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In paper [1] the noncovariant classical and quantum theories of the relativistic string in a constant homogeneous electromagnetic field were constructed. The main difficulty encountered there was the proof of the relativistic invariance of obtained solutions in the quantum case. The usual method used in noncovariant quantization of the free string and based on the check of the Poincare algebra is inapplicable to electromagnetic field because it is unable to construct the conserved operators of the Lorentz rotations. Therefore **an** attempt to consider this problem in the covariant formalism is natural.

The action of the string in the electromagnetic field $F_{\mu\nu}(x)$ has the form

$$S=\int_{\tau_1}^{\tau_2}d\tau\int_0^l d\sigma\left\{-\gamma\left[(\dot{x}x')^2-\dot{x}^2x'^2\right]^{1/2}-g\,\dot{x}_\mu\dot{x}_\nu F_{\mu\nu}(x)\right\}, \tag{1}$$

where g is the absolute value of charges at the ends of the string and γ is a constant with dimension l^{-2}.

The variation of action (1) results in the equations of motion

$$\ddot{x}_\mu(\sigma,\tau)-x''_\mu(\sigma,\tau)=0 \tag{2}$$

and in the boundary conditions

$$\gamma x'_\mu+gF_{\mu\nu}\dot{x}_\nu=0\,,\quad \sigma=0,l. \tag{3}$$

In addition the functions $x_\mu(\sigma,\tau)$ as in the free string case must obey the subsidiary conditions

$$(\dot{x} \pm x')^2 = 0. \tag{4}$$

The solution to Eqs. (2) and (3) can be expressed in terms of Fourier series

$$x_\mu(\sigma,\tau) = \frac{i}{\sqrt{\pi\gamma}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{2n}\left[\alpha_{n\mu} e^{-i\frac{n\pi}{l}(\sigma+\tau)} + (1-f)^{-1}_{\mu\rho}(1+f)_{\rho\beta}\alpha_{n\beta} e^{i\frac{n\pi}{l}(\sigma-\tau)}\right] + \tag{5}$$

$$+ (1-f^2)^{-1}_{\mu\rho}\,\mathring{x}_\rho + \frac{1}{l\gamma}P_\mu \tau - f_{\mu\rho}\frac{P_\rho}{l\gamma}\left(\sigma - \frac{l}{2}\right),$$

where

$$f_{\mu\nu} = \frac{q}{\gamma}F_{\mu\nu}, \quad \alpha_{-n\mu} = \alpha^*_{n\mu}, \quad \mu = 0,1,2,3, \quad \alpha_{n4} = i\alpha_{n0}.$$

This solution describes the motion of the relativistic string in such fields that

$$\det\|(1-f)_{\mu\nu}\| = \det\|(1+f)_{\mu\nu}\| = 1 + \left(\frac{q}{\gamma}\right)^2(H^2 - E^2) - \left(\frac{q}{\gamma}\right)^4(\vec{E}\vec{H})^2 \neq 0.$$

Specifically, the expansion (5) is not suitable when $\left(\frac{q}{\gamma}\right)^2 E^2 = 1$ and $\vec{H} = 0$ (see paper /1/).

The expressions of the inverse matrices $(1-f)^{-1}$ and $(1-f^2)^{-1}$ in Eq. (5) are not required below. Nevertheless, we note that these matrices were obtained in paper /2/ .

The substitution of the expansion (5) into (4) gives

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} \alpha_{n-m}\alpha_m = 0, \quad n = 0,1,2,\ldots, \tag{6}$$

where

$$\alpha_{0\mu} = \frac{1}{\sqrt{\pi\gamma}}(1-f)_{\mu\rho}P_\rho, \quad L_{-n} = L^*_n.$$

Thus the subsidiary conditions (4) in terms of normal modes α_n have the same form as in the free string case /3/ .

