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KINEMATICAL RELATIONS FOR TWO-BODY REACTIONS WITH MOVING TARGET PARTICLE

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KINEMATICAL RELATIONS FOR TWO-BODY REACTIONS WITH MOVING TARGET PARTICLE

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I. Introduction

In order to investigate the interactions of fast particles with atomic nuclei, one needs often to solve kinematical problems with moving target particles. Among them two-body and quasi two-body reactions are distinguished as particularly simple and important ones because the collisions with quasi-free intranuclear nucleons can provide valuable information on the structure of nuclei, in particular on the Fermi motion of intranuclear nucleons.

In the present paper kinematical relations for two-body reactions are given in the case of arbitrarily moving target particles. Also, the expressions for extreme values of cosines of emission angles, momenta and total energies of secondary particles are calculated in the form convenient for numerical computations. Similar expressions for target at rest are given in detail, e.g., in refs./1,2/ and follow from our formulae as particular cases.

The dependence of the probability density function of the target particle momentum distribution on that of the distribution of emission angles and momenta of one of the secondary particles has been obtained. These relations may be applied to extract information on the Fermi motion of intranuclear nucleons.

II. Kinematics of two-body reactions

Let us consider the reaction

$$a_1 + a_2 \rightarrow a_3 + a_4 \qquad (1)$$

of spinless particles (or a reaction in which spins can be neglected what is usually assumed in high energy collisions).

Let m_i, $\vec{p}_i$, E_i, θ_i, φ_i, (i = 1, 2, 3, 4) be the mass, momentum, total energy and angular coordinates of $\vec{p}_i$ in the laboratory (Lab.) spherical system. From the principles of momentum and energy conservation we have

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 , \qquad (2)$$

$$E_1 + E_2 = E_3 + E_4 , \qquad (3)$$

where

$$E_i = p_i^2 + m_i^2 , \qquad (4)$$

and

$$p_i^2 \equiv |\vec{p}_i|^2 . \qquad (5)$$

We assume throughout this paper that particle 1 is the incident particle, particle 2 is the target particle and

$$p_1 > p_2 , \qquad (6)$$

and we choose the system of coordinates in which $\theta_1 = 0$.

From (4)-(6) we can get the following four forms of these dependences:

1. Dependence of p_3 on $\cos\theta_3$:

$$p_3 = p_3(\cos\theta_3) = \frac{\beta r \pm E\sqrt{\beta^2 - m_3^2(E^2 - r^2)}}{E^2 - r^2} , \qquad (7)$$

where

$$E = E_1 + E_2 , \qquad (8)$$

$$r = p \cdot \cos\theta_3 + p_2 \cdot \cos\theta_2 , \qquad (9)$$

$$p = p_1 + p_2 \cdot \cos\theta_2 , \qquad (10)$$

$$q = p_2 \cdot \sin\theta_2 \cdot \cos\varphi , \qquad (11)$$

$$\varphi = \varphi_2 - \varphi_3 , \quad k = \operatorname{sgn} q , \qquad (11')$$

$$\beta = \frac{1}{2}(m_1^2 + m_2^2 + m_3^2 - m_4^2) + E_1 E_2 - p_1 p_2 \cos\theta_2 . \qquad (12)$$

It can be proved (see App. 1) that $\beta > 0$ for $p_1 > p_2$.

The function (7) is single - valued for $\beta \geq E m_3$, then in (7) sign + should be taken; for $\beta < E m_3$ the function (7) is double - valued (see Table 1). In table 1 the cosines of the maximum angle $\theta_{3\,max}$ and the minimal angle $\theta_{3\,min}$ are denoted by x_1 and x_2, respectively. The cosine of the angle at which the function $p_3(\cos\theta_3)$ takes its extreme (maximal or minimal) value is denoted by x_0. For the double - valued function the solid lines in Table 1 are given for sign + in (7), those plotted by the dashed curves are for sign - . Table 1 shows the x_1, x_2, x_0 values and the corresponding p_3 values, and the shapes of the curves $p_3 = p_3(\cos\theta_3)$ given by (7) are plotted.

