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SCALAR REPRESENTATIONS
OF CONFORMAL SUPERALGEBRA

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Скалярные представления конформной супералгебры.

В работе построено двухпараметрическое семейство представлений, действующих на скалярные суперполя. Получены генераторы алгебры для таких представлений и конечные преобразования суперполей $\phi(x,\theta)$. Найдено R- преобразование, аналогичное R- инверсии конформной группы. Решаются математические проблемы, связанные с построением супергрупп SOQ(4,2) и SU(2,2;1) и обобщением понятия индуцированных представлений для супергрупп Ли.

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Scalar Representations of Conformal Superalgebra

In the paper a two-parameter representation family acting on the scalar superfields is constructed. The algebra generators for such representations are obtained and the finite transformations of the superfields $\phi(x,\theta)$ are found. An R-transformation, analogous to the R-inversion of the conformal group is obtained.

Some mathematical problems, connected with the construction of the supergroups SOQ (4,2) and SU(2,2;1) and the generalization of the concept of the induced representations for the case of Lie suppergroups are discussed.

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SCALAR REPRESENTATIONS OF CONFORMAL SUPERALGEBRA

Introduction

Recently Wess and Zumino [1] have introduced a Lie superalgebra [3,4] containing the Lie algebra of conformal group (conformal superalgebra) and built some of its representations (scalar and vector multiplets). The explicit form of the conformal superalgebra was found by Ferrara [2]. In papers [5,6] the study of representations of this algebra was started on the basis of the notion of superfield proposed by Salam and Strathdee [7].

Some two-parameter families of representations of the conformal superalgebra acting on scalar superfields $\varphi^{\pm}(x,\theta^{+},\theta^{-})$ were built up in ref. [6]. For some connections between the parameters d (conformal dimension) and z (γ_5-dimension) these representations turn out to be reducible [6,8] and, in particular, contain the subrepresentations which are equivalent to the scalar multiplet of Wess and Zumino [1,2,5] ("chiral" superfields $\varphi^{\pm}(x,\theta^{\pm})$). In paper [8] the spinor superfields $\psi_{\alpha}^{\pm}(x,\theta^{+},\theta^{-})$ were also considered and the two-point functions of these superfields were found as well as invariant subspaces of the corresponding representations.

In this paper another (two-parameter) family of representations acting on scalar superfields $\varphi(x,\theta)$ is constructed (the representations $T_{(d,z)}$). This set contains a one-parameter subset of representations equivalent to the Wess-Zumino vector multiplets.

The paper is organized as follows. §1 contains the necessary information on conformal superalgebra. In §2 the generators of the algebra for the representations $T_{(d,z)}$ are built and finite transformations of the superfields $\varphi(x,\theta)$ are found.

In §3 the transformation R analogous to the R-inversion of conformal group is constructed. By means of R-inversion in §4 the reduction of the representations T_χ $(\chi = (d, z))$ with respect to the conformal subgroup is performed. In Appendix I Lie supergroups[9,10] $SOQ(4,2)$ and $SU(2,2;1)$ are constructed. Appendix II is devoted to the generalization of the notion of induced representation[15] to the case of Lie supergroups.

The presentation does not pretend for mathematical rigour, partially because the theory of infinitedimensional representations of Lie supergroups is practically undeveloped until now. Therefore, we have to follow analogies with the theory of semisimple Lie groups and of their representations rather than strict results.

1. Conformal Superalgebra

The conformal superalgebra[1,2] is a Lie superalgebra[3,4,12] with commutation relations $(g_{11} = g_{22} = g_{33} = g_{55} = -g_{00} = -g_{66} = 1)$:

$$[\mathcal{J}_{AB}, \mathcal{J}_{CD}] = -i(g_{AD}\mathcal{J}_{BC} + g_{BC}\mathcal{J}_{AD} - g_{AC}\mathcal{J}_{BD} - g_{BD}\mathcal{J}_{AC}) \tag{1.1}$$

$$[\pi, \mathcal{J}_{AB}] = 0 \tag{1.2}$$

$$[Q, \pi] = -i\beta_7 Q \tag{1.3}$$

$$[Q, \mathcal{J}_{AB}] = -i\gamma_{AB} Q \tag{1.4}$$

$$[Q_\alpha, Q_\beta]_+ = 2(\gamma_{AB} A \mathcal{J}^{AB} - \tfrac{3}{2}\beta_7 A \pi) \tag{1.5}$$

$$(A, B, C, D = 0, 1, 2, 3, 5, 6,\ \alpha, \beta = 1, 2, \ldots, 8)$$

where

$$[\beta_A, \beta_B]_+ = 2g_{AB},\quad \gamma_{AB} \equiv \tfrac{1}{4}[\beta_A, \beta_B],\quad \beta_7 = \beta_0\beta_1\beta_2\beta_3\beta_5\beta_6, \tag{1.6}$$

A is the "charge conjugation" matrix with the properties:

$$A^{-1}\Gamma A = \Gamma^T \quad \text{for } \Gamma = I, \beta_A, \beta_A\beta_B, \beta_A\beta_B\beta_7\ (A \neq B), \tag{1.7}$$

$$A^{-1}\Gamma A = -\Gamma^T \quad \text{for } \Gamma = \beta_7, \gamma_{AB}, \beta_A\beta_B\beta_C\ (A < B < C). \tag{1.8}$$

The algebra (1.1)-(1.5) contains the Lie algebra $so(4,2)$ as a subalgebra, and is the minimal extension of $so(4,2)$ containing the bispinor generator Q. We will denote this graded algebra as $soq(4,2)$, or as $su(2,2;1)$*.

The algebra $soq(4,2)$ is a real form of the algebra $soq(6, C)$ (which is isomorphic to Kac's $A(3,0)$[12]). This may be verified by replacing $J_{AB} \to iJ_{AB}$, $\pi \to i\pi$, $Q \to \sqrt{i}\,Q$, and using the Majorana representation for β_A, i.e., such a representation that γ_{AB} (the generators of bispinor representation) are real matrices[2]. Then all of the structure constants in (1.1)-(1.5) become real numbers.

In the Euclidean conformal invariant QFT (see, e.g.,[11] and Refs. therein) the algebra $so(4,2)$ is replaced by the algebra $so(5,1)$, the representation theory of which is simpler than that of $so(4,2)$. It turns out, however, that we cannot make a similar replacement in our case. The reason is that the algebra $soq(6, C)$ has no "Euclidean" real form (containing $so(5,1)$ as a subalgebra). This fact follows from Kac's classification of simple real Lie superalgebras[12].

