СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 10212

L.G.Zastavenko

THE ZERO-MASS GOLDSTONE PARTICLES IN THE MODELS $[(\varphi^{x}\varphi)^{2}]_{2}$ AND $[(\varphi^{x}\varphi)^{2}]_{3}$ WITH DEGENERATED VACUUM

1976

E2 - 10212

L.G.Zastavenko

THE ZERO-MASS GOLDSTONE PARTICLES IN THE MODELS $[(\varphi^x\varphi)^2]_2$ AND $[(\varphi^x\varphi)^2]_3$ WITH DEGENERATED VACUUM

Заставенко Л.Г. **E2 - 10212**

Голдстоуновские частицы нулевой массы покоя в моделях $[(\phi^x\phi)^2]_2$ и $[(\phi^x\phi)^2]_3$ с вырожденным вакуумом

Дано новое доказательство существования частиц нулевой массы покоя в рассматриваемых моделях.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований
Дубна 1976

Zastavenko L.G. **E2 - 10212**

The Zero-Mass Goldstone Particles in the Models $[(\phi^x\phi)^2]_2$ and $[(\phi^x\phi)^2]_3$ with Degenerated Vacuum

The models $[(\phi^x\phi)^2]_2$ and $[(\phi^x\phi)^2]_3$ with the degenerated vacuum are considered. Under the assumptions that:
i) the functional of vacuum (4), (5) and excited state functional (9) are invariant under the gauge transformation, and
ii) the eqs. (10) for the determination of the excited state functional (9) have nontrivial solutions with bounded coefficient functions,
we prove that the functional (9) defines the particle with zero rest mass. So, a new proof is given of the Goldstone theorem for the models considered.

Communication of the Joint Institute for Nuclear Research
Dubna 1976

1. INTRODUCTION

The models in the title are manifestly relativistic covariant and possess continuous gauge symmetry. The ground state in these models is infinitely degenerated (see eq. (7)) in some region of the parameter M^2, $M^2 < M_0^2$. The ground states (7) with $m \neq 0$ are not gauge invariant.

So, according to the Goldstone theorem [1], the models considered contain zero-mass particles, Goldstone bosons.

In this work we give a new proof of this result proceeding directly from the analysis of the Schrödinger eq. solutions.

1.1. We use the method of linked cluster expansions. The possibility of applying this method to the analysis of the Schrödinger eq. solutions in quantum field theory was pointed out by Coester and Haag [3]. Later on we developed this method in detail [2,4,8,9].

1.2. The article is organized as follows. In Sect. 1 we list the necessary formulae of the work [2], where there is studied the model $[(\phi^x\phi)^2]_2$.

The very important suggestions about the gauge transformation properties of the ground and excited state functionals are given in Sect. 2.

In Sect. 3 we study the odd in ψ_2, eq. (3), one-particle state, eq. (9), in the model $[(\phi^x\phi)^2]_2$. It is shown that, if the system (10) has a nontrivial solutions, the functional (9) describes the zero mass particle, Goldstone boson.

The papers [6,7] contain the statement that there are no Goldstone bosons in two space-time dimensions.

In order to avoid this subjection, in Sect. 4 we transform the consideration of Sect. 3 to the case of three-dimensional space-time, where Goldstone bosons are not forbidden.

1.3. The models under consideration are defined by the Hamiltonian

$$H = \frac{1}{2}\int dk\Big[-\frac{\delta^2}{\delta\phi_\rho(k)\,\delta\phi_\rho(-k)} + (k^2+M^2)\phi_\rho(k)\phi_\rho(-k)\Big] + \tag{1}$$

$$+ g\int dk_1 dk_2\, dk_3 dk_4\, \phi_\rho(k_1)\phi_\rho(k_2)\phi_\sigma(k_3)\phi_\sigma(k_4)\,\delta(k_1+k_2+k_3+k_4)$$

and by the Schrödinger equation

$$(H-E)\Omega = 0. \tag{2}$$

Here Ω is the functional describing some physical state; E, the energy of this state.