2. Dependence of E_3 on θ_3:

$$E_3 = E_3(\cos\theta_3) = \frac{E\beta \pm |r|\sqrt{\beta^2 - m_3^2(E^2 - r^2)}}{E^2 - r^2} . \qquad (13)$$

This function is single - valued (with sign + for $\beta \geq E m_3$) and is double - valued for $\beta < E m_3$ (see Table 2). This table gives x_1, x_2 and x_0, the cosine of the angle θ_3, at which the function E_3 takes its extreme values (they have the same values as in Table 1), and also the corresponding E_3 values. The shapes of the curves $E_3 = E_3(\cos\theta_3)$ have quite the same character as those of the curves (7) in the corresponding cases and therefore they are not drawn.

3. Dependence of θ_3 on p_3:

$$\cos\theta_3 = \frac{1}{s^2 p_3}\left\{p(EE_3-\beta) \pm |q|\sqrt{s^2p_3^2-(EE_3-\beta)^2}\right\}. \qquad (14)$$

This function is simple - valued for $\beta > Em_3$ and $k=0$, for $\beta < Em_3$ and $k = 0, -1$ when $\gamma \leq p$, and for $\beta = Em_3$ and $k=0,-1$, then sign + in (14) should be taken. For the remaining cases it is double - valued (see Table 3). Then the functions plotted by solid lines are for sign + in (14), and those plotted by dashed lines are for sign - in (14). The values of x_1, x_2 and x_0 form Table 3, and the corresponding p_3 values can be read from Table 1.

4. Dependence of θ_3 on E_3:

$$\cos\theta_3 = \frac{1}{s^2\sqrt{E_3^2-m_3^2}}\left\{p(EE_3-\beta)^2 \pm |q|\sqrt{s^2(E_3^2-m_3^2)-(EE_3-\beta)^2}\right\}. \qquad (15)$$

The situation is quite the same as for the case 3, and is presented in Table 3; only the notation " p_3 -axis" should be replaced by "E_3 - axis". The values of x_1, x_2 and x_0 from Table 3 and the corresponding E_3 values are given in Table 2.

III. Momentum distribution of target particles

Two-body reactions at high energies may be coveniently applied to obtain some information on the Fermi motion intranuclear nucleons[4]. In this case the particle a_2 of the reaction (1) can be regarded as the intranuclear nucleon. We shall now deduce a relation between the function of the probability

distribution of the momentum density of the particle a_2 from the reaction (1) and the distribution function of one of the secondary particles (a_3). For the two-body reaction (1) we assume the conservation of energy and momentum expressed by equations (2) and (3). Physically this corresponds closely to the scattering of electrons of energy $E_e \gtrsim 100$ MeV on nuclei when the observed particle is the scattered electron.

We start with the theorem concerning mathematical statistics[5] which will be useful for our further considerations. Let $f(x,y)$ be the probability density. Let us consider the reversible and continuous (with the first derivatives) transformation from the variables (x,y) to the variables (u_1,u_2) in some rectangular region of (x,y)

$$\begin{aligned} x &= h_1(u_1,u_2) \\ y &= h_2(u_1,u_2) \end{aligned} \qquad (16)$$

and denote by J the Jacobian of this transformation. Then the probability density in terms of the variables u_1, u_2 equals

$$h(u_1,u_2) = f(h_1(u_1,u_2), h_2(u_1,u_2))|J| \quad . \qquad (17)$$

This theorem can be generalized to probability densities being functions of an arbitrary number of variables.

For our problem we denote by f the probability density as a function of four variables p_3, $\cos\theta_2$, θ_3, φ and consider the reversible transformation to the variables p_2, $\cos\theta_2$, θ_3, φ, namely

$$\begin{aligned} p_2 &= g_1(p_3,\cos\theta_2,\theta_3,\varphi), \\ \cos\theta_2 &= \cos\theta_2, \\ \theta_3 &= \theta_3, \\ \varphi &= \varphi. \end{aligned} \qquad (18)$$

The inverse transformation is

$$
\begin{aligned}
p_3 &= h_1(p_2, \cos\theta_2, \theta_3, \varphi), \\
\cos\theta_2 &= \cos\theta_2, \\
\theta_3 &= \theta_3, \\
\varphi &= \varphi.
\end{aligned} \tag{19}
$$

