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OSCILLATING PARTICLE-LIKE SOLUTIONS OF NONLINEAR KLEIN-GORDON EQUATION

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Осциллирующие частицеподобные решения нелинейного уравнения Клейна-Гордона

Получен счетный набор осциллирующих сферически-симметричных частицеподобных решений уравнения Клейна-Гордона с кубической нелинейностью. Моделируемые ими протяженные частицы оказываются слабоизлучающими и долгоживущими.

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Oscillating Particle-Like Solutions of Nonlinear Klein-Gordon Equation

A denumerable set of oscillating spherically-symmetric particle-like solutions of the Klein-Gordon equation with cubic nonlinearity is found. Extended particles modelled by them turn out to be slightly radiating and long-lived.

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OSCILLATING PARTICLE-LIKE SOLUTIONS OF NONLINEAR KLEIN-GORDON EQUATION

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During the last decades numerous attempts have been undertaken to find a particle-like solution (PLS) of Lorentz-invariant nonlinear field equations (see, for instance, the review[1]). We shall restrict ourselves by investigation of real scalar fields which are described by the Klein-Gordon equation with cubic nonlinearity:

$$u_{tt} - \Delta u + u - u^3 = 0. \qquad (1)$$

The equation (1) possesses nontrivial steady-state solutions, i.e., plane one-dimensional solutions[2] and spherically-symmetric (ss) PLS [3,4]. But they have turned out to be unstable [5,8] . On the other hand, in the framework of Eq. (1) in (x,t) case $(\Delta \to \frac{\partial^2}{\partial x^2})$ stable* self-localized nonlinear oscillations have been described [9] (let us call them "pulsons" for brevity).

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* In computer experiments we have observed the formation of the plane pulsons from the oscillating field bunches close to them.

Of course, spatial PLS are of more interest for the elementary particle physics. For the first time long-lived ss-pulsons were found by investigation of the Higgs field equation (see paper[10]). Their amplitude $c(t)$ decreases slowly as a result of slight radiation and their life-time τ is $\sim 10^3$. In the present paper we shall find and investigate ss-pulsons of Eq. (1) applying the Fourier method in the presence of small parameter ($u^2 \ll 1$)[9] and using a computer.

We shall look for the solution of Eq.(1) in the following form

$$u(r,t) = a(r) \cdot \cos\omega t + b(r) \cdot \cos 3\omega t + \dots \qquad (2)$$

Substituting the expression (2) into Eq.(1) we obtain the nonlinear eigenvalue problem

$$a_{rr} + \frac{2}{r} a_r + \frac{3}{4} a^3 = \lambda a, \quad \lambda = 1 - \omega^2, \qquad (3)$$
$$a_r(0) = 0, \quad a(\infty) = 0.$$

Let $y(r)$ be its solution at $\lambda = 1$. One can easily see that then $y_\lambda = \sqrt{\lambda}\, y(\sqrt{\lambda}\, r)$ is the solution of Eq. (3) at given $\lambda = 1-\omega^2$. Introduce $A = \sqrt{\frac{3}{4}}\, a$. . The equation which one obtains for the variable A,

$$A_{rr} + \frac{2}{r} A_r - A + A^3 = 0 \qquad (4)$$

possesses at boundary conditions $A_r(0) = 0$, $A(\infty) = 0$ the denumerable set of solutions $A_i(r)$, $i = 1,2,\dots,n,\dots$, with the solution number i has $(i-1)$ nodes, and $A_1(0) \simeq 4.34 < A_2(0) \simeq 14.10 < A_3(0) \simeq 29.13 < \dots < A_n(0) < \dots$ [3,4].

Thus functions

$$u_i(r,t) = \sqrt{\frac{4}{3}}\,u_0 \cdot A_i(ku_0 r)\cdot \cos(\sqrt{1-u_0^2}\cdot t) = \\ = u_m \cdot \frac{A_i(ku_0 r)}{A_i(0)} \cdot \cos(\sqrt{1-u_0^2}\cdot t),\ \ k=1 \qquad (5)$$

to the approximation of order $u_m^2 \ll 1$ are solutions of Eq. (1) and describe ss-pulsons. One can easily obtain an expression for $b(r)$ at $u_m^2 \ll 1$:

$$b(r) = -\frac{1}{12\sqrt{3}}\,u_0^3 A_i^3(u_0 r)\,. \qquad (6)$$

The dynamics of PLS (5) has been investigated by computer. The first three modes ($i = 1,2,3$) of solutions (5) have been studied at amplitudes $u_m = 0.2; 0.4; 0.7$. At $u_m \le 0.4$ the results of computations are described by formula (5) with high accuracy (errors are less than 1%). We should especially note, that, in any case, at $u_m^2 \ll 1$ radiation of pulson to infinity is very slight, and its life-time $\tau \to \infty$ at $u_m^2 \to 0$. When one sets a larger value $u_m = 0.7$ in the formula (5), then pulsation amplitude $c(t)$ slowly decreases down to $c(t) = 0.63$ at $t = 80$, and the characteristic radius R_c grows.

