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REMARKS ON THE OPERATOR OF SECOND QUANTIZATION

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In this paper we give a complete classification of the domains of the operator of second quantization. The classification is based on papers [1,2], but in section 2 we will repeat and extend the basic definitions to make this paper independent of [1,2].

1. By $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{R}, \boldsymbol{C}$ we denote the set of naturals, integers, real and complex numbers resp., assume $0 \in \boldsymbol{N}$.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{F}(\mathcal{H})$ the associated Fock space over $\mathcal{H}$, i.e.,

$$\mathcal{F}(\mathcal{H}) = \sum_{n=0}^{\infty} \oplus \mathcal{H}^{(n)} \qquad \text{(Hilbert direct sum)}$$

where $\mathcal{H}^{(0)} = \boldsymbol{C}$

$$\mathcal{H}^{(n)} = \mathcal{H} \otimes \ldots \otimes \mathcal{H} \qquad \text{(n-fold complete tensor product).}$$

Besides $\mathcal{F}(\mathcal{H})$ we use the symmetric or Boson Fock space $\mathcal{F}_s(\mathcal{H})$ and the antisymmetric or Fermion Fock space $\mathcal{F}_a(\mathcal{H})$.

Let A be a selfadjoint operator on the dense domain $\mathfrak{D}(A) \subset \mathcal{H}$ and $\mathfrak{D}_1 = \mathfrak{D}_1(A) \subset \mathfrak{D}(A)$ so that the restriction of A to $\mathfrak{D}_1$, $A|\mathfrak{D}_1$, is essentially selfadjoint. Then (/3/)

i) the operator $A^{(n)} = A\otimes I\otimes\ldots\otimes I + I\otimes A\otimes I\otimes\ldots\otimes I + \ldots + I\otimes\ldots\otimes I\otimes A$ is essentially selfadjoint on the domain

$\mathfrak{D}_1^{(n)} = \mathfrak{D}_1\otimes\ldots\otimes\mathfrak{D}_1 \subset \mathcal{H}^{(n)}$

ii) putting $A^{(0)} = 0$ on $\mathfrak{D}_1^{(0)} = \mathbf{C}$ the operator

$d\Gamma(A)' = \sum_{n=0}^{\infty} A^{(n)}$

is essentially selfadjoint on

$\mathfrak{D}(d\Gamma(A)') = \sum_{n=0}^{\infty} \mathfrak{D}_1^{(n)}$ (algebraic direct sum).

The operator $d\Gamma(A)'$ is called the <u>operator of second quantization of A</u>. As we use first of all the selfadjoint operator $d\Gamma A = \overline{d\Gamma(A)'}$, we will call it also second quantization of A. Besides $d\Gamma(A)$ we regard the corresponding "symmetric" and "antisymmetric" second quantizations of A, namely

$$d\Gamma(A)_s = d\Gamma(A)|\mathcal{F}_s(\mathcal{H}) \quad \text{and} \quad d\Gamma(A)_a = d\Gamma(A)|\mathcal{F}_a(\mathcal{H})$$

which are also selfadjoint.

The classification of $d\Gamma(A)$ is partly based on the following

<u>Theorem 1 (/3/)</u>

Let $\{A_k\}, k = 1,\ldots,n$ be a family of selfadjoint operators on $\mathcal{H}_k$. The closure of $I\otimes\ldots\otimes A_k\otimes\ldots\otimes I$ on $\otimes\mathfrak{D}(A_k)$ will be denoted by A_k also. Let $P(x_1,\ldots,x_n)$ be a polynomial with real coefficients of degree n_k in x_k and suppose that $\mathfrak{D}_k^e$ is a domain of essential selfadjointness for A^{n_k}. Then

i) $P(A_1,\ldots,A_n)$ is essentially selfadjoint on

$\mathfrak{D}^e = \bigotimes_{k=1}^{n} \mathfrak{D}_k^e$

ii) The spectrum of $P(A_1,\ldots,A_n)$ is the closure of the range of the product of the spectra of A_k, i.e.,

$\sigma(P(A_1,\ldots,A_n)) = \overline{P(\sigma(A_1),\ldots,\sigma(A_n))}$:

First of all we use the following special case of this Theorem.

