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CONNECTION BETWEEN LIGHT CONE SINGULARITIES AND THE ASYMPTOTIC BEHAVIOUR OF THE MOMENTS OF THE STRUCTURE FUNCTIONS

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1. Introduction

It is now a well-known fact that the asymptotic behaviour of the deep inelastic scattering is connected with the light cone singularities of the current products. At first such a connection has been established between the leading light cone singularities and the leading terms of the structure functions [1,2].

In connection with asymptotical free field theories [9] and conformal invariant theories operator product expansions, valid near the light cone, have been postulated [3]. For scalar currents $j(x)$ it looks like

$$[j(x), j(0)] \approx \sum C_n(x^2)\, x_{\mu_1} \cdots x_{\mu_n}\, O^{\mu_1 \cdots \mu_n}, \tag{1.1}$$

where the operators $O^{\mu_1 \cdots \mu_n}$ are characterized by spin n and scale dimension d_n. The generalized functions C_n carry light cone singularities in decreasing order. For one particle matrix elements occurring in the description of deep inelastic scattering (1.1) takes the form

$$\langle p | [j(x), j(0)] | p \rangle \underset{x^2 \approx 0}{\approx} \sum C_n(x^2)\, P_n(x_0, x^2) \tag{1.2}$$

Here P_n are polynomials in x_0 of order n, $p = (1, \vec{0})$. For the leading terms a connection to the moments [4] $\mu_{2n}(Q^2) = \int_0^1 d\xi\, \xi^{2n-1}\, W(\xi, Q^2)$ (1.3), W structure functions, has been established [5]

$$\mu_{2n}(Q^2) \approx (Q^2)^{2n} \left(\frac{\partial}{\partial q^2}\right)^{2n} \tilde{C}_{2n}(q^2), \quad Q^2 = -q^2 \to \infty . \tag{1.4}$$

It should be kept in mind, however, that up to now a light cone expansion (LCE) (1.1) has not yet been proved from general principles of quantum field theory. It is also unclear in which mathematical sense the series represents the operator product (e.g., asymptotical series or series for generalized functions). Each point of view bears its own difficulties. Leaving aside the problem of LCE for operator products itself we introduce on the basis of a Dyson-Jost-Lehmann (DJL) representation the globally valid expansion for the matrix elements of currents /6/

$$\langle P|[j(x),j(0)]|P\rangle = \sum (\vec{x}^2)^n f_{2n}(x^2). \qquad (1.5)$$

In spite of some similarities with eq. (1.2) the difference is that there is no apriori reason for an ordering of the light cone singularities of the generalized functions $f_{2n}(x^2)$ because eq. (1.5) is primarily a Taylor expansion with respect to $(\vec{x})^2$. Nevertheless, it turns out that this expansion has a unique asymptotic connection with the slightly modified moments

$$\hat{\mu}_{2n}(Q^2) = \int_0^1 d\xi\, \xi^{2n-1}\, W(\xi,Q^2)\left(1+\frac{4\xi^2}{Q^2}\right)^{-n} + O\left(\frac{1}{Q^2}\right). \qquad (1.6)$$

More difficult is the investigation of the one-to-one correspondence between the leading light cone singularities of f_{2n} and the asymptotic behaviour of the moments $\hat{\mu}_{2n}$. It turns out that such a relation can be established if further conditions on the spectral functions are imposed.

2. Integral representations for $f_{2n}(x^2)$ and $\hat{\mu}_{2n}(Q^2)$ following from the DJL representation

In the following $W(q,P)$ denotes an invariant structure function connected by Fourier transform to the invariants $C(x,P)$ of the one-particle matrix elements of the current commutator. For technical reasons it is more convenient to deal with the symmetrized commutator $\bar{C}(\vec{x},x^2)$ defined by

$$(C(x),\varphi(x)) = \left(\bar{C}(\vec{x},x^2),\ \frac{\varphi(\vec{x},\sqrt{\vec{x}^2+x^2})-\varphi(\vec{x},-\sqrt{\vec{x}^2+x^2})}{2\sqrt{\vec{x}^2+x^2}}\right). \qquad (2.1)$$