Return now to the algebra $soq(4,2)$. Writing in the Majorana representation[2] the spinor Q as

$$Q = \begin{pmatrix} S \\ -T \end{pmatrix} \tag{1.9}$$

and introducing the four-dimensional generators

$$M_{\mu\nu} \equiv J_{\mu\nu}\quad (\mu, \nu = 0, 1, 2, 3),\qquad D \equiv J_{56},$$
$$P_\mu \equiv J_{6\mu} + J_{5\mu},\qquad K_\mu = J_{6\mu} - J_{5\mu}, \tag{1.10}$$

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*) V.G.Kac denotes the same algebra as $su(4,1;2,1)$[12]

we can obtain the commutation relations between the generators $M_{\mu\nu}, K_\mu, P_\mu, D, \pi, S_\alpha$ and T_α (see [2]). From these commutation relations it follows that the algebra soq (4.2) is actually a Z -graded Lie algebra [4]:

$$L = \overset{\infty}{\underset{n=-\infty}{\oplus}} L_n \tag{1.11}$$

where*)

$$L_0 = \{D\ \pi, M_{\mu\nu}\}, \quad L_1 = \{S_\alpha\}, \quad L_{-1} = \{T_\alpha\}$$
$$L_2 = \{P_\mu\}, \quad L_{-2} = \{K_\mu\}, \quad L_k = 0 \quad if\ |k| > 2 \tag{1.12}$$

and

$$[L_i, L_j] \subset L_{i+j} \tag{1.13}$$

The generalized commutator is determined by the identity

$$[M_i, M_j] \equiv M_i M_j - (-1)^{ij} M_j M_i \quad (M_i \in L_i,\ M_j \in L_j). \tag{1.14}$$

The grading is given by the eigenvalue of the dilatation operator:

$$[D, M] = \tfrac{i}{2} j M, \qquad M \in L_j. \tag{1.15}$$

Note that the operator

$$C_2 = J_{AB} J^{AB} - \tfrac{3}{2}\pi^2 - \tfrac{i}{4}\bar{Q}Q = M_{\mu\nu}M^{\mu\nu} - 2KP - 2D^2 - 4iD - \tfrac{3}{2}\pi^2 + i\bar{T}S \tag{1.16}$$

is the Casimir operator of the algebra soq (4,2). The conjugated spinors entering into expression (1.16) are determined as follows (in the Majorana representation):

$$\bar{Q} = QA = Q\gamma_{60} \tag{1.17}$$

for the eight-dimensional spinor Q and

$$\bar{T} = T\gamma^0 \tag{1.17'}$$

for the four-dimensional spinor T.

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*) The brackets $\{a, \ldots, b\}$ will denote, throughout this and the following papers, the set of elements a,...,b; or the vector space with the basis a,...,b; or the algebra generated by a,...,b.

For every four-dimensional spinor θ we can introduce the Weil spinors θ^+ and θ^- :

$$\theta^\pm \equiv \frac{1 \pm i\gamma_5}{2}\theta, \quad \bar{\theta}^\pm \equiv \bar{\theta}\frac{1 \pm i\gamma_5}{2}. \tag{1.18}$$

The complex conjugation for anticommuting spinors θ must be determined in such a way that [7]

$$(\theta_\alpha \ldots \theta_\beta)^* = \theta_\beta^* \ldots \theta_\alpha^*. \tag{1.19}$$

For Majorana spinors θ ($\theta^* = \theta$ in the Majorana representation) the identity

$$(\theta^\pm)^* = \theta^\mp \tag{1.20}$$

holds.

2. The Representations of Conformal Superalgebra

The algebra soq (4,2) is a simple Lie superalgebra [12]. Consider an outer automorphism R of the algebra soq (4,2):

$$M_{\mu\nu} = M_{\mu\nu}, \quad \dot{D} = -D, \quad \dot{\pi} = -\pi, \quad \dot{P}_\mu = K_\mu, \quad \dot{K}_\mu = P_\mu, \quad \dot{S}_\alpha = T_\alpha, \quad \dot{T}_\alpha = S_\alpha, \tag{2.1}$$

where, by definition, for every element M of soq (4,2)

$$\dot{M} \equiv R(M). \tag{2.2}$$

The automorphism R is involutory

$$R^2 = 1. \tag{2.3}$$

It is an analogue of the R-inversion of conformal algebra so (4,2).

We have the following "Cartan" decomposition of the algebra soq(4.2) with respect to the automorphism R:

$$soq(4,2) = \mathcal{R} \oplus \mathcal{P}, \tag{2.4}$$

where

$$\mathcal{R} = \{ M_{\mu\nu},\ K_\mu + P_\mu,\ S_\alpha + T_\alpha \},\quad \mathcal{P} = \{ D, \pi, K_\mu - P_\mu, S_\alpha - T_\alpha \} \tag{2.5}$$

and

$$\dot{M} = M \ (M \in \mathcal{R}), \qquad \dot{M} = -M \ (M \in \mathcal{P}) \tag{2.6}$$

$$[\mathcal{R},\mathcal{R}] \subset \mathcal{R},\ [\mathcal{R},\mathcal{P}] \subset \mathcal{P},\ [\mathcal{P},\mathcal{P}] \subset \mathcal{R}. \tag{2.7}$$

Note that $\mathcal{R}_v \equiv \{ M_{\mu\nu},\ K_\mu + P_\mu \}$ is not a compact subalgebra.

We associate with "Cartan" decomposition (2.4) the following expansion (cf.[13,16]):

$$soq(4,2) = \dot{n} \oplus \mathcal{a} \oplus m \oplus n, \tag{2.8}$$

where

$$\mathcal{a} = \{ D, \pi \},\ m = \{ M_{\mu\nu} \},\ n = \{ K_\mu, T_\alpha \},\ \dot{n} = \{ P_\mu, S_\alpha \}. \tag{2.9}$$

Note that $\mathcal{a}$ is the maximal abelian subalgebra in $\mathcal{P}$; m is the centralizer of $\mathcal{a}$ in $\mathcal{R}$; $\dot{n}\,(n)$ is the direct sum of subspaces of soq (4,2) with positive (negative) gradings (see eq. (1.12)).

Consider the "Bruhat" decomposition of the Lie supergroup SOQ (4,2) (the definition of this supergroup see in App. I) generated by decomposition (2.8):

$$SOQ(4,2) = e^{\dot{n}} e^{\mathcal{a}} e^{m} e^{n}. \tag{2.10}$$

Here $e^{\dot{n}}$ stands for a subsupergroup of SOQ(4,2) generated by the superalgebra $\dot{n}$, etc.,the supergroup $e^{\dot{n}}$ is a supermanifold[10] of the kind:

$$e^{\dot{n}} = \{ X;\ C^\infty(X) \otimes \Lambda_4(\theta) \} \equiv (x,\theta), \tag{2.11}$$

where X is the Minkowski space and $\Lambda_4(\theta)$ Grassman algebra generated by the components of the anticommuting Majorana spinor θ.