<u>Corollary 2 (/3/)</u>

Let $A_1,\ldots,A_n$ be selfadjoint operators on $\mathcal{H}_1,\ldots,\mathcal{H}_k$ and suppose that, for each k, $\mathfrak{D}_k$ is a domain of essential selfadjointness for A_k. Then,

i) $A = A_1 + \ldots + A_n$ is essentially selfadjoint on $\mathfrak{D} = \bigotimes_{k=1}^{n} \mathfrak{D}_k$.

ii) $\sigma(\bar{A}) = \overline{\sum_{k=1}^{n} \sigma(A_k)}$.

2. In this section we repeat and extend the basic definitions of /1/.

We recall some simple facts. Let A be a selfadjoint operator and $P(.)$ the associated family of spectral projections. We say that $\lambda\in\sigma_{ess}(A)$, , the <u>essential spectrum of A</u>, if and only if $P(\lambda-\epsilon,\lambda+\epsilon)$ is infinite dimensional for all $\epsilon > 0$, $\lambda\in\sigma_d(A)$, the <u>discrete spectrum of A</u>, if and only if $P(\lambda-\epsilon,\lambda+\epsilon)$ is finite dimensional for some $\epsilon > 0$. Then $\sigma(A) = \sigma_{ess}(A)\cup\sigma_d(A)$ is a disjoint decomposition of the spectrum of A.

From the spectral theorem it follows that to any selfadjoint operator A we can associate a diagonal operator $T = T\{(t_n), (\phi_n)\}$ such that

$$T\phi_n = t_n \phi_n, \quad \phi_n \in \mathfrak{D}(A) \qquad \text{for all } n,$$

(ϕ_n) is an orthonormal basis of $\mathcal{H}$,

$$\mathfrak{D} = \mathfrak{D}(T) = \mathfrak{D}(A) = \{\phi = \Sigma y_n \phi_n : \Sigma |y_n|^2 t_n^2 < \infty\}$$

(t_n) can be chosen $\subset \mathbf{Z}$, T commutes with A. We use the notation $\mu(t_n)$ for the multiplicity of t_n. If necessary we write $\mu_T(t_n)$ to indicate the dependence on T. Now let us repeat the classification of operators and domains in a more general form as in /1/ .

An arbitrary selfadjoint operator $T = T^*$ (bounded or not) is said to be of

Class I
if and only if $\sigma(T) = \sigma_d(T)$
especially, T is of

Class I_A if and only if $\sigma(T) = \sigma_d(T)$ is a finite set,

Class I_B if and only if $\sigma(T) = \sigma_d(T)$ is an infinite set;

Class II
if and only if $\sigma_{ess}(T)$ is bounded, non-empty, especially, T is of

Class II_A if and only if $\sigma_d(T)$ is bounded,

Class II_B if and only if $\sigma_d(T)$ is unbounded;

Class III
if and only if $\sigma_{ess}(T)$ is unbounded.

Class III splits into three subclasses. To give the precise definitions we introduce the following notations. For any $\lambda \in \sigma_d(T)$ let $\rho(\lambda) \in \sigma_{ess}(T)$ be such a point of the essential spectrum of T that

$$|\lambda - \rho(\lambda)| = \operatorname{dist}(\lambda, \sigma_{ess}(T)).$$

A subset $\mathfrak{M} \subset \sigma_d(T)$ is said to be non-essential with respect to $\sigma_{ess}(T)$ (or simply: non-essential) if and only if there are constants $C_1, C_2 > 0$ such that

$$C_1 |\lambda| \le |\rho(\lambda)| \le C_2 |\lambda| \qquad \text{for all } \lambda \in \mathfrak{M}$$

It is clear that any finite subset of $\sigma_d(T)$ is non-essential. A non-essential subset $\mathfrak{M} \subset \sigma_d(T)$ is said to be maximal if there is no non-essential subset $\mathfrak{N} \subset \sigma_d(T)$ such that $\mathfrak{M} \subset \mathfrak{N}$ and $\mathfrak{N} - \mathfrak{M}$ is an infinite set.

