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* Address after September 1, 1976:
Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria.

S u m m a r y

The relativistic n-particle dynamics is studied as a problem with constraints of the type

$$(2\phi_i \equiv)\; m_i^2 - p_i^2 + \Phi_i = 0, \quad i = 1, \dots, n, \qquad \text{(C)}$$

where Φ_i are Poincaré invariant functions of the particles' coordinates, momenta and spin components; Φ_i is assumed to vanish asymptotically when the i-th particle coordinates tend to infinity. In the two particle case we assume in addition that the Poisson bracket $\{\phi_1, \phi_2\}$ vanishes on the surface (C). That allows us to give a formulation of the theory, invariant with respect to the choice of the time-parameter on each trajectory. The quantization of the relative two-particle motion is also discussed. It is pointed out that the stationary Schrödinger equation obtained in this manner is a local quasipotential equation.

Introduction

It is often argued that there could be no consistent theory of relativistic interacting particles without the presence of a field which mediates the interaction. Although the field mediation is indeed physically more realistic, the direct interaction between particles is also logically possible (see, e.g., /8,7,26,18/ the last reference also contains an extensive bibliography). However, there seems to be no generally accepted dynamical framework to deal with such a problem. Moreover, too straightforward attempts to generalize the non-relativistic canonical formalism to the relativistic case have led to embarrassing no-interaction theorems (see the review /7/ where earlier papers of Currie, Jordan and Sudarshan are also cited). The non-Hamiltonian approach of ref. /28/ (started in earlier work of Van Dam and Wigner cited there) is clearly self-consistent, but nevertheless, does not seem to stir much enthusiasm among students of the field, presumably, because of its somewhat unconventional appearance. On the other hand, the quantum mechanical (say, off energy shell) dynamics of n relativistic particles has been treated with some success, e.g., in refs. /5,22,23/ . The idea has even come to mind /11,6/ to derive the correct formulation of the classical relativistic two-body problem as a limit (for $\hbar \to 0$) of some relativistic quasipotential equation in the quantum framework (see, e.g., /15,16,20/ ; the last reference also contains a bibliography up to 1974)). No unanimity on the right choice of the theory has been achieved in that approach either. The off-energy shell formulation (of the type used, say, in [5,15,22,23]) lacks manifest covariance; the treatment in ref. [11] is not symmetric with respect to the two particles (see, in particular, Secs.VI and VIII of that reference).

In the present paper we propose a manifestly covariant formulation of classical relativistic mechanics of point particles, treated as a dynamical theory with non-holonomic constraints [x)] . The constraints are defined as generalizations of the mass-shell conditions. (The strict mass-shell relation $p^2 = m^2$ for a given particle is only recovered asymptotically, when its distance to all other particles tends to infinity.) In the special case of the two-body problem we deal with two constraints $\varphi_1 = 0 = \varphi_2$ which are assumed to have weakly vanishing Poisson brackets (i.e., $\{\varphi_1, \varphi_2\} = 0$ for $\varphi_1 = 0 = \varphi_2$). An invariant characteristic of the particles' motion is

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[x)] The idea of formulating the entire classical mechanics as a theory with constraints (excluding the anthropomorphic notion of force) is a rather old one. It has reached a high point in the posthumous book of Herz [13] . The most fundamental difference is that in the 19th century physicists tried to go further and "explain" the constraints (by inventing ingeneous mechanisms with hidden masses,etc.). The modern mind is satisfied to find that the equations reflect the underlying symmetry of the problem.

given by their space-time trajectories. The arbitrariness of the choice of the (time)-parameter on each trajectory gives rise to a "gauge" freedom. A gauge invariant formulation of the two-body problem is given, employing the technics of ref. /10/. The quantization of the two-particle relative motion leads to a (local) quasipotential Schrödinger equation of the type applied recently to the bound state problem in quantum electrodynamics (see /24,25,20,21/).

In Sec.1 we present a Lorentz and gauge invariant description of free relativistic classical particles with arbitrary spin. Section 2.A contains a general discussion of relativistic n-particle dynamics as a problem with an n constraints. The two-particle case is treated in more detail under some additional assumptions in Secs. 2B and 2C. A brief discussion of the quantization of relative (two-particle) motion is given in Sec.3.

1. <u>Covariant description of relativistic 1-particle phase space.</u>

A. <u>Positive energy orbits of the Poincaré group</u>

According to the general group theoretical approach of Kirillov /14/ the phase space of a (free) relativistic point particle can be identified with an orbit in the co-adjoint representation of the (proper) Poincaré group $\mathcal{P}=\mathcal{P}_+^\uparrow$. We shall present here (for the reader's convenience) Reyman's description /19/ of the positive energy orbits (see also /1-3/) in a manifestly covariant form.