We shall study the (ray) representations of the supergroup

SOQ (4,2) acting on "superfields"[7] $\Phi(x,\theta)$ determined on the supermanifold (X,θ) which is supposed to be imbedded into the homogeneous space SOQ $(4,2)/e^{\mathcal{a}} e^{m} e^{n}$ in accordance with decomposition (2.10).

Let some linear representation B of the subgroup $H = e^{\mathcal{a}} e^{m} e^{n}$ in Z_2-graded space V be given, with the generators

$$B(M_{\mu\nu}) = \Sigma_{\mu\nu},\ B(D) = -id,\ B(\pi) = z,\ B(K_\mu) = k_\mu,\ B(T_\alpha) = t_\alpha. \tag{2.12}$$

The representation B induces the representation T_B of the group SOQ (4,2) on some linear subspace E of the space of functions on X with values in $V \otimes \Lambda_4(\theta)$ (i.e., the representation T_B acts on superfields). We may construct the generators of the representation T_B by applying the standard procedure of the induced representation method generalized to the case of Lie supergroups (see Appendix II)

$$T_B(S) = i\partial_\theta + \hat{\partial}\theta;\ T_B(P) = i\partial;\ T_B(D) = -i(d + x\partial + \tfrac{1}{2}\bar\theta\partial_\theta);\ T_B(\pi) = z - i\bar\theta\gamma_5\partial_\theta \tag{2.13}$$

$$T_B(T) = \hat{x}\,T_B(S) + t + 2i\,\Sigma_{\mu\nu}\gamma^{\mu\nu}\theta + 2(d + \tfrac{3}{2}iz\gamma_5)\theta + i\bar\theta\theta\hat{\partial}\theta + (\bar\theta\theta + \bar\theta\gamma^\rho\theta\gamma_5 \ [illegible] \tag{2.14}$$

$$T_B(M_{\mu\nu}) = \Sigma_{\mu\nu} - i(x_\mu\partial_\nu - x_\nu\partial_\mu + \bar\theta\gamma_{\mu\nu}\partial_\theta) \tag{2.15}$$

$$\begin{aligned} T_B(K_\mu) = {} & k_\mu + 2\Sigma_{\mu\nu}x^\nu - i\bar\theta\gamma_\mu\gamma^{\rho\sigma}\theta\,\Sigma_{\rho\sigma} - \bar\theta\gamma_\mu t + \tfrac{3}{2}i\bar\theta\gamma_5\gamma_\mu\theta z - 2ix_\mu d - \\ & - i[x_\mu\bar\theta + 2\bar\theta\gamma_{\mu\nu}x^\nu - i\bar\theta\theta\,\bar\theta\gamma_\mu]\partial_\theta - i[2x_\mu x^\nu - g_\mu^\nu(x^2 + \tfrac{1}{2}(\bar\theta\theta)^2)]\partial_\nu . \end{aligned} \tag{2.16}$$

If the representation B is such that $k_\mu = t_\alpha = 0$ and, besides, $\Sigma_{\mu\nu}$ form an irreducible representation of the Lorentz group, then the operators d and z turn out to be multiples of the unit operator*). The Casimir operator C_2 (see eq. (1.16)) for the induced representation T_B is expressed in this case by the operators $\Sigma_{\mu\nu}, d, z$:

*) So that in this case the representation T_B is determined by a character of the subgroup $e^{\mathcal{a}}$ and by a representation of the subgroup e^{m} (cf./13,16/).

$$C_2 = \Sigma_{\mu\nu}\Sigma^{\mu\nu} + 2d^2 - 4d - \tfrac{3}{2}z^2. \tag{2.17}$$

Note that the (eigen) value of this operator is the same for the representations $T_{(\Sigma_{\mu\nu}, d, z)}$ and $T_{(\Sigma_{\mu\nu}, 2-d, \pm z)}$.

From now on we will study only the scalar representations

$$\Sigma_{\mu\nu} = k_\mu = l_\alpha = 0, \quad d \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{2.18}$$

Denote the corresponding induced representations as T_χ , where

$$\chi \equiv (d, z). \tag{2.19}$$

The representation T_χ acts in the space E_χ of scalar superfields:

$$\Phi(x,\theta) = A(x) + \bar\theta\Psi(x) + \tfrac{1}{4}\bar\theta\theta F(x) + \tfrac{1}{4}\bar\theta\gamma_5\theta G(x) +$$

$$+ \tfrac{1}{4}\bar\theta\gamma_5\gamma_\mu\theta A^\mu(x) + \tfrac{1}{4}\bar\theta\theta\bar\theta\chi(x) + \tfrac{1}{32}(\bar\theta\theta)^2 D(x). \tag{2.20}$$

The fields $A(x), \Psi(x), \ldots, D(x)$ entering into the decomposition (2.20) will be called the coordinates of the superfield $\Phi(x,\theta)$.

A more complete definition of the spaces E_χ is given below (see §4). Note, that the fields $A(x), \Psi(x), \ldots, D(x)$ in the decomposition of the superfield $\Phi(x,\theta) \in E_{(d,0)}$ form the "vector multiplet" of Wess and Zumino[1].

Finite transformations of the superfields:

$$(T_\chi\Phi)(x,\theta,g) = B_\chi(x,\theta;g)\,\Phi(g^{-1}x, g^{-1}\theta) \tag{2.21}$$

can be obtained with the help of the generators (2.13)-(2.16): For example:

$$e^{i\rho D}(x,\theta) = (e^{-\rho}x, e^{-\rho/2}\theta), \qquad B_\chi(x,\theta;e^{i\rho D}) = e^{\rho d}\ ; \tag{2.22}$$

$$e^{i\beta\Gamma}(x,\theta) = (x, (\cos\beta - \sin\beta\gamma_5)\theta), \quad B_\chi(x,\theta;e^{i\beta\Gamma}) = e^{i\beta z}\ ; \tag{2.23}$$

$$e^{i\bar\varepsilon S}(x,\theta) = (x - i\bar\varepsilon\gamma\theta, \theta+\varepsilon), \; B_\chi(x,\theta;e^{i\bar\varepsilon S}) = 1\,; \tag{2.24}$$

$$e^{iyP}(x,\theta) = (x+y, \theta), \qquad B_\chi(x,\theta;e^{iyP}) = 1\,. \tag{2.25}$$

However, to find the finite special conformal transformations in this way seems to be very tedious. The latter may be calculated by means of Eq.(2.25) and the R-inversion transformation (see §3):

$$e^{icK}(x,\theta) = \left(\frac{x+cx^2}{1+2cx+c^2x^2} - \frac{c(1+2cx) - xc^2}{2(1+2cx+c^2x^2)^2}(\bar\theta\theta)^2,\ \frac{(1+\hat c\hat x)\theta - i\bar\theta\theta\hat c\theta}{1+2cx+c^2x^2}\right) \tag{2.26}$$

$$B_\chi(x,\theta;e^{icK}) = (1-2cx+c^2x^2)^{-d}\exp\left\{\frac{-3z(\bar\theta\gamma_5\hat c\theta - c^2\bar\theta\gamma_5\hat x\theta) + dc^2(\bar\theta\theta)^2}{2(1-2cx+c^2x^2)}\right\} \tag{2.27}$$

Eqs.(2.26)-(2.27) have, of course, only formal meaning now, because the representations T_χ as determined here are only infinitesimal representations of SOQ(4,2). In §4 global representations are constructed (denoted by the same symbol T_χ) for which Eqs.(2.26)-(2.27) as well as Eqs.(3.17)-(3.18) of §3 become correct.