Now we can give the definitions. T is said to be of

Class III_A if and only if $\sigma_d(T) = \emptyset$ or $\sigma_d(T)$ is non-essential

Class III_B if and only if there exists a maximal non-essential subset $\mathfrak{M} \subset \sigma_d(T)$ such that $\sigma_d(T) \setminus \mathfrak{M}$ is an infinite set (i.e., $\sigma_d(T)$ is not non-essential)

Class III_C if and only if there is no maximal non-essential subset of $\sigma_d(T) \ne \emptyset$.

We write $T \in (I_A), (I_B), \ldots, (III_C)$.

Notice the following facts (cf. also /1/):

i) we say, the dense domain $\mathfrak{D} \subset \mathcal{H}$ is of class $I_A, \ldots, III_C$ if and only if there is a selfadjoint operator

$T\in(I_A),\dots,(III_C)$ with $\mathfrak{D}=\mathfrak{D}(T)$. This definition is independent of the special choice of T on $\mathfrak{D}$.

ii) T is bounded if and only if $T\in(I_A)$ or $\in(II_A)$.

$T\in(I_A)$ if and only if $\dim\mathfrak{D}<\infty$.

iii) all classes are non-empty and disjoint.

In /1/ and /2/ there was given a more detailed classification in the language of diagonal operators which allows also a geometrical interpretation. Because we do not use this here, we refer to these papers.

To give the classification of $\mathfrak{D}(d\Gamma(A))$ we need the following two simple Lemmata.

Lemma 3

For the spectrum of $d\Gamma(A)=\overline{\sum_{n\ge0} A^{(n)}}$ there is

$$\sigma(d\Gamma(A)) \supset \bigcup_{n\ge1} \overline{\Big(\sum_{k=1}^{n} \sigma(A)\Big)} = \bigcup_{n\ge1}\sigma\big(\overline{A^{(n)}}\big) \supset \bigcup_{n\ge1} \Big(\sum_{k=1}^{n}\sigma(A)\Big).$$

Proof:

The last inclusion is trivial; the representation of the spectrum of $\overline{A^{(n)}}$ (i.e., the equality above) is given by Corollary 2 for $A_i = A$, , $1\le i\le n$.

We prove the first inclusion. Let $\lambda\in\sigma(\overline{A^{(n)}})$, $\phi_k^{(n)}\in\mathfrak{D}(\overline{A^{(n)}})$ with $\|\phi_k^{(n)}\|=1$, $\|(\overline{A^{(n)}}-\lambda)\phi_k^{(n)}\|\to 0$ for $k\to\infty$. Then for $\hat\phi_k^{(n)}=(0,\dots,\phi_k^{(n)},0,\dots)$ we have $\hat\phi_k^{(n)}\in\mathfrak{D}(d\Gamma(A))$, $\|\hat\phi_k^{(n)}\|=1$ and of course

$$\|(d\Gamma(A)-\lambda)\hat\phi_k^{(n)}\| = \|(\overline{A^{(n)}}-\lambda)\phi_k^{(n)}\|\to 0 \text{ for } k\to\infty$$

which says $\lambda\in\sigma(d\Gamma(A))$. Q.E.D.

In the case of diagonal operators we have a more explicit representation of the spectra, namely:

Lemma 4

Let $T=T\{(t_n),(\phi_n)\}$ be a diagonal operator. Then

i) The domain $\mathfrak{D}^e$ of essential selfadjointness for T can be taken to be the set of all finite linear combinations of (ϕ_n), and therefore $d\Gamma(T)'$ (and consequently also $d\Gamma(T)$) has the eigenvalues

$t_{k_1}+\dots+t_{k_m}$, $m=1,2,\dots$; $k_i\in N$ for all i,

or equivalently, the eigenvalues have the form

$m_1t_1+\dots+m_rt_r$, $r=1,2,\dots$; $m_i\ge 0$, integers,

$\sum m_i\ne 0$ in any case.