The DJL representation of $\bar{C}$ reads

$$\bar{C}(\vec{x},x^2) = \frac{1}{4i\pi^2}\frac{\partial}{\partial x^2}\left\{\theta(x^2)\int_0^\infty d\lambda^2\, J_0(\lambda\sqrt{x^2})\int_{|\vec{u}|\le 1} d\vec{u}\; e^{i\vec{u}\vec{x}}\,\psi(\vec{u},\lambda^2)\right\}. \qquad (2.2)$$

It is essential to remark, that the functional $(C(\vec{x},x^2),\varphi(x^2))$ $\varphi(x^2)\in S_+$ is an entire function of the variable $\vec{x}$. To show this one has to take into account that

$$\varphi_B(\lambda^2) = \left(\frac{\partial}{\partial x^2}\{\theta(x^2)J_0(\lambda\sqrt{x^2})\},\ \varphi(x^2)\right) \qquad (2.3)$$

belongs to $S_+(R_1)$ too /2/, so that

$$(\bar{C}(\vec{x},x^2),\varphi(x^2)) = \frac{1}{4i\pi^2}\int_{|\vec{u}|\le 1} d\vec{u}\; e^{i\vec{u}\vec{x}}\,(\psi(\vec{u},\lambda^2),\varphi_B(\lambda^2)). \qquad (2.4)$$

Therefore it is justified that the exponential in (2.2) can be expanded as follows

$$\int d\vec{u}\; e^{i\vec{u}\vec{x}}\,\psi(\vec{u},\lambda^2) = \sum \frac{i^{2n}}{(2n)!}(\vec{x})^{2n}\frac{1}{2n+1}\int d\vec{u}\,(\vec{u})^{2n}\psi(\vec{u},\lambda^2). \qquad (2.5)$$

This leads to the expansion of $\bar{C}$

$$\bar{C}(\vec{x},x^2) = \frac{1}{4i\pi^2}\sum\frac{i^{2n}}{(2n)!}(\vec{x})^{2n} f_{2n}(x^2), \qquad (2.6)$$

where

$$f_{2n}(x^2) = \frac{\partial}{\partial x^2}\left(\theta(x^2)\int_0^\infty d\lambda^2\, J_0(\lambda\sqrt{x^2})\, h_{2n}(\lambda^2)\right) \tag{2.7}$$

and

$$h_{2n}(\lambda^2) = \frac{1}{2n+1}\int d\vec{u}\,(\vec{u})^{2n}\,\psi(u,\lambda^2) \quad . \tag{2.8}$$

A few comments concerning expansion (2.6). It is valid in the usual sense after integration with a test function $\varphi(x^2) \in S_+$ and appears as a global expansion not restricted to the neighbourhood of the light cone $x^2 = 0$. It is simply the Taylor expansion of an entire function and for this reason there is primarily no ordering of the strength of the light cone singularities in eq. (2.6). It turns out that the coefficients f_{2n} by integral transforms are connected to appropriately defined moments of the structure function $W(q,p)$. We define such moments for space-like momenta $q^2 = -Q^2 < 0$ by

$$\hat{\mu}_{2n}(Q^2) = \int_0 d\eta\, \eta^{2n-1}\, W(Q^2,\eta) \tag{2.9}$$

with the new scaling variable η

$$\eta = \frac{Q^2}{2|\vec{q}|} = \xi\left(1 + \frac{4\xi^2}{Q^2}\right)^{-\frac{1}{2}} \quad . \tag{2.10}$$

As usual $\xi = Q^2/2q_0$. The integration runs over all positive η where $W \neq 0$.