The last thing to be found in this section is the representation conjugated to the representation T_χ i.e. the representation $\tilde T_\chi$ acting on the space $\tilde E_\chi$ of continuous linear functionals over E_χ as follows:

$$\langle \tilde T_\chi(M)F\,|\,\varphi\rangle + (-)^{ij}\langle F\,|\,T_\chi(M)\varphi\rangle = 0,\ M \in (soq(4,2))_i,\ F \in (\tilde E_\chi)_j\ (i,j = \bar 0, \bar 1) \tag{2.28}$$

To get the generators of the representation $\tilde T_\chi$ we use the properties of the generalized Jacobian (Berezinean)[17] and obtain that if

$$T_\chi(M_i) = B_i(y) + i\varphi_i^{\,j}(y)\partial_j\ , \tag{2.29}$$

then

$$\tilde T_\chi(M_i) = -B_i^T(y) + i\varepsilon_{jj}\varepsilon_{ji}\partial_j\varphi_i^{\,j}(y), \tag{2.30}$$

where

$$\varepsilon_{ij} = \begin{cases} +1, & \text{if } [y_i, y_j] = [\partial_i, \partial_j] = 0 \\ -1, & \text{if } [y_i, y_j]_+ = [\partial_i, \partial_j]_+ = 0 \end{cases} \tag{2.31}$$

Here both commuting variables x_i and anticommuting variables θ_α are combined in a unique set of variables $\{y_i\}$ for the compactness of expressions (2.29)-(2.31).

Applying Eq. (2.30) to the generators (2.13)-(2.16) we find immediately

$$\check{\tilde{T}}_\chi \equiv \tilde{T}_{(d,\chi)} \sim T_{(\ell-d,-\chi)} \equiv T_{\tilde{\chi}} . \qquad (2.32)$$

The sign "$\sim$" in Eq. (2.32) denotes the weak equivalence of the representations /14/ under consideration, i.e., roughly speaking, the fact that the transformations (2.21) are identical for both representations ($\hat{T}_\chi$ and $T_{\tilde{\chi}}$).

The representations conjugated to the scalar representations $R_d^\dagger$ and R_d^- acting on "chiral" superfields $\Phi(x,\theta')$ and $\Phi(x,\theta^\cdot)$ (see /5,6/) can be found similarly:

$$\widetilde{R_d^\dagger} \sim \dot{R}_{3-d} , \quad \widetilde{R_d^-} \sim \ddot{R}_{3-d} \qquad (2.33)$$

§ 3. R-inversion

In this section the transformations of the space (X,θ) and of the scalar superfields $\Phi(x,\theta)$ under the action of R-inversion (see (2.1), (2.2)) are found.

Given a (graded) Lie algebra L having an outer automorphism R. Denote for every $K \in L$

$$\dot{K} \equiv R(K). \qquad (3.1)$$

Suppose that the automorphism R is involutory, i.e.,

$$R^2 = 1. \qquad (3.2)$$

The automorphism R of the algebra L induces the automorphism of the corresponding group $G = e^L$:

$$R^*(e^{i\gamma K}) \equiv e^{i\gamma\dot{K}} \qquad (3.3)$$

and the automorphism of the space of the (classes of equivalent) representations of the group G, which maps every representation T on the representation $\dot{T}$ defined as follows:

$$\dot{T}(g) \equiv T(\dot{g}). \qquad (3.4)$$

The representation $\dot{T}$ is called the "mirror image" (or R-image) of the representation T. Note that in general $\dot{T}$ is not equivalent to T.

The automorphism R^* is supposed to be generated by an element R of the group G (because G can always be extended, if necessary, so as to include R):

$$RgR = \dot{g}. \qquad (3.5)$$

Let $X = G/H$, where H is a subgroup of G and determine the function $X'(x)$ by the expression

$$X'(x) \equiv R(x) , \quad x \in X . \qquad (3.6)$$

From Eq. (3.5) it follows that

$$gX'(x) = X'(\dot{g}x), \qquad (3.7)$$

or in an infinitesimal form:

$$\varphi_j^k(x)\,\partial_k X'^i(x) = \dot{\varphi}_j^{\,i}(x'), \qquad (3.8)$$

where the functions $\dot{\varphi}_j^{\,i}$ are determined by the generator $\dot{M}_j \equiv R(M_j)$ (see Eq.(A.II. 7)).

Equations (3.8) are equivalent to the operator equations:

$$M_i^\circ(x',\partial') = \dot{M}_i^\circ(x,\partial). \qquad (3.9)$$

The number of independent equations among equations (3.8) is equal to the number of different pairs $\{M_i, \dot{M}_i\}$ of the generators. As a rule, one of the two equivalent equations of each pair turns out to be a linear differential equation.

Given a representation T_χ of the group G. Let

$$(T_\chi(R)f)(x) \equiv B_\chi^R(x) f(Rx). \tag{3.11}$$

We are looking for the representation $T_{\dot\chi}$ with the property

$$T_{\dot\chi}(R)\, \dot{T}_{\dot\chi}(g)\, T_\chi(R) = T_\chi(g), \tag{3.12}$$

where $\dot{T}_{\dot\chi}$ is the R-image of $T_{\dot\chi}$ (see (3.4)).

The identity (3.2) (or (3.12)) implies

$$T_{\dot\chi}(R)\, T_\chi(R) = 1. \tag{3.13}$$

In other words, we are looking for a representation $T_{\dot\chi}$ such that there exists an intertwining operator $T_\chi(R): T_\chi \longrightarrow \dot{T}_{\dot\chi}$ establishing the equivalence of the representations T_χ and $\dot{T}_{\dot\chi}$. Taking into account the definitions (3.11) and (A.II.4) equations (3.12)-(3.13) can be rewritten in terms of the multipliers of the representations as

$$B_\chi(x,g) = B_{\dot\chi}^R(x)\, B_{\dot\chi}(Rx, \dot g)\, B_\chi^R(\dot g^{-1} Rx), \tag{3.14}$$

$$B_{\dot\chi}^R(Rx)\, B_\chi^R(x) = 1, \tag{3.15}$$

or in an infinitesimal form

$$i\,(B_\chi^R(x)\, B_i^{\dot\chi}(Rx) - B_i^{\dot\chi}(x)\, B_\chi^R(x)) = -\varphi_i^j(x)\, \partial_j B_\chi^R(x), \tag{3.16}$$

where the functions $B_i^\chi(x)$ and $\varphi_i^j(x)$ are determined by the generator M_i in the representation T_χ (see Eq. (A.II.5) of Appendix II).