ii) $d\Gamma(T)_s$ has the same eigenvalues as $d\Gamma(T)$ and for the multiplicities $\mu_s=\mu_{d\Gamma(T)_s}$, $\mu=\mu_{d\Gamma(T)}$ we have $\mu_s(\lambda)\le\mu(\lambda)$; $\mu_s(\lambda)=\infty$ if and only if $\mu(\lambda)=\infty$.

iii) $d\Gamma(T)_a$ has the eigenvalues

$m_1t_1+\dots+m_rt_r$, $r=1,2,\dots$, $0\le m_i\le\mu_T(t_i)$,

$\sum m_i\ne 0$ in any case.

Proof:

i) is clear. ii) and iii) follow because any eigenvector $\phi=\phi_{n_1}\otimes\dots\otimes\phi_{n_r}\ne0$ of $d\Gamma(T)$ can be "symmetrized" and gives an eigenvector ϕ_s of $d\Gamma(T)_s$ with the same eigenvalue. But ϕ gives a non-vanishing antisymmetric vector ϕ_a only if $n_i\ne n_j$ for $i\ne j$ (and then the same eigenvalue).

Q.E.D.

The next Theorem deals with the classification of $\mathfrak{T}(d\Gamma(T))$ in dependence of $\mathfrak{T}(T)$.

Theorem 5

Let $T = T^* \neq 0$ be a selfadjoint operator. Then for $\mathfrak{T}(d\Gamma(T))$ we have

i) $\mathfrak{T}(T) \in (I)$ and T definite, i.e., $T > 0$ or $T < 0$ implies $\mathfrak{T}(d\Gamma(T)) \in (I_B)$.

ii) Each of the following conditions implies $\mathfrak{T}(d\Gamma(T)) \in (III_A)$:

a) $\mathfrak{T}(T) \in (I)$ and T indefinite or semidefinite (i.e., $0 \in \sigma(T)$))

b) $\mathfrak{T}(T) \in (II)$, $\mathfrak{T}(T) \in (III)$.

Proof:

ad i) Because $\mathfrak{T}(d\Gamma(T)) = \mathfrak{T}(d\Gamma(-T))$ we may assume $T > 0$. Lemma 4 gives

$$\sigma(d\Gamma(T)) = \{\lambda = \sum_{r=1}^{i} m_r t_r,\ i = 1,2,\dots;\ m_r \geq 0 \text{ integers}, \lambda \neq 0\}$$

$T \in (I)$, $T > 0$ implies $\mu(\lambda) < \infty$ for all $\lambda \in \sigma(d\Gamma(T))$ and $\sigma(d\Gamma(T))$ unbounded, i.e., $\mathfrak{T}(d\Gamma(T)) \in (I_B)$.

ad ii) First we prove that a) and b) imply $\mathfrak{T}(d\Gamma(T)) \in (III)$ and then it will be shown that from $\mathfrak{T}(d\Gamma(T)) \in (III)$ it follows $\mathfrak{T}(d\Gamma(T)) \in (III_A)$. $T \in (I)$ and T indefinite means that there are $a, b \in \sigma(T) = \sigma_d(T)$ with $a < 0 < b$. From Lemma 4 it follows that $\{ma + nb;\ m,n \geq 1\} \subset \sigma(d\Gamma(T))$. It is easy to see that the set of accumulation points of $\{ma + nb\}$ contains a sequence (d_n) with

$$\lim d_n = \infty,\ |d_{n+1} - d_n| \leq r \quad \text{for a fixed } r > 0 \qquad (1)$$

Therefore, clearly, $\mathfrak{T}(d\Gamma(T)) \in (III)$. The case where T semidefinite is also not complicated.