An integral representation for $\hat{\mu}_{2n}$ can be obtained by inserting the DJL representation

$$W(q,p) = \varepsilon(q_0)\int d\vec{u}\int d\lambda^2\, \delta\!\left(q_0^2 - (\vec{q}-\vec{u})^2 - \lambda^2\right)\psi(\vec{u},\lambda^2) \tag{2.11}$$

into eq. (2.9). From an investigation of the integral

$$\int d\eta\, W(Q^2,\eta)\,\phi(\eta) = 2\pi Q^2\int d\rho\,\rho^2\int dz\, d\lambda^2\, \frac{\psi(\rho,\lambda^2)\,\rho z}{[Q^2+\rho^2+\lambda^2]^2}\,\phi\!\left(\frac{Q^2\rho z}{Q^2+\rho^2+\lambda^2}\right)$$

we learn that this expression is well defined for convergent λ^2 integrations only. By the evaluation of the moments, however, according to eq. (2.9) η^{2n-1} plays the role of the test function $\phi(\eta)$ and gives convergent λ^2 integrals for sufficiently high n. We get

$$\hat{\mu}_{2n}(Q^2) = (Q^2)^{2n}\int_0^\infty d\lambda^2\, \frac{\hat{h}_{2n}(\lambda^2)}{(Q^2+\lambda^2)^{2n+1}} \tag{2.12}$$

with

$$\hat{h}_{2n}(\lambda^2) = \frac{1}{2n+1}\int d\vec{u}\,(\vec{u})^{2n}\,\psi(\vec{u},\lambda^2-\vec{u}^2) \quad . \tag{2.13}$$

Another possible derivation of these relations starts from the DJL representation of the T-product, uses dispersion relations at fixed values of Q^2 and expresses the moments as suitable derivatives of the amplitude T

$$\hat{\mu}_{2n} = \frac{\pi}{(2n)!\,2^{2n}}\,(Q^2)^{2n}\,\Delta_q^n\, T\,\Big|_{Q^2 \text{ fix},\, \vec{q}=0} \quad . \tag{2.14}$$

From this approach it is clear that the above-mentioned difficulty is connected with the problem of subtractions. In the following it is always assumed that unsubtracted representations are valid, omitting the lowest moments if necessary.

3. Connection between light cone singularities and the asymptotic behaviour of the moments

The problem is now to prove a unique connection between the asymptotic behaviour of $\hat{\mu}(Q^2)$ for $Q^2 \to \infty$ and the light cone behaviour of $f(x^2)$ for $x^2 \to 0$ which would give a generalization of the well known results /1,2/ concerning the leading terms in the Bjorken region and the light cone, respectively.

A generalized function $f(x)$ has the q-limit of order k /2/ for $x \to \infty$ if

$$(t^{\kappa} L(t))^{-1} \, (f(tx), \varphi(x)) \underset{t\to\infty}{\rightarrow} (f_{\infty}(x), \varphi(x)) \tag{3.1}$$

where $L(t)$ is a suitable chosen weakly rising function with the property $L(at)/L(t) \to 1$ for $t \to \infty$. Definition (3.1) characterizes in a general way asymptotic power behaviour modified by weakly rising functions (e.g.,logarithms). A corresponding definition will be used to describe the behaviour of generalized functions as $x \to 0$.

In our case the asymptotic connection of the four functions $f_{2n}(x^2)$, $h_{2n}(\lambda^2)$, $\hat{h}_{2n}(\lambda^2)$ and $\hat{\mu}_{2n}(Q^2)$ has to be investigated. The starting point are the relations (2.7), (2.8), (2.12), (2.13).

$$f_{2n}(x^2) = \frac{\partial}{\partial x^2} \left(\theta(x^2) \int_0^{\infty} d\lambda^2 \, J_0(\lambda \sqrt{x^2}) \, h_{2n}(\lambda^2) \right), \tag{3.2a}$$

$$h_{2n}(\lambda^2) = \frac{1}{2n+1} \int d\vec{u} \, (\vec{u})^{2n} \, \Psi(\vec{u}, \lambda^2) \quad , \tag{3.2b}$$

$$\hat{h}_{2n}(\lambda^2) = \frac{1}{2n+1} \int d\vec{u} \, (\vec{u}^2)^{n} \, \Psi(\vec{u}, \lambda^2 - \vec{u}^2), \tag{3.2c}$$

$$\hat{\mu}_{2n}(Q^2) = (Q^2)^{2n} \int d\lambda^2 \, \frac{\hat{h}_{2n}(\lambda^2)}{(Q^2+\lambda^2)^{2n+1}} \; . \tag{3.2d}$$

We study the chain of connections in each direction separately.