Equations (3.16) permit to find the representation $\dot\chi$ and the multiplier $B_\chi^R(x)$ of the intertwining operator $T_\chi(R)$.

Return now to the case of $L = so(4,2)$ and R determined by eqs. (2.1)-(2.2). Solving equations (3.8) and (3.16) we get

$$R(x,\theta) = \left(\frac{x}{x^2}\, e^{\frac{(\bar\theta\theta)^2}{2x^2}}, -\frac{1}{x^2}(\hat{x}\theta - i\theta\,\bar\theta\theta)\right), \tag{3.17}$$

$$B_\chi^R(x,\theta) = (x^2)^{-d} \exp\left\{\frac{-3z\,\bar\theta\gamma_5\hat{x}\theta + d(\bar\theta\theta)^2}{2x^2}\right\}. \tag{3.18}$$

Besides, if $\chi = (d,z)$, then

$$\dot\chi = (d,-z). \tag{3.19}$$

Note that the expression

$$T(x,\theta) \equiv \frac{\bar\theta\gamma_5\hat{x}\theta}{x^2} \tag{3.20}$$

is R-invariant: $T(x,\theta) = T(Rx, R\theta)$.

Now we are in a position to establish, for example, the transformation law of the superfield $\Phi(x,\theta)$ under the action of special conformal transformations:

$$e^{i c K}(x,\theta) = R\, e^{i c P} R(x,\theta) = R(Rx + c, R\theta), \tag{3.21}$$

$$B_\chi(x,\theta; e^{icK}) = B_{\dot\chi}^R(x,\theta)\, B_\chi^R(Rx - c, R\theta). \tag{3.22}$$

Substituting expressions (3.17) and (3.18) into equalities (3.21) and (3.22) we obtain Eqs. (2.26)-(2.27).

It should be noted that formulae (3.17) and (3.18) are valid only for the Majorana spinors θ. For arbitrary complex spinors Eqs. (3.17), (3.18) must be generalized as follows:

$$R(x,\theta) = \left\{\left[\frac{x}{x^2}\, e^{\frac{(\bar\theta\theta)^2}{2x^2}}\right]^*, -\frac{1}{x^{*2}}(\hat{x}\theta - i\theta\,\bar\theta\theta)^*\right\}, \tag{3.23}$$

$$B_\chi^R(x,\theta) = (x^{*2})^{-d}\left[\exp\frac{3z^*\bar\theta\gamma_5\hat{x}\theta + d^*(\bar\theta\theta)^2}{2x^2}\right]^*, \tag{3.24}$$

where $*$ denotes complex conjugation. Taking Majorana spinors θ and Hermitean x we recover Eqs. (3.17), (3.18) from Eqs. (3.23), (3.24) taking into account Eq. (1.19).

Now, in particular, we can obtain the R-transformation of the manifold (X,θ') and of chiral superfields $\Phi(x,\theta')$ /5,6/:

$$R(x,\theta') = \left(\frac{x^{\mu}}{x^{2}}, -\frac{\tilde{x}\theta^{-}}{x^{2}}\right), \quad B_{d}^{R}(x,\theta') = (x^{2})^{-d}. \tag{3.25}$$

Moreover, the following identity:

$$R e^{i\bar{\varepsilon}S} R = e^{i\bar{\varepsilon}' T} \tag{3.26}$$

is valid. For example, $R e^{i\bar{\varepsilon}' S'} R = e^{i\bar{\varepsilon}^{-} T^{-}}$ (we recall that $(\varepsilon')' = \varepsilon^{-}$ if ε is a Majorana spinor as in our case).

§4. The Spaces E_{χ} of Superfields

In this section we construct another realization of the representations T_{χ} and diagonalize them with respect to the conformal subgroup.

The supergroup SOQ (4,2) acts on the complex "Minkowski" space, i.e.,on the supermanifold $\{z, C^{\omega}(z)\otimes\Lambda_{4}(\tau)\}$, where z is a complexification of the space x and $\tau=\theta+i\theta'$ is a complex anticommuting spinor.

It can be observed that the submanifold $\{T^{+}, C^{\omega}(T^{+})\otimes\Lambda_{4}(\tau)\}$ (where T^{+} is a forward tube $T^{+}=\{x+iy;\ y^{2}<0,\ y_{0}>0\}$) is invariant with respect to the supergroup SOQ (4,2), so that we can realize the representation T_{χ} in the space of analytic functions on the forward tube domain with entries in the complex Grassman algebra $\Lambda_{4}(\tau)$ (cf./18-20/). In close analogy with the case of conformal group/18-20/ we can consider the boundary values instead of analytic functions, i.e., realize the representation T_{χ} in some subspace E_{χ} of the space $S_{+}'(X)\otimes\Lambda_{4}(\theta)$. Here $\mathcal{F}S_{+}'(X)$ is the space of tempered distributions having supports within the forward light cone/18-20/ Eqs. (2.15) - (2.16) and (2.22) - (2.27) remain unchanged but x^{2} must be replaced by $x^{2}+i\varepsilon x_{0}$. From now on the representations T_{χ} are supposed to act on these subspaces of tempered distributions.

The only thing which remains unclear now is the asymptotic behaviour of the superfields $\Phi(x,\theta)$ entering into the space E_{χ} . Instead of direct computation of the asymptotic behaviour of the superfields we can diagonalize the representation T_{χ} with respect to the conformal subgroup applying afterwards the results of Rühl concerning the representations of the interpolated discrete series of conformal group/18-20/. It may be observed that the special conformal transformations and R-inversion are nondiagonal operators in the "basis" consisting of the fields $A(x),\dots,D(x)$, but rather triangular, because in transformations of each of the fields $A^{\mu}(x)$, $D(x)$ and $\chi(x)$ one of the fields $A(x)$ or $\Psi(x)$ takes part as well. The situation differs in this respect from what we have in the case of chiral superfields $\Phi(x,\theta')$ and $\Phi(x,\theta^{\cdot})$ /5,6/.

Our task now is to find such an operator $\mathcal{D}_{\chi}:E_{\chi}\to E_{\chi}'$ (where $E_{\chi}'\subset S_{+}'(X)\otimes\Lambda_{4}(\theta)$) transforming the coordinates $A(x),\dots,D(x)$ of the superfield $\Phi(x,\theta)$ into the coordinates $A'(x),\dots,D'(x)$ of the superfield

$$\Phi'(x,\theta) = (\mathcal{D}_{\chi}\Phi)(x,\theta), \tag{4.1}$$

that the fields $A'(x),\dots,D'(x)$ transform independently under the action of the conformal subgroup SO (4,2), i.e.,that the elements of this subgroup (and R-inversion in particular) be diagonal in the basis consisting of the primed fields $A'(x),\dots,D'(x)$.