Now we regard the case b). $T \in (II)$ or $\in (III)$ means that there is a $\lambda \in \sigma_{ess}(T)$, i.e., there is an orthonormal set $\{\psi_k\}$ such that

$$\|(T - \lambda)\psi_k\| \to 0 \quad \text{for} \quad k \to \infty.$$

First suppose $\lambda \neq 0$. Then $\phi_k = \psi_k \otimes \dots \otimes \psi_k$ (n factors) is an orthonormal set in $\mathfrak{T}(T^{(n)})$ and

$$\|(T^{(n)} - n\cdot\lambda)\phi_k\| \to 0 \quad \text{for} \quad k \to \infty,$$

i.e., $\lambda n \in \sigma_{ess}(T^{(n)})$ for all $n \geq 1$ and therefore $\{n\cdot\lambda\} \subset \sigma_{ess}(d\Gamma(T))$ which means $\mathfrak{T}(d\Gamma(T)) \in (III)$. Now let $\lambda = 0$ the only point of $\sigma_{ess}(T)$ (which can happen only if $T \in (II)$). Because $T \neq 0$, there is a $\mu \in \sigma(T)$, $\mu \neq 0$, and consequently there are $\{\chi_k\} \subset \mathfrak{T}(T)$ with $\|\chi_k\| = 1$,

$$\|(T - \mu)\chi_k\| \to 0 \text{ for } k \to \infty.$$

Regard the vectors $\phi_k = \psi_k \otimes \chi_k \otimes \dots \otimes \chi_k$ (n factors χ_k), we see that $(\phi_k) \subset \mathfrak{T}(T^{(n+1)})$ is an orthonormal set and

$$\|(T^{(n+1)} - n\mu)\phi_k\| \to 0 \quad \text{for } k \to \infty, \text{ i.e.,}$$

$n\cdot\mu \in \sigma_{ess}(T^{(n+1)})$, consequently, $\{n\mu, n \geq 1\} \subset \sigma_{ess}(d\Gamma(T))$. But that means $\mathfrak{T}(d\Gamma(T)) \in (III)$. It remains to prove that $\mathfrak{T}(d\Gamma(T)) \in (III)$ implies $\mathfrak{T}(d\Gamma(T)) \in (III_A)$.

The considerations above show that if $\mathfrak{T}(d\Gamma(T)) \in (III)$ so there is either a sequence $(d_n) \subset \sigma_{ess}(d\Gamma(T))$ with (1) or there is a $\lambda \neq 0$ such that $\{n\cdot\lambda,\ n \geq 1,\ n \in N\} \subset \sigma_{ess}(d\Gamma(T))$. Without loss of generality we may suppose that $\lambda > 0$ (if not, regard $-T$ instead of T). Regard the decomposition $d\Gamma(T) = d\Gamma(T)_- \oplus d\Gamma(T)_+$ with

$d\Gamma(T)_+ \geq 0$, $d\Gamma(T)_- \leq 0$. Then $\mathfrak{T}(B) = \mathfrak{T}(d\Gamma(T))$ with $B = (-d\Gamma(T)_-) \oplus d\Gamma(T)_+$ and $(d_n)(\{n\cdot\lambda$; $n \geq 1$, $n \in N\}$ resp.) $\subset \sigma_{ess}(B)$. The associated diagonal operator D (see the beginning of section 2) can be taken to have the eigenvalues $(|d_n|)$ $((|n\lambda|)$ resp.) each with multiplicity $= \infty$. That means of course

$\mathfrak{T}(D) = \mathfrak{T}(d\Gamma(T)) \in (III_A)$.

Q.E.D.

Remark

The proof of Theorem 5 ii) shows that if $\mathfrak{T}(d\Gamma(T)) \in (III)$, then $\mathfrak{T}(d\Gamma(T)) = \mathfrak{T}(N)$ where N is a diagonal operator with the eigenvalues $\lambda_n = n$, $\mu(\lambda_n) = \infty$ for all $n = 1, 2, \ldots$

Consequently we obtain the following result: if S, T are arbitrary selfadjoint operators such that $\mathfrak{T}(d\Gamma(S))$ and $\mathfrak{T}(d\Gamma(T)) \in (III)$, then $\mathfrak{T}(d\Gamma(S))$ and $\mathfrak{T}(d\Gamma(T))$ are unitary equivalent (cf. also /1/). To give the classification of $\mathfrak{T}(d\Gamma(T)_s)$ we remark that all considerations of the proof of Theorem 5 go through also in the symmetric case, because we can, if necessary, symmetrize the vectors regarded there. Consequently we have

Theorem 5

All statements of Theorem 5 are true also for $\mathfrak{T}(d\Gamma(T)_s)$.