1.4. Let us produce the transformation of the variables [2]

$$\phi_1(k) = \beta\,\delta(k) + \psi_1(k),$$
$$\phi_2(k) = \psi_2(k). \tag{3}$$

After this transformation the Hamiltonian (1) will contain only even degrees of ψ_2,

so that one can search for the ground state Ω_0 in the form

$$\Omega_0(\phi) = \exp[-\kappa(\phi)],\quad \phi(k) = [\phi_1(k) + i\phi_2(k)]/\sqrt{2}, \tag{4}$$

$$2\kappa = \int dk_1 dk_2\,\delta(k_1+k_2)[C_{20}(k_1,k_2)\psi_1(k_1)\psi_1(k_2) +$$

$$+ C_{02}(k_1,k_2)\psi_2(k_1)\psi_2(k_2)] +$$

$$+ \int dk_1 dk_2 dk_3\,\delta(k_1+k_2+k_3)[C_{30}(k_1,k_2,k_3)\psi_1(k_1)\psi_1(k_2)\psi_1(k_3) +$$

$$+ C_{12}(k_1;k_2,k_3)\psi_1(k_1)\psi_2(k_2)\psi_2(k_3)] +$$

$$+ \ldots\ . \tag{5}$$

Substitution of (4), (5) into (2) gives the system of equations

$$C^2_{20}(k,-k) = k^2 + 8g\beta^2 + \int ds[6C_{40}(s,-s,k,-k) + C_{22}(k,-k;s,-s) -$$

$$- \frac{3}{2\beta}C_{30}(s,-s,0) - \frac{1}{2\beta}C_{12}(0;s,-s)], \tag{6}$$

$$C^2_{02}(k,-k) = k^2 + \int ds[C_{22}(s,-s;k,-k) + 6C_{04}(s,-s,k,-k) -$$

$$- \frac{3}{2\beta}C_{30}(s,-s,0) - \frac{1}{2\beta}C_{12}(0;s,-s)],$$

$$C_{30}(k_1,k_2,k_3)[C_{20}(k_1,-k_1) + C_{20}(k_2,-k_2) + C_{20}(k_3,-k_3)] = 8g\beta +$$

$$+ \int ds[10C_{50}(k_1,k_2,k_3,s,-s) + C_{32}(k_1,k_2,k_3;s,-s)],$$

$$C_{12}(k_1;k_2,k_3)[C_{20}(k_1,-k_1)+C_{02}(k_2,-k_2)+C_{02}(k_3,-k_3)] = 8g\beta +$$

$$+\int ds[3C_{32}(k_1,s,-s;k_2,k_3)+6C_{14}(k_1;s,-s,k_2,k_3)],$$

$$\ldots \quad .$$

for the determination of the coefficient functions C_{nm}.

1.5. Apart from the functional (4), (5) to the lowest eigenvalue of the Hamiltonian (1) there belong all the functionals [2]

$$\Omega_m(\phi) = e^{im\theta}\,\Omega_0(\phi), \quad m = 0, \pm 1, \pm 2, \ldots \qquad (7)$$

$$\theta = \mathrm{arctg}[\phi_2(0)/\phi_1(0)] = \mathrm{arctg}\,\phi(0).$$

1.6. One has to search for the excited state functionals in the form

$$\Omega_{mp}(\phi) = U_p(\phi)\,\Omega_m(\phi), \qquad (8)$$

where p is the momentum; Ω_m, the functional (7); and U_p, the expansion in powers of ψ_1 and ψ_2 which contains only odd or only even degrees of ψ_2, e.g.,

$$U_{p2}(\phi) = \Gamma^p_{01}\psi_2(p) +$$

$$+\int \Gamma^p_{11}(k_1;k_2)\psi_1(k_1)\psi_2(k_2)\,dk_1 dk_2\,\delta(k_1+k_2-p) +$$

$$+\int[\Gamma^p_{21}(k_1,k_2;k_3)\psi_1(k_1)\psi_1(k_2)\psi_2(k_3) +$$

$$+\Gamma^p_{03}(k_1,k_2,k_3)\psi_2(k_1)\psi_2(k_2)\psi_2(k_3)]\,dk_1 dk_2 dk_3\,\delta(k_1+k_2+k_3-p)+$$