The last requirement determines the operator $\mathcal{D}_{\chi}$ uniquely. It turns out that

$$\mathcal{D}_\chi^{-1} = 1 + N_\chi , \qquad (4.2)$$

where

$$(N_\chi \Phi')(x,\theta) = -\frac{3z}{4d}\bar\theta\gamma_5\hat\partial\theta A'(x) - \frac{i}{2(d-1)}\bar\theta\theta\bar\theta(d + \frac{3z}{2}i\gamma_5)\hat\partial\Psi'(x) +$$

$$+\frac{(\bar\theta\theta)^2}{16}\left\{\frac{2(d-(\frac{3}{2}z)^2)}{d(d-1)}\partial^2 A'(x) + \frac{3z}{d-2}\partial_\mu A'^\mu(x)\right\}. \qquad (4.3)$$

The operator N_χ is a nilpotent operator so that

$$\mathcal{D}_\chi = 1 - N_\chi + N_\chi^2 \qquad (4.4)$$

The explicit expression of the operator $\mathcal{D}_\chi$ can be found from (4.3). It is not difficult to see that the operator $\mathcal{D}_\chi$ turns out to be a polynomial of the operators θ, ∂ and ∂_θ .

Under the action of the conformal group and R-inversion the fields $A'(x), \Psi'(x), \ldots, D'(x)$ transform independently as if they belong to the representation $C_{(0,d)}, C_{(\frac{1}{2},d+\frac{1}{2})}, \ldots, C_{(0,d+2)}$, respectively, of the conformal group $SO(4,2)$ ($C_{(s,d)}$ is the representation of the interpolated discrete series of $SO(4,2)$ with spin s and anomalous dimension d [18-20]). Thus we obtain the reduction of the representation T_χ with respect to the conformal subgroup of the group $SOQ(4,2)$ (if $\chi \notin \{(1,z), (0,z\neq 0), (2,z\neq 0)\}$):

$$T_\chi = C_{(0,d)} \oplus C_{(\frac{1}{2},d+\frac{1}{2})} \oplus 2C_{(0,d+1)} \oplus C_{(1,d+1)} \oplus C_{(\frac{1}{2},d+\frac{3}{2})} \oplus C_{(0,d+2)}. \qquad (4.5)$$

The space E_χ is completely determined by Eqs. (4.1)-(4.5). if $\chi \notin \{(1,z), (0,z\neq 0), (2,z\neq 0)\}$. But if $\chi = (1,z)$ or $\chi = (0,z\neq 0)$ or $\chi = (2,z\neq 0)$ then the operator-valued function $\mathcal{D}_\chi$ has singularities.

This gives evidence to the fact that the representation T_χ is not decomposable with respect to the conformal subgroup at these exceptional values of χ , and,in particular, at canonical dimension.

The representation $T'_\chi \equiv \mathcal{D}_\chi T_\chi \mathcal{D}_\chi^{-1}$ determined on the set of superfields $E'_\chi \equiv \mathcal{D}_\chi E_\chi$ is equivalent to the representation T_χ and, besides, the coordinates $A'(x), \ldots, D'(x)$ of the superfield $\Phi' \in E'_\chi$ transform in the usual way[16,18-20] with respect to the conformal subgroup $SO(4,2)$. This is one of the reasons why the basis consisting of the fields $A'(x), \Psi'(x), \ldots, D'(x)$ may be called the physical one. The other is the "diagonality" of the two-point $SO(4,2)$-invariant Green functions with respect to this basis, i.e., the absence of the nondiagonal terms of the type $\langle A'(x_1) D'(x_2)\rangle$ etc.,in the decomposition of the two-point function in powers of θ_1 and θ_2 (owing to the usual law of "conservation of dimensions"[11]). Therefore, in constructing a nontrivial $SOQ(4,2)$-invariant physical theory it would, presumably, be more convenient to deal with this basis.

Similar considerations can take place for chiral representations R_d^+ and R_d^- . The reduction with respect to the conformal subgroup is:

$$R_d^\pm = C_{(0,d)} \oplus C_{(\frac{1}{2},d+\frac{1}{2})} \oplus C_{(0,d+1)}, \qquad (4.6)$$

so that the conformal subgroup is diagonal with respect to the superfields $\Phi(x,\theta^\pm)$ from the very beginning[6]. In a subsequent paper the invariant two- and three-point functions of scalar superfields belonging to the representations T_χ will be found. These functions will be used then to construct invariant bilinear and Hermitean forms of the representations T_χ , as well as the intertwining operators of these representations which, in turn, will permit to give a more complete treatment of the reducibility of the representations T_χ . The unitary scalar representations will be also found.

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Appendix I. The Supergroups SU (2,2;1) and SOQ(4,2)

The notion of Lie supergroup and of supermanifold are introduced in the paper of Berezin and Leites /10/ (see also an earlier work of Berezin and Kac /9/).

Consider the complex Lie supergroup $\mathcal{GL}_{4,1}$ /9,21/. Let be

$$\mathcal{O}\!\mathcal{l} \equiv C^{\omega}(GL(4)\times \mathbb{C}^{*})\otimes \Lambda_{8}^{c}(\theta) \tag{AI.1}$$

the algebra of global section of this supergroup and

$$T \equiv (T)_{mn} \quad (m,n = 1,2,3,4,5) \tag{AI.2}$$

the matrix of generating system of the algebra.

The algebra $\mathcal{O}\!\mathcal{l}$ is a complex Z_2-graded algebra with involution. The relations

$$\Delta(T) - I = 0\ , \quad TST^{+} - S = 0, \tag{AI.3}$$

where $\Delta(T)$ is the Berezinean /17/ of the matrix T and

$$S = \begin{pmatrix} I_2 & & \\ & -I_2 & \\ & & 1 \end{pmatrix} \tag{AI.4}$$

determine some factoralgebra $\mathcal{O}\!\mathcal{l}/\mathcal{J}$ and some real subsupermanifold of the supermanifold $\mathcal{GL}_{4,1}$:

$$SU(2,2;1) \equiv \{ U(2,2), C^{\omega}(U(2,2))\otimes \Lambda_{8}^{R}(\theta)\} \tag{AI.5}$$

and even more, the supermanifold $SU(2,2;1)$ is a subsupergroup of the supergroup $\mathcal{GL}_{4,1}$.