Now we go on to the classification of $\mathfrak{T}(d\Gamma(T)_a)$ where some interesting moments shall arise.

Theorem 6

Let $T = T^* \neq 0$ be a selfadjoint operator.

i) If $T \in (I_A)$ and $\dim \mathcal{H} = n$, then

a) $T^{(k)} = 0$ for $k > n$

b) $d\Gamma(T)_a = \sum_{r=0}^{n} T^{(r)}$

c) $\mathfrak{T}(d\Gamma(T)_a) \in (I_A)$.

ii) If $T \in (I_B)$ and there are only finite many negative (or positive) eigenvalues, then $\mathfrak{T}(d\Gamma(T)_a) \in (I_B)$.

iii) If $T \in (II_A)$, $\sigma_{ess}(T) = (0)$ and for the sequence (t_n) of non-zero eigenvalues of T (each eigenvalue being repeated a number of timesequal to its multiplicity) there is

$\Sigma|t_n| < \infty$, i.e., T is nuclear,

then $\mathfrak{T}(d\Gamma(T)_a) \in (II_A)$, i.e., $d\Gamma(T)_a$ is bounded.

iv) If one of the following conditions is fulfilled then

$\mathfrak{T}(d\Gamma(T)_a) \in (III_A)$:

a) $T \in (II_A)$ but iii) does not hold

b) T belongs to class II_B or class III.

Proof:

The statements i) and ii) are immediate consequences of Lemma 4 iii).

ad iii) $\Sigma|t_n| < \infty$ is equivalent to the statement that

$$\{t_{k_1} + \ldots + t_{k_m},\ m = 1, 2, \ldots;\ k_i \neq k_j \text{ for } i \neq j\} \quad (2)$$

is a bounded set, i.e., $d\Gamma(T)_a$ is bounded. Because $T \in (II_A)$, it is clear that $d\Gamma(T)_a \notin (I_A)$, i.e., $\mathfrak{T}(d\Gamma(T)_a) \in (II_A)$.

ad iv)

a) If $\Sigma|t_n| = \infty$, then there can arise two cases:

α) $\Sigma t_n < \infty$; β) Σt_n is divergent

In case α) it is a result of classical analysis that the set of accumulation points of (2) is equal to R, but this means $\mathfrak{T}(d\Gamma(T)_a) \in (III)$.

Let $(t_n^+) = \{t_j \in (t_k)$ with $t_j > 0\}$; $(t_n^-) = \{t_j \in (t_k)$ with $t_j < 0\}$. Then in case β) at least one of the sums Σt_n^+, Σt_n^- is infinite, say Σt_n^+. This and the boundedness of T give $\mathfrak{T}(d\Gamma(T)_a) \in (III)$; more exactly, the set of accumulation points of

$$\{t^+_{k_1} + \dots + t^+_{k_m},\ m = 1,2,\dots,\ k_i \neq k_j \text{ for } i \neq j\}$$

contains a sequence (d_n) with (1).

b) Let $\lambda \in \sigma_{ess}(T)$, $\lambda \neq 0$ and $\{\phi_k\}$ an orthonormal set such that

$$\|(T - \lambda)\phi_k\| \to 0 \qquad \text{for } k \to \infty .$$

Regard $\hat{\psi}_r = \phi_{r\cdot n+1} \otimes \dots \otimes \phi_{(r+1)\cdot n}$, $r = 1,2,\dots$ and let ψ_r be the corresponding antisymmetrized vector. Then $\{\psi_r\}$ is an orthonormal set and

$$\|(T^{(n)} - n\cdot\lambda)\psi_r\| \to 0 \qquad \text{for } r \to \infty$$

that is, $\{n\cdot\lambda;\ n = 1,2,\dots\} \subset \sigma_{ess}(d\Gamma(T)_a)$ and therefore $\mathfrak{T}(d\Gamma(T)_a) \in (III)$. Now let $\lambda = 0$ be the only point of the essential spectrum of T, $\mu_n \in \sigma_d(T)$, $\mu_n \to \infty$ $(T \in (II_B))$. As above let $\{\phi_n\}$ be an orthonormal set such that