The field bunch obtained by compressing (5) along r-axis ($k > 1$) at fixed amplitude gradually gets wider, so that $R_c \to \infty$ at $t \to \infty$ and $c(t)$ monotonously decreases (Fig. 1a). On the contrary, the bunch which is wider than the pulson (5) ($k < 1$) begins collapse to the centre with $c(t)$ increasing (the closer k to unity, the slower) up to value $u_{cr} \sim 1$. Then the "explosive" (more fast than the exponential one) appearance of the field singularity takes place,

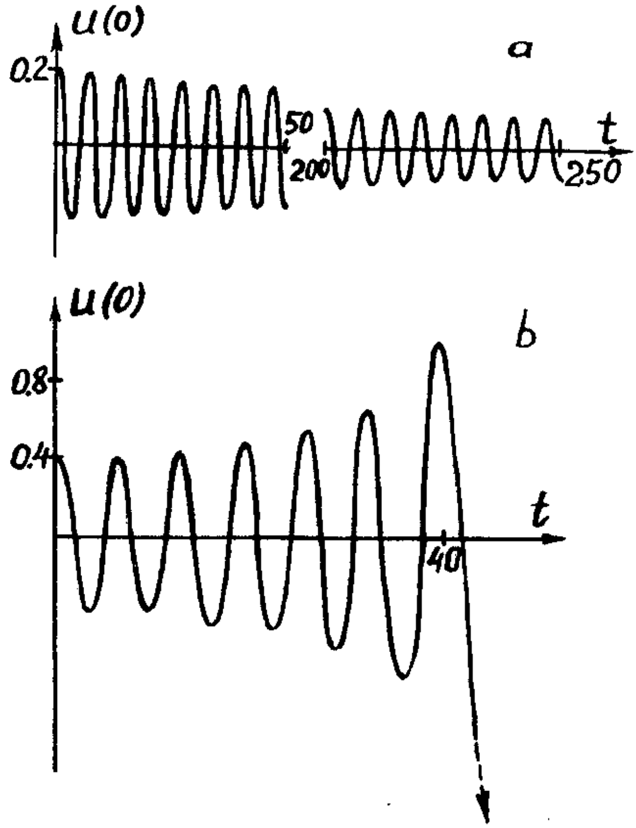

Fig. 1. Dependence $u(0,t)$ when the field function $u(r,0)$ is given by the formula (5) a) $k = 1.3$, b) $k = 0.8$.

$|u(0,t)| \to \infty$ (Fig. 1b). This effect possibly can be explained by the dependence $U(u) = = u^2 - \frac{u^4}{2}$ of the field "potential energy". The pulson amplitude u_{max} of Eq.(1) is apparently limited from above by the constant $u^* \sim 1$: $u_{max} < u^* \sim 1$. It is necessary to take into account more terms of function

$a(r)$, , $b(r)$,...: expansion in u_m powers /9/ to describe these PLS at $u_m^2 \lesssim 1$. It is convenient to calculate pulson (5) energy

$$E = \frac{1}{2}\int_0^\infty [u_r^2 + u_t^2 + u^2 - \frac{u^4}{2}] r^2 dr = \int_0^\infty H r^2 dr = \int_0^\infty \mathcal{H}\, dr \qquad (7)$$

at moments, when $u_t = 0$. Substituting (5) into (7) we obtain for the mode number i :

$$E_i = I_1^{(i)}(u_0) + I_2^{(i)}(u_0) - I_3^{(i)}(u_0) = u_0(I_1^{(i)} - I_3^{(i)}) + u_0^{-1} I_2^{(i)}, \qquad (8)$$

$$I_1^{(i)} = \frac{1}{2}\int_0^\infty (\frac{dy_i}{dr})^2 r^2 dr;\; I_2^{(i)} = \frac{1}{2}\int_0^\infty y_i^2 r^2 dr;\; I_3^{(i)} = \frac{1}{4}\int_0^\infty y_i^4 r^2 dr.$$