Consider now the subalgebra in $\mathcal{O}\!\mathcal{l}/\mathcal{J}$ consisting of even functions of the generators T_{ij} . This subalgebra defines a factor supergroup

$$SOQ(4,2) \simeq SU(2,2;1)/Z_2 \tag{AI.6}$$

which is isomorphic as a supermanifold to

$$SOQ(4,2) \simeq \{ SO(4,2)\times U(1), C^{\omega}(SO(4,2)\times U(1))\otimes \Lambda_{8}^{R}(\theta)\}. \tag{AI.7}$$

Appendix II. Induced Representations of Lie Supergroups

Definition 1. Let $\mathcal{M}=\{M,\mathcal{O}_M\}$ be a supermanifold. The vector bundle over the supermanifold $\mathcal{M}$ is a sheaved space $\mathcal{V}=\{M,\mathcal{B}_M\}$ where the sheaf $\mathcal{B}_M$ is a locally free Z_2-graded $\mathcal{O}_M$-Module of the finite type.

In other words, $\mathcal{V}$ is a vector bundle over $\mathcal{M}$ if there exists an open covering $M=\bigcup_{i\in I} U_i$ and the set $\{V_i\}_{i\in I}$ of Z_2-graded finite dimensional K-vector spaces (K is a basic field) such that for every $i\in I$ $\mathcal{B}_M(U_i)\simeq\mathcal{O}_M\otimes V_i$. If all of the V_i are isomorphic to the vector space V then V is said to be a fibre of the vector bundle $\mathcal{V}$.

Definition 2. Let $\mathcal{M},\mathcal{M}'$ be the supermanifolds, $\mathcal{V}$ and $\mathcal{V}'$ the vector bundles over $\mathcal{M}$ and $\mathcal{M}'$, respectively, $\mathcal{M}\xrightarrow{f}\mathcal{M}'$ morphism of the supermanifolds[10].

The morphism $\alpha:\mathcal{V}\longrightarrow\mathcal{V}'$ of the sheaved spaces is referred to as f-°comorphism of $\mathcal{V}$ into $\mathcal{V}'$ if it is consistent (in an obvious sense) with the morphism f (cf.[22]). Vector bundles with comorphisms form a category denoted further as Buncom. Comorphisms would be denoted by waved arrows in order not to mix them with the ordinary morphisms (Def. 4) .

Let $\mathcal{M},\mathcal{M}'$ be the supermanifolds, $\mathcal{V}$ the vector bundle over $\mathcal{M}$, $\mathcal{M}'\xrightarrow{f}\mathcal{M}$ a morphism of the supermanifolds. Consider the category $Arr(f,\mathcal{V})$ of arrows $\mathcal{V}'\overset{\alpha}{\rightsquigarrow}\mathcal{V}$, where α is f-°comorphism and morphisms in this category are by definition $Id_{m'}$-°comorphisms such that the corresponding diagram be commutative.

Definition 3. The universal repelling object in the category $Arr(f,\mathcal{V})$ is referred to as the inverse image of the vector bundle $\mathcal{V}$ with respect to the morphism f . It is denoted as $f^*\mathcal{V}\overset{f^\circ}{\rightsquigarrow}\mathcal{V}$.

Lemma 1. The inverse image exists for every f and $\mathcal{V}$. If $\mathcal{V}$ is a vector bundle with fibre V , then $f^*\mathcal{V}$ is also a vector bundle with fibre V .

Definition 4. Let $\mathcal{M}\xrightarrow{f}\mathcal{M}'$ be morphism of the supermanifolds, $\mathcal{V}$ and $\mathcal{V}'$ vector bundles over $\mathcal{M}$ and $\mathcal{M}'$, respectively. Id_m-°comorphism $f^*\mathcal{V}'\overset{\alpha}{\rightsquigarrow}\mathcal{V}$ is said to be f-morphism of the vector bundle $\mathcal{V}$ into the vector bundle $\mathcal{V}'$. Morphisms would be denoted further by the straight arrows. The °comorphism $f^*\mathcal{V}'\overset{Id}{\rightsquigarrow}f^*\mathcal{V}'$ defines, in particular, some f-morphism $f^*\mathcal{V}'\xrightarrow{f^\wedge}\mathcal{V}'$.

Lemma 1'. Every f-morphism $\mathcal{V}\xrightarrow{\varphi}\mathcal{V}'$ can be uniquely decomposed as $\varphi=f^*\circ\varphi^\wedge$, where $\varphi^\wedge$ is some Id_m-morphism.

Lemma 2. Bundles and morphismes of bundles form a category (Bun).

Definition 5. Given the supermanifolds $\mathcal{M}_1,\mathcal{M}_2$ and the vector bundle $\mathcal{V}$ over $\mathcal{M}_2$. Denote as $\mathcal{M}_1\times\mathcal{V}$ the vector bundle $\pi_2^*\mathcal{V}$ over $\mathcal{M}_1\times\mathcal{M}_2$ (π_2 is a canonical projection $\mathcal{M}_1\times\mathcal{M}_2\to\mathcal{M}_2$). It is said to be a product of the supermanifold $\mathcal{M}$ and the vector bundle $\mathcal{V}$.

Lemma 3. Let $\mathcal{M}\xrightarrow{f}\mathcal{M}'$ is a morphism of supermanifolds, $g:\mathcal{V}\to\mathcal{V}'$ morphism of vector bundles. There exists one and only one (f,g)-morphism $(f,g):\mathcal{M}\times\mathcal{V}\longrightarrow\mathcal{M}'\times\mathcal{V}'$ such that the diagram

$$\begin{array}{ccc}\mathcal{M}\times\mathcal{V} & \longrightarrow & \mathcal{V}\\ {\scriptstyle(f,g)}\downarrow & & \downarrow{\scriptstyle g}\\ \mathcal{M}'\times\mathcal{V}' & \longrightarrow & \mathcal{V}'\end{array}$$

is commutative (_ denotes here the canonical functor $Bun\to$ Supermanifolds, namely, transition to the bases and to their morphisms).

Let $\mathcal{G}$ be the supergroup, $\mathcal{M}$ the supermanifold with some action $\mu:\mathcal{G}\times\mathcal{M}\to\mathcal{M}$ of $\mathcal{G}$ (see[9,10]), $\mathcal{V}$ the vector bundle over $\mathcal{M}$.

Definition 6. The isomorphism $\mathcal{G}\times\mathcal{V}\xrightarrow{\lambda}\mu^*\mathcal{V}$ is said to be the continuation of the action μ on the vector bundle $\mathcal{V}$ if some diagram of morphisms of vector bundles over $\mathcal{G}\times\mathcal{G}\times\mathcal{M}$ is commutative (this diagram can be seen in chap.3, §12 of Ref.[23]). Given such an action we define the quotient vector bundle $(\mathcal{G},\lambda)\backslash\mathcal{V}$ as a cokernel of the pair of morphisms[24]:

$$\mathcal{G}\times\mathcal{V} \underset{\pi_2^*}{\overset{\mu^*\lambda}{\rightrightarrows}} \mathcal{V} \qquad (A.II.1)$$

$(\mathcal{G},\lambda)\backslash\mathcal{V}$, if it exists, is the vector bundle over $\mathcal{G}\backslash\mathcal{M}$.