$$+ \ldots \quad . \qquad (9)$$

Substituting (8), (9) into (2), (1) gives the equations

$$[C_{02}(p,-p) - \Lambda(p)]\Gamma^p_{01} = \int ds[\Gamma^p_{21}(s,-s;p)+3\Gamma^p_{03}(s,-s,p)],$$

$$[C_{02}(p_2,-p_2)+C_{20}(p_1,-p_1)-\Lambda(p)]\Gamma^p_{11}(p_1;p_2)+$$

$$+\frac{2\cdot 1}{2}C_{12}(p_1;p_2,-p)\Gamma^p_{01} = \int ds[3\Gamma^p_{31}(s,-s,p_1;p_2)+3\Gamma^p_{13}(p_1;s,-s,p_2)],$$

$$[C_{20}(p_1,-p_1)+C_{20}(p_2,-p_2)+C_{02}(p_3,-p_3)-\Lambda(p)]\Gamma^p_{21}(p_1,p_2;p_3)+$$

$$+\frac{2\cdot 1}{2}C_{22}(p_1,p_2;p_3,-p)\Gamma^p_{01}+\frac{3\cdot 1}{2}C_{30}(p_1,p_2,-p_1-p_2)\Gamma^p_{11}(p_1+p_2;p_3)+$$

$$+\frac{2\cdot 1}{2}\frac{1}{2}[C_{12}(p_1;p_3,-p_1-p_3)\Gamma^p_{11}(p_2;p_1+p_3)+$$

$$+C_{12}(p_2;p_3,-p_2-p_3)\Gamma^p_{11}(p_1,p_2+p_3)] =$$

$$= \int ds[6\Gamma^p_{41}(p_1,p_2,s,-s;p_3)+3\Gamma^p_{23}(p_1,p_2;s,-s,p_3)],$$

$$\ldots\ldots\ldots\ldots \quad . \qquad (10)$$

which determine functions Γ_{nm} and eigenvalue $\Lambda(p)$; $\Lambda(p)$ in (10) is the excitation energy, i.e., the energy of the state (8) minus the energy of the ground state (4).

1.7. There exist the excited states of three types:

1) the one-particle excited states, 2) the bound states and 3) the states of scattering.

While the coefficient functions of states of types 1) and 2) are continuous and bounded, the coefficient functions of states of type 3) are singular ones.

The one-particle excited states differ from the bound states because for the former states in the limit $\beta\to\infty$ there disappear all the coefficient functions Γ_{nm}, except for Γ_{01} (odd in ψ_2 one-particle state) or Γ_{10} (even in ψ_2 one-particle state).

Perturbation theory for $\beta\to\infty$ shows the odd in ψ_2 functional U_{p2} with continuous and bounded coefficient functions to describe the particle with zero or vanishing in the limit $\beta\to\infty$ mass.

1.8. We shall denote this functional and its coefficient functions by $\tilde{U}_{p2}(\phi)$ and $\tilde{\Gamma}_{nm}$.

2. CONJECTURES ABOUT THE TRANSFORMATIONAL PROPERTIES OF THE EIGENFUNCTIONALS UNDER GAUGE TRANSFORMATION

In this section we shall study the transformational properties of the functionals (5) and (9) under gauge transformation

$$\phi(k)\to\phi(k)e^{i\alpha}. \qquad (11)$$

We state that there hold the formulae (for any real constant α)

$$\kappa(\phi e^{i\alpha})=\kappa(\phi), \qquad (12)$$

$$\tilde{U}_{p2}(\phi e^{i\alpha})=\tilde{U}_{p2}(\phi), \quad p\neq 0, \qquad (13)$$

$$\tilde{U}_{02}(\phi e^{i\alpha})=\alpha\tilde{\Gamma}_{01}\beta\delta(0)+\tilde{U}_{02}(\phi). \qquad (14)$$

2.1. For small values of α transformation (11) in variables ψ is

$$\begin{aligned}&\psi_1(k)\to\psi_1(k)-\alpha\psi_2(k)+\dots\\&\psi_2(k)\to\psi_2(k)+\alpha[\beta\delta(k)+\psi_1(k)]+\dots\ .\end{aligned} \qquad (15)$$