In the limit $u_0 \to 0$ $E_i \simeq u_0^{-1} I_2^{(i)}$ and the main part of field energy density $H(r,t)$ is connected with terms u_t^2 and u^2. Their sum $u_t^2 + u^2 = u_i^2(r,0)\cdot(\cos^2 \omega t + \omega^2 \sin^2 \omega t)$ to the approximation of order of u_0^2 is constant at all r because $\omega^2 = (1-u_0^2) \to 1$, when $u_0^2 \to 0$. Therefore, $H(r)$ and $\mathcal{H}(r)$ distributions in the same approximation do not depend on time (Fig. 2, a,b,c). Note, that because of the relationship $u_0 = u_m / \sqrt{\frac{4}{3}} A_i(0)$ the "mass" distribution with respect to the radius at fixed amplitude is constant in time the more precisely, the more the mode number i .

Thus, refusing from demand that the field function $u(r,t)$ should be stationary, one has the possibility of constructing the denumerable set of PLS of Eq. (1) being one-field models of zero-spin long-lived particles. In the limit $u \to 0$ at equal u_0 masses of these particles are proportional to $I_2^{(i)}$ and at equal u_m are proptional to $I_2^{(i)} A_i^{-1}(0)$ ($\approx$1:2:3:4:9...).

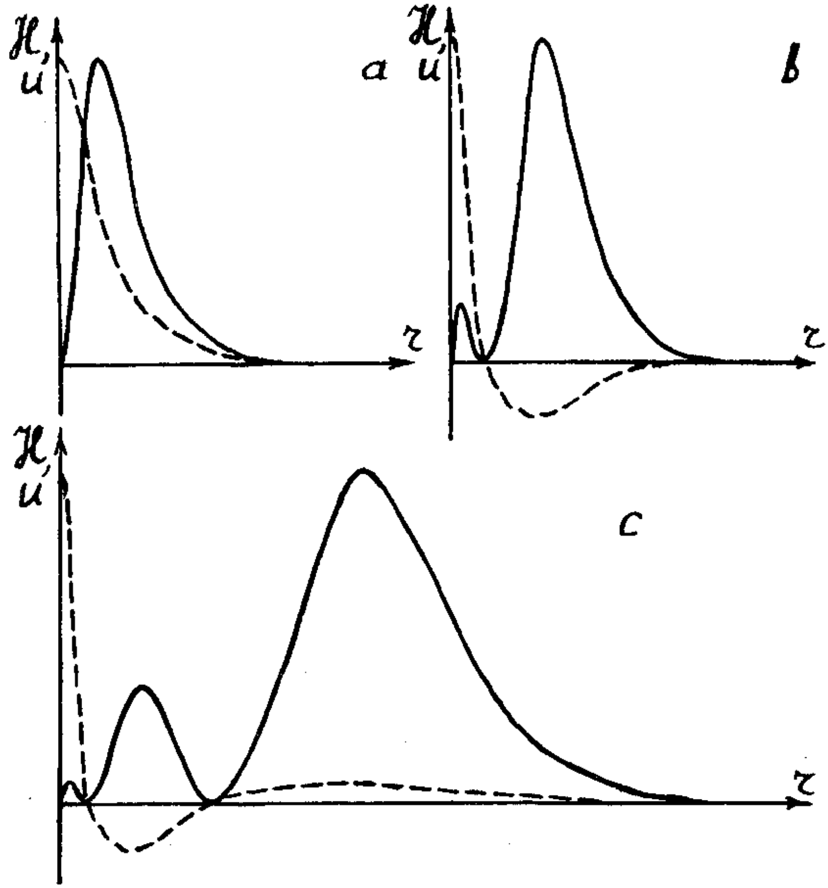

Fig.2. The structure of the first three pulson (5) modes. a) i = 1 b) i = 2 c) i = 3 ; - - - - the function $u_i(r,0)=\sqrt{\frac{4}{3}}\,u_0 A_i(u_0 r)$; ——— the distribution $\mathcal{H}(r)$.

It is possible, that the pulson solutions will be useful for the description of ψ--bosons (the first soliton model of these particles had been suggested in papers /11/ , where the one-dimensional Higgs field equation was studied).

The results obtained at $u^2 \ll 1$ may be applied to ss-sine-Gordon equation

$$u_{tt} - \Delta_{rr} u + \sin u = 0. \qquad (9)$$

But unlike Eq. (1) in the framework of Eq. (9) long-lived ss-pulsons having the amplitudes $c(t) > 1$, $c(t) \sim 2\pi$ are found to exist. Note, that the pulsons of paper /10/ may be also described at the amplitude $c(t) \ll 1$ by the Fourier method.

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