СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 10178

J.M.Charap, I.Bloomer

SPIN, TORSION AND THE SINGULARITY THEOREMS OF GENERAL RELATIVITY

1976

E2 - 10178

J.M.Charap*, I.Bloomer

SPIN, TORSION AND THE SINGULARITY THEOREMS OF GENERAL RELATIVITY

* Permanent address: Queen Mary College, Mile End Rd, London EI 4NS, England.

Чарап Дж.М., Блумер И. **E2 - 10178**

Спин, кручение и теоремы о сингулярностях в общей теории относительности

Приводятся аргументы в пользу того, что тензор спинового углового момента должен быть полностью антисимметричным. Как следствие этого спин-спиновое взаимодействие в общей теории относительности с кручением не может предотвратить появление сингулярностей.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований
Дубна 1976

Charap J.M., Bloomer I. **E2 - 10178**

Spin, Torsion and the Singularity Theorems of General Relativity

It is argued that the spin angular momentum tensor should be totally antisymmetric. As a consequence the spin-spin contact interaction of the torsion-modified version of general relativity cannot prevent the formation of singularities.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research
Dubna 1976

A number of authors[1] have considered the possibility that the modification to general relativity which admits torsion as well as curvature to the space-time manifold may so change the Hawking-Penrose[2] singularity theorems as to be able to prevent the occurrence of singularities. We wish to bring arguments against this hypothesis and in support of the conclusion reached by Kerlick[3].

The ECSK[4] field equations for this theory are, in the notation of Hehl et al.[1],

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R_k{}^k = k\,\Sigma_{ij}\,, \qquad (1)$$

$$S_{ij}{}^k + \delta_i^k S_{j\ell}{}^\ell - \delta_j^k S_{i\ell}{}^\ell = k\,\tau_{ij}{}^k\,, \qquad (2)$$

where $R_{ij} = R_{kij}{}^k$ is the Ricci tensor constructed from the non-symmetric connection $\Gamma^k{}_{ij}$, and $S_{ij}{}^k = \Gamma^k{}_{[ij]}$ is the torsion tensor. The tensors Σ_{ij} and $\tau_{ij}{}^k$ are respectively the canonical energy-momentum and spin angular-momentum tensor for matter, and k is the usual gravitational constant, $k = 8\pi G/c^4$. Still following Hehl et al., it is possible to obtain the equation

$$G^{ij}(\{\ \}) = k\,\tilde{\sigma}^{ij} \qquad (3)$$

for the Einstein tensor G^{ij} constructed from the Christoffel connection. The tensor $\tilde{\sigma}^{ij}$ is not the metric energy-momentum tensor σ^{ij} of the Einstein-Hilbert theory. Rather we have after the use of (2),

$$\tilde{\sigma}^{ij} - \sigma^{ij} \equiv \Delta^{ij} = \qquad (4)$$

$$= k\,[-4\tau^{ik}{}_{[\ell}\,\tau^{j\ell}{}_{k]} - 2\tau^{ik\ell}\,\tau^{j}{}_{k\ell} +$$

$$+\tau^{k\ell i}\,\tau_{k\ell} + \frac{1}{2}\,g^{ij}\,(4\tau_{m}{}^{k}{}_{[\ell}\,\tau^{m\ell}{}_{k]} + \tau^{mk\ell}\,\tau_{mk\ell})].$$

The modification introduced by admitting torsion is thus seen to be, in effect, to have generated a spin-spin contact interaction(5).

The Hawking-Penrose theorems are concerned with the focusing of geodesics, which being metrically defined are related to the Christoffel part of the connection, not to the torsion. The energy condition for the theorem arises from the requirement that for all time-like vectors ξ^{i} the quantity

$$R^{ij}(\{\ \})\,\xi_i\,\xi_j$$

should be non-negative. From Eq. (3) this means

$$(\tilde{\sigma}^{ij} - \frac{1}{2}\,g^{ij}\,\tilde{\sigma}_{k}{}^{k})\,\xi_i\,\xi_j \geq 0. \qquad (5)$$

Hehl et al. /1/, who give this inequality, make the important point that if it is violated for some time-like vector ξ^{i} a singularity may be prevented. They also show how just such a violation may arise in certain models, all of which are based on a semi-classical spin fluid for which $\tau_{ij}{}^{k} = s_{ij}\,u^{k}$ with the spin density s_{ij} transverse, $s_{ij}\,u^{j} = 0$. They warn that the

identification of $\tau_{ij}{}^{k}$ so defined with the canonical spin angular momentum tensor may be at fault. This is the point to which we wish to direct attention, as did Kerlick /3/.