3.1. Asymptotic behaviour of $f_{2n}(x^2)$ near the light cone implies the large Q^2 behaviour of $\hat{\mu}_{2n}(Q^2)$

Suppose $f_{2n}(x^2)$ to have a q-limit or order $-(\kappa_n - 2)$ or

$$\lim_{t\to\infty} t^{2-\kappa_n} L_n^{-1}(t) \left(f_{2n}\left(\tfrac{x^2}{t}\right), \varphi(x^2) \right) = \left(f^{0}_{2n}(x^2), \varphi(x^2) \right) \tag{3.3}$$

then

$$\lim_{t\to\infty} t^{-\kappa_n} L_n^{-1}(t) \left(h_{2n}(t\lambda^2), \varphi(\lambda^2) \right) = \left(h^{0}_{2n}(\lambda^2), \varphi(\lambda^2) \right), \tag{3.4}$$

which means that $h_{2n}(\lambda^2)$ has the q-limit of order κ_n as $\lambda^2 \to \infty$. As it has been shown /2/ the B-transformation (2.3) maps the test function space S_+ onto S_+ which leads to a corresponding mapping of the dual spaces $(f_{2n}(x^2), \varphi(x^2)) = (h_{2n}(\lambda^2), \varphi_0(\lambda^2))$. From this it follows that the existence of the limit in one space implies the same in the transformed space.

The next task is to show the existence of the q-limit of $\hat{h}_{2n}(\lambda^2)$ knowing the q-limit of $h_{2n}(\lambda^2)$. For this reason we consider the difference of two functionals

$$
\begin{aligned}
& t^{-\kappa_n} L_n^{-1}(t) \left(h_{2n}(t\lambda^2) - \hat{h}_{2n}(t\lambda^2), \varphi(\lambda^2) \right) \\
&= t^{-\kappa_n} L_n^{-1}(t) \int_{|\vec{u}|\le 1} d\vec{u} \int d\lambda^2 \, \Psi(\vec{u}, t\lambda^2) \left(\varphi(\lambda^2) - \varphi\left(\lambda^2 + \tfrac{\vec{u}^2}{t}\right) \right) \vec{u}^{2n} \\
&= -t^{-\kappa_n} L_n^{-1}(t) \frac{1}{2n+1} \frac{1}{t} \int_{|\vec{u}|\le 1} d\vec{u} \, (\vec{u})^{2n+2} \int d\lambda^2 \, \Psi(u, t\lambda^2) \, \varphi'\left(\lambda^2 + \theta \tfrac{\vec{u}^2}{t}\right) \\
&\underset{t\to\infty}{\longrightarrow} -t^{-\kappa_n - 1} L_n^{-1}(t) \left(h_{2n+2}(t\lambda^2), \varphi'(\lambda^2) \right).
\end{aligned} \tag{3.5}
$$

Sufficient for the vanishing of this functional and thereby for the existence of the q-limit of $\hat{h}_{2n}$ is the assumption that the differences of the q-limit orders of successive coefficients f_{2n} is smaller than one. This is in accordance with experimental indications and theoretical models /5,9/. In the course of the proof in the opposite direction the same conclusions could be done invoking positivity of W and consequently ordering $\hat{\mu}_{2n+2}(Q^2) \le \hat{\mu}_{2n}(Q^2)$. /7/

As last step it remains to derive the asymptotic behaviour of $\hat{\mu}_{2n}(Q^2)$ knowing the q-limit of $\hat{h}_{2n}(\lambda^2)$.