The canonical momentum of the string $\pi_\mu = \partial\mathcal{L}/\partial\dot{x}_\mu$ according to Eq. (4) is

$$\pi_\mu = \gamma(\dot{x}_\mu + f_{\mu\nu}x'_\nu). \tag{7}$$

Inserting (5) into (7) we get

$$\pi_\mu(\sigma,\tau) = \frac{\sqrt{\pi\gamma}}{l}\sum_{n=-\infty}^{+\infty}(1+f)_{\mu\rho}\alpha_{n\rho}e^{-i\frac{n\pi}{l}\tau}\cos\left(\frac{n\pi}{l}\sigma\right). \tag{8}$$

The canonical momentum of the string as a whole is

$$\Pi_\mu = \int_0^l d\sigma\, \pi_\mu(\sigma,\tau) = \sqrt{\pi\gamma}\,(1+f)_{\mu\rho}\alpha_{0\rho} = (1-f^2)_{\mu\rho}P_\rho.$$

It was shown in paper /1/ , that Π_μ is conserved if $F_{\mu\nu}$ = const. Therefore defining the rest mass of the string with Π_μ we obtain

$$M^2 = -\Pi^2 = -P_\mu(1-f^2)^2_{\mu\nu}P_\nu. \tag{9}$$

The matrix polynomial $[(1-f^2)^2]_{\mu\nu} = (1-2f^2+f^4)_{\mu\nu}$ can be reduced to the squared polynomial with respect to the tensor of the electromagnetic field $f_{\mu\nu}$. For this purpose we have to take into account that $f_{\mu\nu}$ obeys its characteristic equation (the Hamilton-Cayley theorem /4/). This equation has the form /5/

$$\lambda^4 + I_1\lambda^2 - I_2 = 0,$$

where

$$I_1=\frac{1}{2}f_{ik}f_{ik}=\frac{1}{2}\left(\frac{q}{\gamma}\right)^2F_{ik}F_{ik}=\left(\frac{q}{\gamma}\right)^2(H^2-E^2),$$
$$I_2=\left(\frac{q}{\gamma}\right)^4(\vec{E}\vec{H})^2.$$

Consequently, $$f^4+I_1f^2-I_2=0$$

and Eq. (9) reads

$$M^2=-P^2(1+I_2)+(2+I_1)P_\mu(f^2)_{\mu\nu}P_\nu.$$

Now we use the subsidiary condition (6) for $n=0$

$$P_\mu(f^2)_{\mu\nu}P_\nu=P^2+\gamma\pi\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\alpha_{-m}\alpha_m. \tag{10}$$

Finally the squared mass of the string is

$$M^2=(1+I_1-I_2)P^2+\gamma\pi(2+I_1)\sum_{\substack{m=-\infty\\ m\neq 0}}^{m=+\infty}\alpha_{-m}\alpha_m. \tag{11}$$

It is interesting to observe in this equation the transition to the free string case. After that the invariants I_1 and I_2 in Eq. (11) are equated to zero it is necessary to use again the condition (10) which for $f_{\mu\nu}=0$ has the form

$$P^2=-\gamma\pi\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\alpha_{-m}\alpha_m.$$

Only now Eq. (11) reduces to the usual expression for the squared mass of the free string

$$M^2=\gamma\pi\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\alpha_{-m}\alpha_m. \tag{11'}$$

Let us go to the construction of the Hamiltonian formalism. The Lagrangian of the string in electromagnetic field (1) is singular. So there are constraints between canonical variables $x_\mu(\sigma,\tau)$ and $\pi_\mu(\sigma,\tau)$

$$\begin{aligned}\varphi_1&=(\gamma^{-1}\pi_\mu-f_{\mu\nu}x'_\nu)^2+x'^2_\mu=0,\\ \varphi_2&=(\gamma^{-1}\pi_\mu-f_{\mu\nu}x'_\nu)x'_\mu=0.\end{aligned} \tag{12}$$

The Hamiltonian constructed in accordance with the usual rules vanishes identically

$$\mathcal{H}=\pi\dot{x}-\mathcal{L}\equiv 0.$$

For the constrained systems the Hamiltonian formalism and the transition to the quantum theory were developed by Dirac /6/. The constraints (12) are primary constraints according to Dirac. Their Poisson bracket vanishes weakly so these are first class constraints. There are no other constraints following from the Lagrangian (1). As the Hamiltonian we have to take the linear combination of constraints (12)

$$H=\frac{\gamma}{2}\int_0^l d\sigma\left[\lambda_1(\sigma,\tau)\varphi_1(\sigma,\tau)+\lambda_2(\sigma,\tau)\varphi_2(\sigma,\tau)\right],$$

where λ_1 and λ_2 are arbitrary functions. This freedom can be used so that the equations of motion would be the most simple. As in the free string case we put $\lambda_1=1,\ \lambda_2=0$

$$H = \frac{T}{2}\int_0^l d\sigma \left[(\gamma^{-1}\pi_\mu - f_{\mu\nu} x'_\nu)^2 + x'^2_\mu \right]. \tag{13}$$