If at the microscopic level we admit that matter should be represented by local fields, it seems reasonable to restrict attention to fields of spin zero, gauge fields of spin one, and Dirac fields of spin one-half. In the first two cases the canonical spin angular-momentum tensor vanishes. This is not unexpected for spin zero, but it is also true for a gauge-field description of spin one. The reason is that the derivatives of the potentials $A^{a}{}_{\mu}$ only enter through the covariant curl, and if the potentials are correctly treated as I -forms $\underset{\sim}{A}^{a} = A^{a}_{\mu}\,dx^{\mu}$ rather than as vectors, the fields $G^{a}_{\mu\nu}$ are the components of a 2-form obtained by taking the exterior derivative of $\underset{\sim}{A}^{a}$, thus

$$\underset{\sim}{G}^{a} \equiv \frac{1}{2}\,G^{a}_{\mu\nu}\,dx^{\mu}\wedge dx^{\nu} = \qquad (6)$$

$$= d\underset{\sim}{A}^{a} - \frac{1}{2}\,f^{a}{}_{bc}\,\underset{\sim}{A}^{b}\wedge\underset{\sim}{A}^{c}.$$

This means that the covariant curl is independent of the connection, even in the presence of torsion.

In the case of the Dirac field (3), we have

$$\tau_{ijk} = \frac{1}{4}\,\bar{\psi}\,\gamma_{[i}\,\gamma_{j}\,\gamma_{k]}\,\psi \qquad (7)$$

$$= \frac{1}{4}\,\epsilon_{ijk\ell}\,\bar{\psi}\,\gamma_{5}\,\gamma^{\ell}\,\psi$$

$$= \frac{1}{4}\,\epsilon_{ijk\ell}\,a^{\ell},$$

where a^ℓ is the axial vector current density.

It should be noted that the tensor τ_{ijk} defined by Eq. (7) is totally antisymmetric. We would suggest that this is a very desirable feature for at least two reasons. First, it means that the spin angular-momentum tensor is irreducible, and can be given as in Eq. (7) in terms of an axial vector. Second, it means that the torsion likewise is totally antisymmetric, and as a consequence the antiparallels

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \tag{8}$$

coincide with the geodesics

$$\ddot{x}^i + \{^{\,i}_{jk}\}\dot{x}^j\dot{x}^k = 0. \tag{9}$$

It seems to us a credible assumption to impose in general. And it excludes the spin-fluid model discussed by Hehl et al./1/.

If we admit Eq. (7), the tensor Λ^{ij} defined by Eq. (4) is simply given by

$$\Lambda^{ij} = \frac{k}{16}(2a^i a^j + g^{ij}a^k a_k)$$

so that

$$(\Lambda^{ij} - \frac{1}{2}g^{ij}\Lambda^k{}_k)\xi_i\xi_j = \frac{k}{8}[(\xi\cdot a)^2 - \xi^2 a^2], \tag{10}$$

from which it follows that the expression in (10) is positive-definite for time-like ξ^k. Thus the effect of the spin-spin contact interaction can only enhance, not prevent the formation of the singularity. The same result has also been obtained by direct consideration of the Raichaudhuri equation

under the same hypothesis of totally antisymmetric torsion (6).

One of us (JMC) would like to thank the Directorate of the Laboratory of Theoretical Physics of the Joint Institute for Nuclear Research and his many friends there for their generous hospitality during the period when this work was completed.

REFERENCES

1. F.W.Hehl, P. von der Heyde, and G.D.Kerlick. Phys.Rev., D10, 1066 (1974) and references therein.
2. S.W.Hawking and R.Penrose. Proc. Roy. Soc.Lond., A314, 529 (1969).
   See also S.W.Hawking and G.F.R.Ellis. The Large Scale Structure of Space-Time (CUP, Cambridge), 1973.
3. G.D.Kerlick. Phys.Rev., D12, 3005 (1975).
4. A.Einstein. Sber. preuss. Akad. Wiss., 844 (1915); Ann. der Phys., (4), 49, 769 (1916).
   E.Cartan. Ann.Ec.Norm.Sup., (3), 40, 325 (1923); ibid. 41, 1 (1924); ibid. 42, 17 (1925).
   D.W.Sciama. In Recent Developments in General Relativity (Pergamon, Oxford), 1962; cf. J.Math.Phys., 2, 472 (1961).
   T.W.B.Kibble. J.Math.Phys., 2, 212 (1961).
   For a review see F.W.Hehl. Gen.Rel.Grav. 4, 333 (1973); ibid. 5, 491 (1974).
   Cf. also G.D.Kerlick. Ph.D.Thesis. Princeton University (unpublished),1975.
5. T.W.B.Kibble, op cit (1961).
   Cf. also D.Ivanenko. Acta Phys.Hung.,32, 341 (1972).

6. I.Bloomer. Ph.D.Thesis, London University (unpublished), 1976.

Received by Publishing Department
on October 21, 1976.