is the only solid result of studying the sum (5). Thus, in what follows by $\tilde{\psi}(g)$ we shall mean $\tilde{\psi}_B(g)$ defined by Eq.(6).

So, we have now to find the sum of the series

$$\Psi(x) = \alpha f(2x), \quad f(t) = \sum (-t)^k (k/e)^k k^4 / k!$$

converging in the unit circle. We have been able to find the function f approximately in the form of a new series

$$f(t) = f_0(t) + f_1(t) + \ldots$$

Here the first term f_0 describes correctly the leading contribution of the coefficients

$$(k/e)^k k^4/k! = (2\pi)^{-1/2} k^{3.5}\left[1 + O(1/k)\right].$$

The second term corresponds to the first correction $\sim k^{2.5}$, etc. Such a type of approximation completely corresponds to the origin and degree of accuracy of the Lipatov results as emphasized by Eq. (3).

Omitting details of calculations we give the final result for $\tilde{\psi}(g) = \psi_0(g) + \psi_1(g) + \dots$

$$\tilde{\psi}_0(g) = \frac{\alpha}{16\sqrt{2}}\Big[16g - \frac{15}{g} - \frac{45}{g^2} - \frac{15}{g^3} - \frac{1}{g^4} + \left(1 + \frac{40}{g} + \frac{58}{g^2} + \frac{16}{g^3} + \frac{1}{g^4}\right)\sqrt{\frac{\pi}{2g}}\, e^{1/2g}\, \mathrm{Erf}\left(\frac{1}{\sqrt{2g}}\right)\Big], \tag{7}$$

$$\tilde{\psi}_1(g) = \frac{\alpha}{192\sqrt{2}}\Big[16g - 1 + \frac{5}{g} + \frac{7}{g^2} + \frac{1}{g^3} - \left(1 + \frac{10}{g^2} + \frac{8}{g^3} + \frac{1}{g^4}\right)\sqrt{\frac{\pi g}{2}}\, e^{1/2g}\, \mathrm{Erf}\left(\frac{1}{\sqrt{2g}}\right)\Big]. \tag{8}$$

A detailed analysis of such expressions reveals that the contributions of nonanalytical terms ($\sim \mathrm{Erf}$) at large g become more and more important with growing m

$$\tilde{\psi}_m(g) \sim g^{m-1/2}, \quad g \to \infty .$$

This means that approximations of the considered type (2) in general cannot give any reliable information on the $\psi(g)$ behaviour at large g .

At the same time the sequence $\psi_0, \psi_1, \psi_2, \dots$ in the region $g \lesssim 1$ diminishes only due to the numerical smallness of the correction terms. In other words, if we detail Eq. (3)

$$\psi_k = \tilde{\psi}_k\left[1 + \frac{a_1}{k} + \frac{a_2}{k} + \dots\right] = \psi_0(k) + a_1\psi_1(k) + a_2\psi_2(k) + \dots$$

then by summing we get

$$\psi(g) = \psi_0(g) + a_1\psi_1(g) + a_2\psi_2(g) + \dots \tag{9}$$

The terms of this series in the region $g \lesssim 1$ diminish only due to the numerical smallness of numbers $a_1, a_2, \dots$ and for $g \gg 1$ grow like $\psi_m \sim g^{m-1/2}$.

Now we can formulate the answer:

1. The sum of the series (5), i.e., the sum of the "leading" contributions of the initial series (2) leads to the expression with the essential singularity at the origin of the g -plane

$$\frac{1}{g^4\sqrt{g}}\exp\left(\frac{1}{2g}\right)\mathrm{Erf}\left(\frac{1}{\sqrt{2g}}\right) = \sqrt{\frac{2}{\pi g^9}}\, e^{1/2g}\int_{(2g)^{-1/2}}^{\infty} e^{-\tau^2}\,d\tau \tag{10}$$

and cut along the negative semiaxis. This result in its main features does not depend on the way of summation. This means that the change in the summation prescription may produce an expression with different position of the cut, with additional terms like Eq. (1),etc.,but with the same type of an essential singularity.

2. At the same time the series (9) cannot give any reliable quantitative information on the sum $\psi(g)$ for $g \gtrsim 1$. Even in the region $g \lesssim 1$ where, as we mentioned, the terms of series (9) can diminish rapidly due to the numerical smallness of coefficients a_m , the explicit form of ψ_m depends on the way of summation.

Hence any attempts to deduce physical conclusions, based on quantitative properties of the sums of asymptotic series (compare[3]) must be supported by a serious supplementary analysis.

As a curious illustration of this thesis let us note that the numerical comparison of Eq. (4) for the first few terms

$$\tilde{\psi}_2 = 0.30, \quad \tilde{\psi}_3 = 7.5, \quad \tilde{\psi}_4 = 167 ,$$

with results of exact perturbation calculations[6]

$$\psi_2 = 3, \quad \psi_3 = 34/3, \quad \psi_4 = 153.65 ,$$

shows that the relative error of Eq. (4) diminishes rapidly as k grows and for k =4 is smaller than 10%. If this fact is not occasional one can make an optimistic hypothesis that for

$k \geqslant 5$ the errors are negligible and the true function ψ can be represented as follows

$$\psi(g) = \tilde{\psi}(g) + \Delta(g) \quad , \quad \Delta(g) = 2.7g^2 - 3.8g^3 - 13.3g^4 .$$

As far as Δ (0.33)=0 and $\tilde{\psi}$ (0.33) $\sim 10^{-3}$ we can conclude that

$$\psi(g^*) = 0 \quad , \quad g^* \sim 1/3 . \tag{11}$$

Prior to a more serious discussion of this relation one should first estimate the corrections to the Eq. (4) of order k^{-1} (and possibly k^{-2}) and show that they do improve the correspondence between $\tilde{\psi}_k$ and already known ψ_k , second, calculate the 4-loop contribution ψ_5 and estimate the influence of higher terms in $\Delta(g)$. Finally, it is necessary to find justification of the summation prescription.

Note also that the success in the proof of Borel summation procedure will face us with the problem of the cut along the negative semiaxis in the g-plane. Such a position of the cut contradicts the selfconsistency of the φ^4 model with attraction. It may happen then that the conclusion about the asymptotic freedom of this model is a result of some "analytical illusion".

The singularities (1) and (10) are closely related and in a sense are complementary one to another. While expression (1) "leaves no footprints" in perturbation expansion, singularity (10), after formal expansion (using integration by parts) yields the "leading" terms of asymptotic series.

The success in determining the explicit form of these terms and of singularities gives the impression of essential progress towards the understanding of the UVA structure of the QFT. However, as was argued, the problem of construction of high-

energy behaviour of physical objects is still far from being solved.

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