Keeping in mind that a generalized function is the N -order derivative of a continuous function we perform generalized partial integrations in eq. (3.2d), using $\delta(x-y) = \int dz \frac{(x-z)_+^{N-1}}{\Gamma(N)} \frac{(z-y)_+^{-N-1}}{\Gamma(-N)}$ we obtain

$$\hat{\mu}_{2n}(Q^2) = (-1)^N (Q^2)^{2n} \frac{\Gamma(n+1+N)}{\Gamma(n+1)} \int d\lambda^2 \frac{\hat{h}_{2n}^{(-N)}(\lambda^2)}{(Q^2+\lambda^2)^{n+N+1}} , \tag{3.6}$$

where $\hat{h}_{2n}^{(-N)}$ is given by

$$\hat{h}_{2n}^{(-N)} = \int_0^{\lambda^2} dy \frac{(\lambda^2-y)^{N-1}}{\Gamma(N)} \hat{h}_{2n}(y) . \tag{3.7}$$

Now let us show that N can be chosen sufficiently large so that $\hat{h}^{(-N)}$ has a classical asymptotic behaviour related to the order of the q -limit of $\hat{h}$. For this aim we consider

$$\hat{h}^{(-N)}(tx) = \int_0^{tx} dy \frac{(tx-y)^{N-1}}{\Gamma(N)} \hat{h}(y) = t^N \int_0^x dz \frac{(x-z)^{N-1}}{\Gamma(N)} \hat{h}(tz) . \tag{3.8}$$

For N large enough the function $(x-z)^{N-1}$ is sufficiently smooth so that it plays the role of a test function. Using the definition of the q -limit (3.1) we get

$$\frac{\hat{h}^{(-N)}(tx)}{t^{N+\kappa} L(t)} = \frac{1}{t^{\kappa} L(t)} \left(\hat{h}(tz), \frac{(x-z)_+^{N-1}}{\Gamma(N)} \right) \xrightarrow[t\to\infty]{} C(x) \tag{3.9}$$

or

$$\hat{h}^{(-N)}(x) \longrightarrow c\, x^{N+\kappa} L(x) \quad \text{for } x \to \infty . \tag{3.10}$$

It is now easy to evaluate the expression (3.6) for large Q^2 by inserting the asymptotic form (3.10)

$$\hat{\mu}_{2n}(Q^2) \approx c\, (Q^2)^{2n} (-1)^n \frac{\Gamma(n+N+1)}{\Gamma(n+1)} \int_0^\infty d\lambda^2 \frac{(\lambda^2)^{\kappa+N} L(\lambda^2)}{(Q^2+\lambda^2)^{n+N+1}}$$

$$\hat{\mu}_{2n}(Q^2) \approx c\, Q^{2\kappa_n} L_n(Q^2)\, (-1)^n \frac{\Gamma(n+1+N)}{\Gamma(n+1)} \int_0^\infty d\varsigma \frac{\varsigma^{\kappa_n+N}}{(1+\varsigma)^{n+N+1}} \tag{3.11}$$

Therefore supposing for $f_{2n}(x^2)$ the q -limit (3.3) on the light cone the asymptotic behaviour of the corresponding moment

$$\hat{\mu}_{2n}(Q^2) \sim Q^{2\kappa_n} L_n(Q^2) \tag{3.12}$$

is derived.

3.2. Asymptotic behaviour for large Q^2 of the moments determines the light cone behaviour

Looking at the relations (3.2) and taking into account the foregoing considerations it is obvious that the existence of a q -limit for $\hat{h}_{2n}$ implies the existence of a corresponding q -limit for $f_{2n}(x^2)$. The main problem is to draw conclusions for $\hat{h}_{2n}$ from the known asymptotic behaviour of $\hat{\mu}_{2n}$. Let us write representation (3.2d) in the form

$$F(x) = \int_0^\infty d\tau \frac{g(\tau)}{(x+\tau)} \tag{3.13a}$$

$$= (-1)^N N! \int_0^\infty d\tau \frac{g^{(-N)}(\tau)}{(x+\tau)^{N+1}} . \tag{3.13b}$$

Appropriate partial integrations have been performed so that $g^{(-N)}$ is a continuous function. As starting point we know the behaviour of $F(x)$ for $x \to \infty$

$$\lim_{x\to\infty} \frac{F(x)}{x^{\kappa} L(x)} = c \quad , \; \kappa<0 \; . \tag{3.14}$$