Then the probability density as a function of the variables p_2, $\cos\theta_2$, θ_3, φ can be expressed as follows

$$h(p_2, \cos\theta_2, \theta_3, \varphi) = f(h_1(p_2, \cos\theta_2, \theta_3, \varphi), \cos\theta_2, \theta_3, \varphi)/J/, \tag{20}$$

where

$$J = \frac{\partial p_3}{\partial p_2}. \tag{21}$$

From (2), (3), (8), (9) we get

$$EE_3 - rp_3 = \beta \tag{22}$$

Then

$$\frac{\partial p_3}{\partial p_2} = \frac{(\cos\theta_2 \cdot \cos\theta_3 + \sin\theta_2 \sin\theta_3 \cos\varphi)p_3 - \frac{Ep_2}{p_3} - p_1\cos\theta_2}{E\frac{p_3}{E_3} - r}, \tag{23}$$

where p_3 is given by (7).

Starting from general physical considerations, we can assume the function f as the product

$$f(p_2, \cos\theta_2, \theta_3, \varphi) = f_1(\varphi) \cdot f_2(\cos\theta_2) \cdot f_{33}(p_3, \theta_3). \tag{24}$$

Now we introduce the supplementary function $\tilde{h}$ defined for $p_2 \geq 0$, $-1 \leq \cos\theta_2 \leq 1$, $0 \leq \theta_3 \leq \pi$, $0 \leq \varphi \leq 2\pi$ as follows

$$\tilde{h}(p_2, \cos\theta_2, \theta_3, \varphi) = \begin{cases} h(p_2, \cos\theta_2, \theta_3, \varphi) & \text{when } p_3 \text{ exists} \\ 0 & \text{when } p_3 \text{ does not exist.} \end{cases} \tag{25}$$

Then the probability density of the distribution of momentum p_2 of particle a_2 from the reaction (1) is equal to

$$W(p_2) = \int_{-1}^{+1} d\cos\theta_2 \int_0^{\pi} d\theta_3 \int_0^{2\pi} \tilde{h}(p_3, \cos\theta_2, \theta_3, \varphi)\, d\varphi. \tag{26}$$

Appendix I

We prove that $\beta > 0$.

From (2) we have

$$p_2 \sin\theta_2 = \pm(p_3 \cdot \sin\theta_3 - p_4 \sin\theta_4), \tag{A1}$$

$$p_1 + p_2\cos\theta_2 = p_3 \cdot \cos\theta_3 + p_4\cos\theta_4. \tag{A2}$$

Squaring (A1) and (A2) and adding, we get after short calculations

$$p_3^2 + p_4^2 - p_1^2 - p_2^2 - 2p_1p_2\cos\theta_2 = -2p_3p_4\cos(\theta_3 + \theta_4). \tag{A3}$$

From (3) and (4) we have

$$p_1^2 + m_1^2 + p_2^2 + m_2^2 + 2E_1E_2 = p_3^2 + m_3^2 + p_4^2 + m_4^2 + 2E_3E_4,$$

or

$$m_1^2 + m_2^2 + m_3^2 - m_4^2 = p_3^2 + p_4^2 + 2E_3E_4 - 2E_1E_2 + 2m_3^2.$$

Inserting (A4) into (12) and using (A3), we get

$$\beta = E_3E_4 - p_3p_4\cos(\theta_3 + \theta_4) + m_3^2. \tag{A4}$$

Since $E_3E_4 > p_3p_4$, we see that

$$\beta > 0.$$

Appendix 2

Notations used in Tables 1 ÷ 3 :

$$p = p_1 + p_2 \cos\theta_2 ,$$
$$q = p_2 \sin\theta_2 \cos\varphi ,$$
$$\varphi = \varphi_2 - \varphi_3 ,$$
$$r = p \cos\theta_3 + q \sin\theta_3 ,$$
$$\beta = \tfrac{1}{2}(m_1^2 + m_2^2 + m_3^2 - m_4^2) + E_1E_2 - p_1p_2\cos\theta_2 ,$$
$$\gamma = \sqrt{E^2 - \beta^2/m_3^2} ,$$
$$s = \sqrt{p^2 + q^2} ,$$
$$x_1 = \cos\theta_3 / \min ,$$
$$x_2 = \cos\theta_3 / \max ,$$

$x_0 = \cos\bar{\theta}_3$ for $p_3(E_3)$ maximal or minimal,

$$P = E^2 - p^2 ,$$
$$S = E^2 - s^2 ,$$
$$X = \sqrt{\beta^2 - m_3^2 P} ,$$
$$Y = \sqrt{\beta^2 - m_3^2 S} ,$$
$$Z = \sqrt{s^2 - \gamma^2} .$$

Table I

Analysis of the dependence (7) for $p > 0$.