Substituting (5) and (15) into (12) gives infinite number of the relations between functions C_{nm}:

$$\begin{aligned}&C_{02}(0,0)=0,\\&C_{20}(k,-k)-C_{02}(k,-k)=\beta C_{12}(k;-k,0),\\&\dots\ .\end{aligned} \qquad (16)$$

Analogously, substituting (9) and (15) into (13) gives infinite number of the relations between functions $\tilde{\Gamma}_{nm}$ $(p\neq 0)$:

$$\tilde{\Gamma}^{p}_{01}+\beta\tilde{\Gamma}^{p}_{11}(p;0)=0,$$

$$-\frac{1}{2}\tilde{\Gamma}^{p}_{11}(k_1;k_2)-\frac{1}{2}\tilde{\Gamma}^{p}_{11}(k_2;k_1)+3\beta\tilde{\Gamma}^{p}_{03}(k_1,k_2,0)=0, \qquad (17)$$

$$\frac{1}{2}\tilde{\Gamma}^{p}_{11}(k_1;k_2)+\frac{1}{2}\tilde{\Gamma}^{p}_{11}(k_2;k_1)+\beta\tilde{\Gamma}^{p}_{21}(k_1,k_2;0)=0,$$

$$\dots\ .$$

2.2. The physical meaning of formulae (12), (13) is quite clear: vacuum (4) has no charge, vacuum (7) has charge m, the excited state $\tilde{U}_p(\phi)\Omega_m(\phi), p\neq 0$ has the same charge as the corresponding vacuum Ω_m.

2.3. We do not know how to prove formulae (16) and (17). Suppose, however, the value β^2 to be large. Then it is possible, neglecting integrals, to determine C_{20}, C_{02}, C_{30}, $C_{12}\dots$ from eqs. (6):

$$C_{20}(k,-k) = \sqrt{k^2 + 8g\beta^2 + \dots} \quad ,$$

$$C_{02}(k,-k) = |k| + \dots \; ,$$

$$C_{12}(k;-k,0) = 8g\beta/(\sqrt{k^2+8g\beta^2} + |k|) + \dots$$

$\dots$.

These functions satisfy relations (16). One can "check" relations (17) analogously.

2.4. Finally, formula (14) is a consequence of formulae (13), (15) with account of the coefficient functions $\tilde{\Gamma}_{nm}$ continuity.

Formula (14) implies the functional $\tilde{U}_{02}(\phi)$ to be not singlevalued one. Therefore the state

$$\Omega_{m0}(\phi) = \tilde{U}_{02}(\phi)\,\Omega_m(\phi) \tag{18}$$

is to be excluded from the set of the eigenstates of the Hamiltonian: particularly it must not be taken into account at the summation over intermediate states.

3. THE ZERO MASS PARTICLE IN THE $[(\phi^{x}\phi)^2]_2$ MODEL

We shall prove here that the odd in ψ_2 one-particle state in this model, if it exists (it is defined by the functional $U_{p2}(\phi)$) describes the particle of zero mass, so that

$$\tilde{\Lambda}(p) \to 0 \qquad \text{as} \quad p \to 0. \tag{19}$$

For the proof let us tend p to zero in the first equation of system (10). Taking into account the first relation of system (16) and second and third relations of system (17), we shall reduce the first equation of (10) as $p \to 0$ to the following one:

$$\tilde{\Lambda}(p)\tilde{\Gamma}^{p}_{01} \to 0, \quad p \to 0. \tag{20}$$

System (10) defines the functions $\tilde{\Gamma}_{nm}$ up to a common multiplier, and $\tilde{\Gamma}^{\circ}_{01} \neq 0$. Therefore (20) implies (19).