The line of reasoning goes as follows. In order to deal with the q -limit of $g(\tau)$ we consider the sequence of functions

$$F_t(z) = \int_0^\infty d\tau \frac{g_t(\tau)}{z+\tau} \; , \quad F_t(z) = \frac{F(tz)}{t^{\kappa} L(t)} \; , \quad g_t(\tau) = \frac{g(t\tau)}{t^{\kappa} L(t)} \; . \tag{3.15}$$

The representation guaranties that $F_t(\tau)$ is a sequence of functions holomorphic in the complex z -plane with a cut along the real negative axis. Because of (3.14) the limit on the positive real axis exists

$$\lim_{t\to\infty} F_t(x) = c\, x^{\kappa} = F_\infty(x) \quad , \quad \text{for } x>0 \; . \tag{3.16}$$

For the investigation of the q -limit of g the functional $(g_t(x), \varphi(x))$ has to be considered taking into account the known properties of $F(x)$. For this purpose the complex inversion formula /11/ is applied

$$(g_t(x), \varphi(x)) = \frac{1}{2\pi i} \lim_{y\to 0} \left(F_t(-x+iy) - F_t(-x-iy), \varphi(x) \right) \tag{3.17}$$

and afterwards the limit $t\to\infty$ performed

$$\begin{aligned} \lim_{t\to\infty} (g_t(x), \varphi(x)) &= \frac{1}{2\pi i} \lim_{t\to\infty} \lim_{y\to 0} \left(F_t(-x+iy) - F_t(-x-iy), \varphi(x) \right) \\ &= \frac{1}{2\pi i} \lim_{y\to 0} \lim_{t\to\infty} \left(F_t(-x+iy) - F_t(-x-iy), \varphi(x) \right) \\ &= \frac{1}{2\pi i} \lim_{y\to 0} \left(F_\infty(-x+iy) - F_\infty(-x-iy), \varphi(x) \right) . \end{aligned} \tag{3.18}$$

In order to obtain an expression for $\lim_{t\to\infty} (g_t(x), \varphi(x))$ containing the asymptotic behaviour of $F(x)$ only, it is essential to interchange the limiting procedures in eq. (3.18). A sufficient condition allowing this interchange is the existence of the single limits

$$\lim_{t\to\infty} F_t(z) = F_\infty(z) = c\, z^{\kappa} \quad \text{valid in the cut plane} \; , \tag{3.19}$$

$$\lim_{y\to 0} \left(F_t(x\pm iy), \varphi(x) \right) = \left(F_t(x\pm i0), \varphi(x) \right) \quad \text{valid for all } t \, , \tag{3.20}$$

where additionally one limit, say the limit $y\to 0$, must be uniform with respect to t. Therefore at first one has to show that the sequence of analytic functions $F_t(z)$ converges to the analytic function $F_\infty(z) = c\, z^{\kappa}$.

For this reason we list the known properties of $F(z)$ implied by the representation (3.13):

1. $F(z)$ is analytic in the cut plane
2. $F(z)$ fulfills the estimate /13/

$$|F(z)| \le C \frac{1+|z|^n}{|y|^m} \qquad C, n, m \text{ constants.} \tag{3.21}$$

3. If g possess a q -limit, i.e., $t^{-\kappa} L^{-1}(t) (g(t\tau), \varphi(\tau)) \to c$, then by repeating the same reasoning which led to (3.12) one obtains

$$F(z) \sim z^{\kappa} L(z) \quad , \quad |z|\to\infty \quad , \quad \arg z \neq \pi \tag{3.22}$$

consequently

$$\lim_{t\to\infty} F_t(z) = c\, z^{\kappa} \; . \tag{3.19}$$

Here we have assumed that the generalized constant $L_{(z)}$ occuring in the definition of the q-limit can be chosen analytic in the cut plane with $L(tz)/L_{(z)} \to 1$ for $t \to \infty$.
It is important to note that the existence of a q-limit for $g_{(\tau)}$ corresponds to the direction independence of the asymptotic behaviour of the function $F_{(z)}$.