Notations are done in Appendix 2.

Solid curves for the sign +, dashed curves for the sign - in (7).

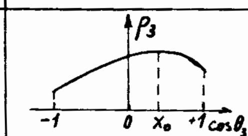
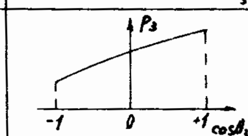
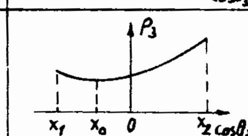
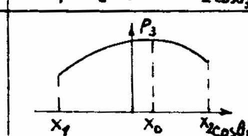
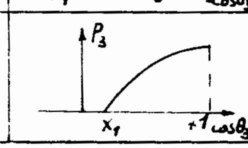
β	k	γ	plot	p_3 for x_1	$\bar{p}_3$ for x_0	p_3 for x_2
$>Em_3$	1 $(\varphi<\frac{\pi}{2})$	*		$(-p\beta+EX)P^{-1}$ for $x_1=-1$	$(s\beta+EY)S^{-1}$ for $x_0=ps^{-1}$	$(p\beta+EX)P^{-1}$ for $x_2=1$
	0 $(\varphi=\frac{\pi}{2})$	*		-״-	*	-״-
	-1 $(\varphi>\frac{\pi}{2})$	*		-״-	$(-s\beta+EY)S^{-1}$ for $x_0=ps^{-1}$	-״-
$=Em_3$	1 $(\varphi<\frac{\pi}{2})$	0		0 for $x_1=-qs^{-1}$	$2s\beta S^{-1}$ for $x_0=ps^{-1}$	$2p\beta P^{-1}$ for $x_2=1$
	0,-1 $(\varphi\geq\frac{\pi}{2})$	0		-״-	*	-״-

Table I (continuation)

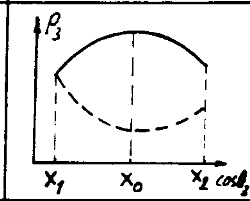
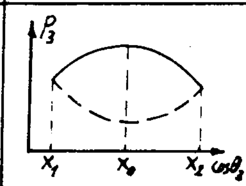
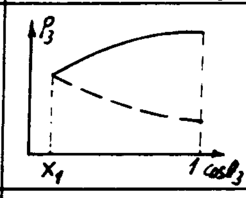
β	k	γ	plot	p_3 for x_1	$\bar{p}_3$ for x_0	p_3 for x_2
$<Em_3$	1 $(\varphi<\frac{\pi}{2})$	$0<\gamma<p$		$\gamma m_3^2\beta^{-1}$	$(s\beta\pm EY)S^{-1}$ for $x_0=pS^{-1}$	$(p\beta\pm EX)P^{-1}$ for $x_2=1$
		$p\leq\gamma\leq s$		—"—	—"—	$\gamma m_3\beta^{-1}$ for $x_2=(p\gamma+qZ)S^{-1}$
		$\gamma>s$	p_3 does not exist			
	0,-1 $(\varphi\geq\frac{\pi}{2})$	$\gamma\leq p$		—"—	*	$(p\beta\pm EX)P^{-1}$ for $x_2=1$
		$\gamma>p$	p_3 does not exist			

Table 2

Analysis of the dependence (13) for $p>0$.
Notations and plots are the same as in Table 1.