$$\|T\phi_n\| \to 0 \qquad \text{for } n \to \infty$$

and let $\{\chi_n\}$ be an orthonormal set of eigenvectors with

$$T\chi_n = \mu_n \chi_n$$

Without loss of generality one may assume that $\phi_n \perp \chi_m \ \forall\, m,n$. Let $\rho = \mu_{n_1} + \dots + \mu_{n_r}$ be an eigenvalue of $T_a^{(r)}$ belonging to the eigenvector $\chi \in \mathfrak{F}(\mathcal{H})_a$, then $\chi \perp \phi_n$ for all n and $\psi_n = \frac{1}{2}(\phi_n \otimes \chi - \chi \otimes \phi_n) \in \mathfrak{F}(\mathcal{H})_a$. The set $\{\psi_n\}$ is an orthonormal one with

$$\|(T_a^{(r+1)} - \rho)\psi_n\| \to 0 \qquad \text{for } n \to \infty ,$$

i.e., $\rho \in \sigma_{ess}(T_a^{(r+1)})$.

Because $\mu_n \to \infty$ it is seen that $d\Gamma(T)_a \in (III)$. Moreover, the same consideration shows that any eigenvalue of $T_a^{(r)}$ is in the essential spectrum of $T_a^{(r+1)}$ in the case regarded above (i.e., $\lambda = 0$ is the only point of $\sigma_{ess}(T)$).) This means not only $\mathfrak{T}(d\Gamma(T)_a) \in (III)$ but even $\mathfrak{T}(d\Gamma(T)_a) \in (III_A)$. The proof that $\mathfrak{T}(d\Gamma(T)_a) \in (III)$ implies $\mathfrak{T}(d\Gamma(T)_a) \in (III_A)$ in the outstanding cases is the same as in Theorem 5.

Q.E.D.

We remark that the Theorem above gives no complete classification of $\mathfrak{T}(d\Gamma(T)_a)$. It remains to consider the case, where $T \in (I_B)$ and $\sigma_d(T)$ contains an infinite number of negative and positive eigenvalues, $s_n < 0$, $t_n > 0$, $\lim s_n = -\infty$, $\lim t_n = +\infty$. We will here only mention that in this case we always have $\mathfrak{T}(d\Gamma(T)_a) \in (I_B)$ or $\mathfrak{T}(d\Gamma(T)_a) \in (III)$ and this always implies $\mathfrak{T}(d\Gamma(T)_a) \in (III_A)$.

3. This section deals with some simple relationships between the domains of two operators of second quantization.

Proposition 7

Let A,B be two selfadjoint operators. Then $\mathfrak{D}(d\Gamma(A)) \subset \mathfrak{D}(d\Gamma(B))$ implies

i) $\mathfrak{D}(A) \subset \mathfrak{D}(B)$

ii) $\mathfrak{N}(A^{(j)}) \subset \mathfrak{N}(B^{(j)})$ for all j, where $\mathfrak{N}(.)$ denotes the nullspace of the operator

iii) $<B^{(j)}\phi,\phi>^2 \le D<A^{(j)}\phi,\phi>^2 \ \forall j,\ \forall \phi \in \mathfrak{D}(A^{(j)})$.