Of course a general function $F_{(z)}$ given by eq. (3.13) must not necessarily have the property that the asymptotic behaviour given on the positive real axis extends to all directions. To isolate functions having this property we restrict the class of allowed generalized functions by the additional condition:

If

$$\lim_{t\to\infty} \frac{1}{t^{\kappa} L_{(t)}} \int_0^{\infty} d\tau \frac{g_{(\tau)}}{(t+\tau)} = c$$

then there exists an integer N_0, so that

$$\left| \frac{1}{t^{\kappa} L_{(t)}} \int_0^{x} d\tau \frac{g^{(-N_0)}_{(\tau)}}{(t+\tau)^{N_0+1}} \right| < M \quad \forall x>0,\ \forall t>t_0 . \tag{3.23}$$

In the appendix it will be shown that this condition implies eq. (3.19). On the other hand this condition is automatically fulfilled if the q-limit of the generalized function $g_{(\tau)}$ exists.

Let us now turn to the limit (3.14). The crucial point in the proof of the uniformity of this limit is the existence of a t independent estimate (3.21).

From the representation (3.13) follows directly $|F_{(z)}| \sim |y|^{-N}$ near the cut. This result together with eq. (3.23) allows one to refine the general estimate (3.21) as

$$|F_{(z)}| \le a \frac{1+|z|^{\kappa+N}}{|y|^N} |L_{(z)}| \tag{3.24}$$

so that

$$|F_{t(z)}| \le b \frac{1+|z|^{\kappa+N}}{|y|^N} \tag{3.25}$$

is independent of the parameter t. This finishes the proof that under condition (3.23) the generalized function has a q-limit. This means the observed asymptotic behaviour of the moments implies the existence of the q-limits for the spectral functions $\hat{h}_{2n}$, h_{2n} and finally for the light cone expansion coefficients f_{2n} if the space of allowed spectral functions is suitable restricted.

Similar problems arise for the two point function

$$G_{(-q^2)} = \int_{m_0^2}^{\infty} dm^2 \frac{\sigma_{(m^2)}}{m^2+q^2} \tag{3.26}$$

and the vacuum polarization (the latter written in once subtracted form)

$$\frac{1}{q^2} \Pi_{(-q^2)} = -\frac{1}{\pi} \int_{m_0^2}^{\infty} dm^2 \frac{\rho_{(m^2)}}{m^2} \frac{1}{m^2+q^2} \tag{3.27}$$

if the connection between asymptotic behaviour at space-like momenta and their imaginary parts is studied /8,10/.
Both formulas are transformations of type (3.13) with positive spectral functions. Obviously condition (3.23) is fulfilled so that a one-to-one connection between the asymptotic behaviour for $q^2 \to \infty$ and the q-limit of the spectral function exists.

4. Conclusions

From general principles of QFT the existence of a globally valid Taylor expansion follows for the symmetrized commutator which bears a great resemblance in structure to a light cone expansion. Furthermore appropriate moments of the structure functions can be defined which are connected with the coefficients of the Taylor expansion by integral transforms. Assuming the q-limit on the light cone the asymptotic behaviour of the moments can be evaluated. The proof in the opposite direction is more complicated. In general no conclusion can be drawn. If however, the space of generalized functions is suitable restricted the existence of the q-limit for the light cone expansion coefficients can be proven. In the case of positive spectral functions (Lehmann representation and vacuum polarization) this condition is fulfilled automatically.