<u>Theorem.</u> Let $\mathcal{G}$ be a supergroup with multiplication $\mu:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, $\mathcal{H}$ a subsupergroup of $\mathcal{G}$ with canonical right action $\mu_{\mathcal{H}}:\mathcal{G}\times\mathcal{H}\to\mathcal{G}$, V $\mathbb{Z}_2$-graded finite-dimensional k-vector space with a linear representation $\ell:\mathcal{H}\times V\to V$ on it (V is the vector bundle over a point $\mathcal{R}=(\cdot;k)$ and hence the object $\mathcal{H}\times V$ is well defined). Let, further, $\Delta:\mathcal{H}\to\mathcal{H}\times\mathcal{H}$ is the diagonal morphism, $\varphi_\ell:(\mathcal{G}\times V)\times\mathcal{H}\to\mathcal{G}\times V$ $\mu_{\mathcal{H}}$-morphism which is the composition of morphisms

$$(\mathcal{G}\times V)\times\mathcal{H}\xrightarrow{(1,\Delta)}(\mathcal{G}\times V)\times(\mathcal{H}\times\mathcal{H})\rightleftarrows(\mathcal{G}\times\mathcal{H})\times(\mathcal{H}\times V)\xrightarrow{(1,(\iota,1))}(\mathcal{G}\times\mathcal{H})\times(\mathcal{H}\times V)\xrightarrow{(\mu_{\mathcal{H}},\ell)}\mathcal{G}\times V$$

where $\iota:\mathcal{H}\to\mathcal{H}$ the inverse morphism of the supergroup $\mathcal{H}$. Then

a) morphism $\varphi_\ell^*:(\mathcal{G}\times V)\times\mathcal{H}\to\mu_{\mathcal{H}}^*(\mathcal{G}\times V)$ is a continuation of the action $\mu_{\mathcal{H}}$ on the vector bundle $\mathcal{G}\times V$.

b) there exists the quotient supermanifold $\mathcal{G}/\mathcal{H}$ and the quotient vector bundle $(\mathcal{G}\times V)/(\mathcal{H},\varphi_\ell^*)$ over $\mathcal{G}/\mathcal{H}$ denoted further as $(\mathcal{G}\times V)^{\mathcal{H}}$. Let $p:\mathcal{G}\times V\to(\mathcal{G}\times V)^{\mathcal{H}}$ the canonical projection and $\mu_{\mathcal{G}/\mathcal{H}}:\mathcal{G}\times\mathcal{G}/\mathcal{H}\to\mathcal{G}/\mathcal{H}$ a canonical action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$.

c) there exists one and only one $\mu_{\mathcal{G}/\mathcal{H}}$-morphism $\mathcal{G}\times(\mathcal{G}\times V)^{\mathcal{H}}\xrightarrow{\tau_\ell}(\mathcal{G}\times V)^{\mathcal{H}}$ such that the diagram

$$\begin{tikzcd}
\mathcal{G}\times(\mathcal{G}\times V) \arrow[r, shift left] \arrow[dr, "{(1,p)}"'] & (\mathcal{G}\times\mathcal{G})\times V \arrow[l, shift left] \arrow[r, "{(\mu,1)}"] & \mathcal{G}\times V \arrow[r, "p"] & (\mathcal{G}\times V)^{\mathcal{H}} \\
 & \mathcal{G}\times(\mathcal{G}\times V)^{\mathcal{H}} \arrow[urr, "\tau_\ell"'] & &
\end{tikzcd} \qquad (A.II.2)$$

is commutative; $\tau_\ell^*:\mathcal{G}\times(\mathcal{G}\times V)^{\mathcal{H}}\to\mu_{\mathcal{G}/\mathcal{H}}^*((\mathcal{G}\times V)^{\mathcal{H}})$ is the continuation of the action $\mu_{\mathcal{G}/\mathcal{H}}$ on the vector bundle $(\mathcal{G}\times V)^{\mathcal{H}}$.

The linear representation T_ℓ of $\mathcal{G}$ arising in the space of global sections of the structure sheaf of $(\mathcal{G}\times V)^{\mathcal{H}}$ is referred to as the representation of $\mathcal{G}$ <u>induced</u> by the representation ℓ of $\mathcal{H}$.

This construction is very convenient to use, as in the case of ordinary Lie groups, if

$$\mathcal{G}=\mathcal{G}_0\mathcal{H} \qquad \mathcal{G}_0\cap\mathcal{H}=1 \qquad (A.II.3)$$

Here $\mathcal{G}_0\mathcal{H}$ denotes the minimal closed supergroup containing both $\mathcal{G}_0$ and $\mathcal{H}$. The supergroup $\mathcal{G}_0$ can be imbedded into the supermanifold $\mathcal{G}/\mathcal{H}$: $\mathcal{G}_0\hookrightarrow\mathcal{G}/\mathcal{H}$ so that the closure of $\mathcal{G}_0$ in $\mathcal{G}/\mathcal{H}$ be $\mathcal{G}/\mathcal{H}$. In particular, the generating system of the algebra of global sections of $\mathcal{G}/\mathcal{H}$ and that of $\mathcal{G}_0$ can be identified.

The induced representation T_ℓ acts in this case on some subspace E of the space $\Gamma(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}_0})$ and the action T_ℓ is given by the ordinary expression:

$$(T_\ell\Phi)(g,x)=B_\ell(x,g)\Phi(g^{-1}x) \qquad (A.II.4)$$

where x is the set of generators of the algebra $\Gamma(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}_0})$ of global sections of $\mathcal{G}_0$, g – the same for $\Gamma(\mathcal{G},\mathcal{O}_{\mathcal{G}})$; $\Phi(x)\in E$; $\Phi(g^{-1}x)\overset{def}{=}(T_{\pi_2}\Phi)(g,x)$ (π_2 is a trivial representation $\mathcal{H}\times V\xrightarrow{\pi_2}V$); $B_\ell(x,g)$ is some element of the space $\Gamma(\mathcal{G},\mathcal{O}_{\mathcal{G}})\otimes\Gamma(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}_0})\otimes V$ depending on ℓ and defining naturally an automorphism of the space $\Gamma(\mathcal{G},\mathcal{O}_{\mathcal{G}})\otimes\Gamma(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}_0})\otimes V$.

Eq.(A.II.4) permits one to find the realization of the generators of Lie superalgebra of the supergroup $\mathcal{G}$ in the representation T_ℓ. The most general expression for this generators is:

$$T_\ell(M_i)=B_i(x)+i\varphi_i^j(x)\partial_j\equiv B_i(x)+M_i^0 \qquad (A.II.5)$$

We conclude this section with one remark concerning the connection between the "ordinary" vector bundles and supermanifolds: the category of supermanifolds is proved to be isomorphic to the category of bundles of Grassman algebras with comorphisms as morphisms.

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