Proof:

From $\mathfrak{D}(d\Gamma(A)) \subset \mathfrak{D}(d\Gamma(B))$ it follows by the closed graph theorem

$$\|d\Gamma(B)\phi\|^2 \le C[\|d\Gamma(A)\phi\|^2 + \|\phi\|^2] \text{ for all } \phi \in \mathfrak{D}(d\Gamma(A)).$$

Consequently

$$\|B^{(r)}\phi\|^2 \le C[\|A^{(r)}\phi\|^2 + \|\phi\|^2] \text{ for all } \phi \in \mathfrak{D}(B^{(r)}), \quad (3)$$
$$r = 1,2,\ldots$$

The statement i) is trivial.

ad ii) Suppose there is a $\phi \in \mathfrak{N}(A^{(r)})$ with

$$A^{(r)}\phi = 0 \qquad \text{but } B^{(r)}\phi \neq 0,\ \|\phi\| = 1.$$

Regard the element $\psi = \phi \otimes \ldots \otimes \phi$ (n factors), then (3) gives

$$\|B^{(n,r)}\psi\|^2 = n\|B^{(r)}\phi\|^2 + n(n-1)<B^{(r)}\phi,\phi>^2 \le C(0+1)^2 = C,$$

which is a contradiction for large n. Therefore $\mathfrak{N}(A^{(r)}) \subset \mathfrak{N}(B^{(r)})$.

ad iii) Suppose, there is an r such that iii) does not hold. Consequently there exists a sequence $(\phi_n) \subset \mathfrak{D}(A^{(r)})$ such that

$$<B^{(r)}\phi_n,\phi_n>^2 \ge d_n<A^{(r)}\phi_n,\phi_n>^2 \quad (4)$$

$$\|\phi_n\| = 1 \text{ for all n and } d_n \to \infty \text{ for } n \to \infty.$$

Regarding $\psi = \phi_n \otimes \ldots \otimes \phi_n$ (m factors) we obtain from (3)

$$\|B^{(m,r)}\psi\|^2 = m\|B^{(r)}\phi_n\|^2 + m(m-1)<B^{(r)}\phi_n,\phi_n>^2 \le$$
$$\le C(m\|A^{(r)}\phi_n\|^2 + m(m-1)<A^{(r)}\phi_n,\phi_n>^2 + 1)$$

By (4) it follows

$$d_n \cdot m(m-1)<A^{(r)}\phi_n,\phi_n>^2 \le C.m\|A^{(r)}\phi_n\|^2 +$$
$$+ C \cdot m(m-1)<A^{(r)}\phi_n,\phi_n>^2 + C$$

which is a contradiction for large m and n.

Q.E.D.

Finally we consider the question under which conditions on A and B we have: $\mathfrak{D}(d\Gamma(A))$ is unitarily equivalent to $\mathfrak{D}(d\Gamma(B))$. One part of the answer is given in Theorem 5, ii) and the Remark following Theorem 5. Therefore it is only interesting to regard such operators A,B which are $\in (I)$ and definite. We may suppose that $A > 0$, $B > 0$.

Proposition 8

Let $A > 0$, $B > 0$ of class I_B with eigenvalues (a_n), (b_n). If

$$C_1 a_n \le b_n \le C_2 a_n \quad \forall_n; \quad C_1, C_2 > 0,$$

then $\mathfrak{D}(d\Gamma(A))$ and $\mathfrak{D}(d\Gamma(B))$ are unitarily equivalent.

Proof:

Lemma 4 says that $d\Gamma(A)$ has the eigenvalues $a_{k_1} + \ldots + a_{k_m}$ $m = 1,2,\ldots; k_i$ naturals for all i,

and $d\Gamma(B)$ has the eigenvalues

$b_{k_1} + \ldots + b_{k_m}$, m and k_i as above.

But then (5) gives

$$C_1(a_{k_1}+\dots+a_{k_m}) \le (b_{k_1}+\dots+b_{k_m}) \le C_2(a_{k_1}+\dots+a_{k_m}),$$

i.e. the sequences of eigenvalues of $d\Gamma(A)$ and $d\Gamma(B)$ are equivalent and the correspondence

$$(a_{k_1}+\dots+a_{k_m}) \leftrightarrow (b_{k_1}+\dots+b_{k_m})$$

implies the unitary equivalence of the corresponding domains (see /1/).

Q.E.D.

Corollary 9

If $A>0$, $B>0$, $\in(I_A)$ and $\dim\mathfrak{D}(A)=\dim\mathfrak{D}(B)$, then $\mathfrak{D}(d\Gamma(A))$ and $\mathfrak{D}(d\Gamma(B))$ are unitarily equivalent.

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