β	k	γ	E_3 for x_1	$\bar{E}_3$ for x_0	E_3 for x_2
$>Em_3$	1 $(\varphi<\frac{\pi}{2})$	*	$(E\beta-pX)P^{-1}$ for $x_1=-1$	$(E\beta+sY)S^{-1}$ for $x_0=pS^{-1}$	$(E\beta+pX)P^{-1}$ for $x_2=1$
	0 $(\varphi=\pi/2)$	*	—"—	*	—"—
	-1 $(\varphi>\frac{\pi}{2})$	*	—"—	$(E\beta-sY)S^{-1}$ for $x_0=pS^{-1}$	—"—
$=Em_3$	1 $(\varphi<\frac{\pi}{2})$	0	m_3 for $x_1=-qS^{-1}$	$(E^2+s^2)m_3S^{-1}$ for $x_0=pS^{-1}$	$(E^2+p^2)m_3P^{-1}$ for $x_2=1$
	0,-1 $(\varphi\geq\pi/2)$	0	—"—	*	—"—
$<Em_3$	1 $(\varphi<\frac{\pi}{2})$	$0<\gamma<p$	$Em_3\beta^{-1}$ for $x_1=(p\gamma-qZ)S^{-1}$	$(E\beta\pm sY)S^{-1}$ for $x_0=pS^{-1}$	$(E\beta\pm pX)P^{-1}$ for $x_2=1$
		$p\leq\gamma\leq s$	—"—	—"—	$Em_3\beta^{-1}$ for $x_2=1$
		$\gamma>s$	E_3 does not exist		
	0,-1 $(\varphi\geq\frac{\pi}{2})$	$\gamma\leq p$	$Em_3\beta^{-1}$ for $x_1=(p\gamma-qZ)S^{-1}$	$(E^2+s^2)m_3S^{-1}$ for $x_0=pS^{-1}$	$(E^2+p^2)m_3P^{-1}$ for $x_2=1$
		$\gamma>p$	E_3 does not exist		

Table 3

Analysis of the dependence (14) /plots/. Notations are the same as in Table 1. Solid curves corresponds to the sign +, dashed ones – to the sign – (14).

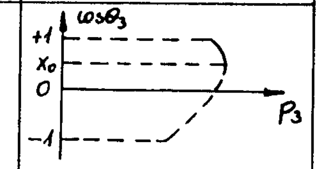
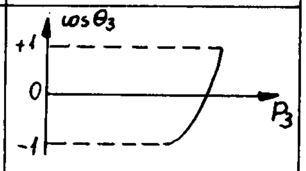
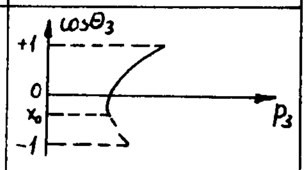
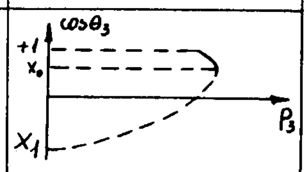
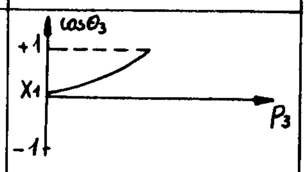
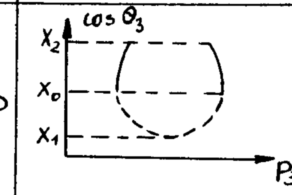
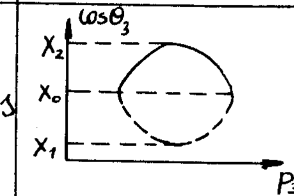
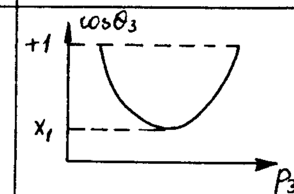
β	k	γ	Plot
$>E_{m_3}$	1 $(\varphi<\frac{\pi}{2})$	$*$	
	0 $(\varphi=\frac{\pi}{2})$	$*$	
	-1 $(\varphi\geqslant\frac{\pi}{2})$	$*$	
$=E_{m_3}$	1 $(\varphi<\frac{\pi}{2})$	0	
	$0,-1$ $(\varphi\geqslant\frac{\pi}{2})$	0	

Table 3 (continuation)

β	k	γ	Plot
$<E_{m_3}$	1 $(\varphi<\frac{\pi}{2})$	$0<\gamma<\rho$	
		$\rho\leqslant\gamma\leqslant s$	
		$\rho>s$	$\cos\theta_3$ does not exist
	$0,-1$ $(\varphi\geqslant\frac{\pi}{2})$	$\gamma\leqslant\rho$	
		$\gamma>\rho$	$\cos\theta_3$ does not exist

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