The Hamilton variational principle

$$\delta S = \delta \int_{\tau_1}^{\tau_2} d\tau \int_0^l d\sigma (\pi \dot{x} - \mathcal{H}) = 0$$

results in the equations of motion

$$\dot{x}_\mu = \partial \mathcal{H} / \partial \pi_\mu , \tag{14}$$

$$\dot{\pi}_\mu = \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{H}}{\partial x'_\mu} \right) \tag{15}$$

and in the boundary conditions

$$\frac{\partial \mathcal{H}}{\partial x'_\mu} = 0, \qquad \sigma = 0, l. \tag{16}$$

Equation (14) establishes the connection between canonical momenta π_μ and co-ordinates x_μ

$$\dot{x}_\mu = \gamma^{-1}\pi_\mu - f_{\mu\nu} x'_\nu . \tag{17}$$

Inserting (13) into (15) we get

$$\dot{\pi}_\mu = \gamma x''_\mu - (\pi_\sigma - \gamma f_{\sigma\nu} x''_\nu) f_{\sigma\mu} . \tag{18}$$

From Eqs. (17), (18) it follows that

$$x''_\mu - \ddot{x}_\mu = 0 .$$

The boundary conditions (16) with (17) take the form

$$x'_\mu + f_{\mu\sigma} \dot{x}_\sigma = 0, \qquad \sigma = 0, l .$$

So the equations of motion and the boundary conditions in the Hamiltonian formalism are the same as those in the Lagrangian method. Consequently, we can use as the solutions for $x_\mu(\sigma,\tau)$ and $\pi_\mu(\sigma,\tau)$ the expansions (5) and (8), respectively.

Substitution of Eqs. (5) and (8) into (13) gives

$$H = \frac{\pi}{l} L_0 = \frac{\pi}{l}\frac{1}{2}\sum_{m=-\infty}^{+\infty} \alpha_{-m}\alpha_m . \tag{19}$$

Using the expansions (5), (8) we see that constraints (12) reduce to Eq. (6) for the normal modes α_n.

In quantum theory we postulate the following commutation relations

$$[\alpha_m, \alpha_n] = m\,\delta_{m+n,0}, \quad m \neq 0, n \neq 0, [\mathring{x}_\mu, P_\nu] = i\,\delta_{\mu\nu}. \tag{20}$$

The commutators between $\mathring{x}_\mu, P_\nu$ and α_p are supposed to equal zero. These requirements are equivalent to the following commutators

$$[x_\mu(\sigma,\tau), \pi_\nu(\sigma',\tau)] = i\,\delta_{\mu\nu}\,\delta(\sigma - \sigma'),$$

$$[x_\mu(\sigma,\tau), x_\nu(\sigma',\tau)] = [\pi_\mu(\sigma,\tau), \pi_\nu(\sigma',\tau)] = 0 .$$

As in the free string case the quantum expressions for L_n have to be taken in the normal product form

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m}\alpha_m: - \delta_{n,0}\,\alpha(0),$$

where $\alpha(0)$ is the constant which appears via transition to the normal product in L_0 . This constant must be introduced obviously into Eqs. (11) and (19) for M^2 and H also.

The Heisenberg equations for operators α_n have the form

$$\frac{d}{d\tau}\alpha_n(\tau)=i\left[H,\alpha_n(\tau)\right]=-i\frac{n\pi}{l}\alpha_n(\tau).$$

Therefore
$$\alpha_n(\tau)=\alpha_n(0)e^{-i\frac{n\pi}{l}\tau}.$$

Thus, the expansions (5) and (8) are valid also in the quantum case.

In quantum theory the subsidiary conditions (12) or (6) have to be imposed onto the state vectors

$$\left[L_n-\delta_{n,0}\alpha(0)\right]|\Psi\rangle=0,\quad n\geq 0. \tag{21}$$

These conditions have the same form as in the free string case /3/ .

Now the problem is reduced to the proof of the absence of the negative norm states if Eq. (21) is satisfied, that is, the "no-ghost" theorem must be proved. The proof of this theorem for the free string /2/ cannot be applied in the case under consideration, as the formula (11) for the squared mass of the string in electromagnetic field and that for the free string (11') are essentially different; also the expansions (5), (8) of the canonical variables $x_\mu(\sigma,\tau)$ and $\pi_\mu(\sigma,\tau)$ differ in these cases. It may be supposed that for the "no-ghost" theorem proof the operators analogous to the DDF-operators for the free string /3,7/ will be useful.

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