3.1. Thus our model contains the state of massless particle. Hence it follows, that it has neither even in ψ_2 one-particle state nor bound states (they are unstable decaying into massless particles). So, all the states in our model, except for the one of zero mass particle, are the states of massless particle scattering. Let $U_{p2}(\phi;p_1,p_2,p_3)$ be the functional, corresponding to the scattering of zero mass particles with momenta p_1, p_2, p_3, $p_1+p_2+p_3 = p$, $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$. This functional satisfies similarly to (13), (14), the relation

$$U_{p2}(\phi e^{i\alpha}; p_1,p_2,p_3) = \beta\alpha\,\Gamma^{\circ}_{01}\delta(p) + U_{p2}(\phi; p_1,p_2,p_3). \tag{13b}$$

Substitute $U_{p2}(\phi;p_1,p_2,p_3)$ for $\tilde{U}_{p2}(\phi)$ in the consideration, which has resulted in (19), (20). Then the relation

$$\Lambda(p)\Gamma^{p}_{01} \to 0 \qquad \text{as} \quad p \to 0 \tag{20a}$$

is valid as before, but (20a) does not imply the limit

$$\Lambda(p) \to 0 \qquad \text{as} \quad p \to 0 \tag{19a}$$

but the limit

$$\Gamma^{p}_{01} \to 0 \qquad \text{as } p \to 0 \tag{21}$$

so that

$$\Gamma^{o}_{01} = 0 \tag{21a}$$

The difference between the behaviour of $\tilde{\Gamma}^{p}_{01}$ and Γ^{p}_{01} as $p \to 0$ has a simple physical interpretation. Namely, provided that

$\tilde{\Gamma}^{o}_{01} = 0$

the functional $\tilde{U}_{02}(\phi)$ would be single-valued one and would define physically meaningless state of zero mass particle with zero momentum. On the contrary, provided that

$\Gamma^{o}_{01} \neq 0$,

the functional $U_{02}(\phi; p_1, p_2, p_3)$, $p_1 + p_2 + p_3 = 0$, would be non-single-valued one and would describe no physical state; this is impossible, if $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$, for the functional $U_{02}(\phi; p_1, p_2, p_3)$ even at $p_1 + p_2 + p_3 = 0$ describes the state of three massless particle scattering.

4. THE ZERO MASS PARTICLE IN THE MODEL $[(\phi^{x}\phi)^{2}]_{3}$

The papers /6/ and /7/ contain statement that there are no Goldstone bosons in two-dimensional space-time. So the consideration of Sect. 3 seems to be useless. Here we, however, extend this consideration to the three-dimensional space-time, where there is no objection of /6,7/.

4.1. In this case one has to add $C_{10}\psi_1(0)$ to the right-hand side of formula (5) (compare /9/); one has also to substitute

$$\beta C_{02}(0,0) = \frac{1}{2} C_{10} \tag{16a}$$

for the first connection of (16), and

$$\begin{aligned} &[C_{02}(p,-p) - \tilde{\Lambda}(p)]\tilde{\Gamma}^{p}_{01} + C_{10}\tilde{\Gamma}^{p}_{11}(0;p)/2 = \\ &= \int ds\, [\tilde{\Gamma}^{p}_{21}(s,-s;p) + 3\tilde{\Gamma}^{p}_{03}(s,-s,p) \end{aligned} \tag{22}$$

for the first equation of the system (10). Then the integral in the right-hand side of eq. (22) vanishes as $p \to 0$ for the same reason as in Sect. 3. Formula (16a) and the first of eqs. (17) imply

$$C_{02}(p,-p)\tilde{\Gamma}^{p}_{01} + C_{10}\tilde{\Gamma}^{p}_{11}(0;p)/2 \to 0 \qquad p \to 0 \tag{23}$$

Thus we again get the limit (19): the model contains zero mass particle.

REFERENCES

1. I.Goldstone, A.Salam, S.Weinberg. Phys. Rev., 127, 965 (1962).
2. L.G.Zastavenko. TMF, 8, 335 (1971).
3. F.Coester, R.Haag. Phys.Rev., 117, 1137 (1970).
4. L.G.Zastavenko. TMF, 7, 20-29 (1971).
5. P.W.Higgs. Phys.Lett., 12, 132 (1964), Phys.Rev., 145, 1156 (1966).
6. S.Coleman. Commun.Math.Phys., 31, 259 (1973).
7. B.G.Conopelchenko. JETP, Pizma, 22, 424 (1975).
8. L.G.Zastanevko. JINR Preprint, E2-7909, Dubna, 1974.
9. L.G.Zastavenko. TMF, 9, 355 (1971).

Received by Publishing Department on November 4, 1976.