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Appendix

Here we show that condition (3.23) implies the limit (3.22). Let us start with (simply denoting N_0 by N)

$$F_{(z)} = \int_0^\infty d\tau \, \frac{g^{(-N)}_{(\tau)}}{(\tau+z)^{N+1}} = \int_0^\infty d\tau \, \frac{g^{(-N)}_{(\tau)}}{(\tau+w)^{N+1}} \cdot \left(\frac{(\tau+w)}{\tau+x+iy} \right)^{N+1} , \qquad w>0$$

$$= \int_0^\infty d\tau \, g^{(-N)}_{(\tau)} \, (\tau+w)^{-N-1} \sum_{k=0}^{N+1} (-i)^k \binom{N+1}{k} A_1^{N+1-k} A_2^k$$

where

$$A_1 = \frac{(\tau+w)(\tau+x)}{(\tau+x)^2+y^2} , \qquad A_2 = \frac{(\tau+w)\,y}{(\tau+x)^2+y^2} .$$

Then the estimate

$$|F_{(z)}| \leq \sum_k \binom{N+1}{k} \left| \int_0^\infty d\tau \, g^{(-N)}_{(\tau)} \, (\tau+w)^{N+1} A_1^{N+1-k} A_2^k \right|$$

is valid. In the following a typical term of the sum will be discussed. With the notation

$$u_{(\tau,w)} = \int_0^\tau d\tau' \, \frac{g^{(-N)}_{(\tau')}}{(\tau'+w)^{N+1}} , \qquad V_{(\tau,w)} = A_1^{n_1} A_2^{n_2}$$

we get

$$I_{n_1,n_2} = \frac{1}{w^k L(w)} \int_0^\infty d\tau \, g^{(-N)}_{(\tau)} \, (\tau+w)^{N+1} A_1^{n_1} A_2^{n_2}$$

$$= \frac{1}{w^k L(w)} \, u_{(\tau,w)} \, v_{(\tau,w)} \Big|_0^\infty - \frac{1}{w^k L(w)} \int_0^\infty d\tau \, u_{(\tau,w)} \frac{dv}{d\tau}$$

We apply

$$\left| \frac{1}{w^k L(w)} \, u_{(\tau,w)} \right| < M \qquad \forall\, w > w_0 , \ \forall\, \tau \qquad (3.23)$$

$$\lim_{\tau\to\infty} v_{(\tau,w)} = \begin{cases} 0 & n_2 \geq 1 \\ 1 & n_2 = 0 \end{cases}$$

so that

$$|I_{n_1 n_2}| \leq C + \frac{1}{w^{\kappa} L(w)} \int_0^{\infty} d\tau \, |u| \left|\frac{dv}{d\tau}\right| \leq C + M \int_0^{\infty} d\tau \left|\frac{dv}{d\tau}\right| .$$

The last integral can be evaluated taking into account the zeros τ_i of $\frac{dv}{d\tau}$

$$\int d\tau \frac{dv}{d\tau} = -v\Big|_{\tau=0} + 2|v|\Big|_{\tau=\tau_1} + \cdots \quad .$$

There exist at least three zeros τ_i . In all cases $v(\tau_i)$ is bounded for all x, y, w so that

$$\frac{1}{w^{\kappa} L(w)} \, |F(z,w)| < \text{const} \quad \text{for } w = |z| \; ,$$

Applying now $|L(w)/L(z)| < \text{const}$
we finally get

$$\left| \frac{1}{z^{\kappa} L(z)} \; F(z) \right| < \text{const}.$$

Now we are able to prove the convergence of $F_{t}(z)$. At first we show that $F_{t}(z)$ is uniformly bounded for $z \in \mathcal{U}$ and all t.

$$F_{t}(z) = |z^{\kappa}| \left|\frac{L(tz)}{L(z)}\right| \left|\frac{1}{L(tz)(tz)^{\kappa}} F(zt)\right| < z^{\kappa} \cdot \text{const} .$$

$\mathcal{U}$ is a compact region of the z plane containing a part of the positive real axis but no points from the negative axis. Consequently, $G_{t}(z) = F_{t}(z) - C z^{\kappa}$ fulfills

$$|G_{t}(z)| < \text{const}.$$

for all $z \in \mathcal{U}$ and all t . Because of $\lim_{t\to\infty} G_{t}(z) = 0$ valid for the submanifold of the positive real axis, a well known theorem /12/ on analytic functions tells us that

$$\lim_{t\to\infty} G_{t}(z) = 0 \quad \text{for all} \quad z \in \mathcal{U} .$$

5. References

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