There are two types of positive mass orbits: most orbits are 8-dimensional and have the topological structure of the direct product $\mathbb{R}^6 \times S^2$ ($\mathbb{R}^6$ being the 6-dimensional real Euclidean space and S^2 standing for the 2-dimensional sphere in $\mathbb{R}^3$): if the radius ρ of the sphere S^2 is zero, then we have a 6-dimensional orbit corresponding to the phase space of a spinless particle. The coadjoint action of the Poincaré transformation $g = (a, \Lambda)$ on the generators P_μ and $M_{\mu\nu} (= -M_{\nu\mu})$ of the Lie algebra of $\mathcal{P}$ is given by

$$
\begin{aligned}
P_\mu &\to {}^g P_\mu = P_\lambda (\Lambda^{-1})^\lambda{}_\mu , \\
M_{\mu\nu} &\to {}^g M_{\mu\nu} = M_{\kappa\lambda} (\Lambda^{-1})^\kappa{}_\mu (\Lambda^{-1})^\lambda{}_\nu + {}^g P_\mu a_\nu - {}^g P_\nu a_\mu
\end{aligned}
\tag{1.1}
$$

(it follows that ${}^g P^\mu = \Lambda^\mu{}_\nu P^\nu$, etc). The Casimir invariants are

$$P^2 = P_\mu P^\mu = P_0^2 - \underline{P}^2 \; (= m^2) \qquad \left(\underline{P} = (P_1, P_2, P_3)\right) \tag{1.2a}$$

$$W = \tfrac{1}{2} P^2 M_{\mu\nu} M^{\mu\nu} - P_\mu M^{\mu\sigma} M_{\nu\sigma} P^\nu \; \left(= m^2 \rho^2\right). \tag{1.2b}$$

In order to give a covariant description of the relativistic phase space $\mathbb{R}^6 \times S^2$, it is convenient to imbed it in a wider space. To this end, we introduce along with the 4-momentum p_μ also the 4-vector x^μ of the particle space-time position and a complex Lorentz 3 vector z_j which is translation invariant and transforms like $\frac{1}{2} \varepsilon_{jkl} M_{kl} - i M_{0j}$ under homogeneous Lorentz transformations. (It is however not covariant under space reflections). We consider the set of infinitely differentiable functions of x and p which are polynomials in z and introduce a Poisson bracket, satisfying the usual conditions (see, e.g., /2/) and such that the only nontrivial brackets of the basic coordinates are

$$\{x^{\mu}, p_{\nu}\} = -\delta^{\mu}_{\nu}, \qquad \{z_j, z_k\} = \varepsilon_{jk\ell} z_\ell \tag{1.3}$$

($\varepsilon_{jk\ell}$ is the 3-dimensional, totally antisymmetric unit tensor). The generators of the Poincaré Lie algebra can be expressed in terms of these coordinates as follows:

$$P_\mu = p_\mu, \quad M_{\mu\nu} = L_{\mu\nu} + s_{\mu\nu}, \tag{1.4}$$

where

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \qquad s_{k\ell} = \varepsilon_{jk\ell} z_j, \quad s_{oj} = i z_j . \tag{1.5}$$

The Poisson brackets among the generators P_μ and $M_{\mu\nu}$ reproduce the known commutation relations in the Lie algebra of the Poincaré group. The symplectic structure, defined by these brackets is non-degenerate on the manifold

$$\Gamma = R^8 \times S^2(C) = \left\{ (x, p; \underline{z}) \in R^8 \times C^3; \ \underline{z}^2 = \rho^2 \right\}. \tag{1.6}$$

(The 2-dimensional complex sphere $S^2(C)$ is regarded here as an analytic manifold.) The Poisson brackets are however degenerate on the mass shell

$$m^2 - p^2 = 0 \tag{1.7}$$

(if for no other reason because the manifold

$$M = \left\{ (x,p) \in R^{8},\ \underline{z} \in S^2(C);\ m^2 - p^2 = 0 \right\} \tag{1.8}$$

is odd dimensional).

For p satisfying (1.7) we introduce the Pauli-Lubanski--Bargmann vector

$$w^{\mu} = \frac{1}{2} \varepsilon^{\kappa\lambda\rho\mu}\, s_{\kappa\lambda} p_{\rho} \qquad (\varepsilon_{0123} = -\varepsilon^{0123} = 1) \tag{1.9a}$$

with components

$$w^0 = \underline{p}\,\underline{z}, \quad \underline{W} = p_0 \underline{z} + i\, \underline{z} \wedge \underline{p} \tag{1.9b}$$

($p_0 > 0$, $(\underline{z} \wedge \underline{p})^j = \varepsilon_{jkl} z^k p^l$). We notice that the 4-vector character of p and w agrees with the 3-dimensional (complex) transformation law for $\underline{z}$. For example, for a Lorentz boost along the third axis we have

$$p_0' = p_0 \operatorname{ch}\alpha - p_3 \operatorname{sh}\alpha, \quad p_1' = p_1, \quad p_2' = p_2, \quad p_3' = -p_0 \operatorname{sh}\alpha + p_3 \operatorname{ch}\alpha, \tag{1.10a}$$

$$z_1' = z_1 \operatorname{ch}\alpha - i z_2 \operatorname{sh}\alpha, \quad z_2' = i z_1 \operatorname{sh}\alpha + z_2 \operatorname{ch}\alpha, \quad z_3' = z_3, \tag{1.10b}$$

which imply $w_0' = w_0 \operatorname{ch}\alpha - w_3 \operatorname{sh}\alpha$, etc.

We shall assume that the 4-vector w is real (it is sufficient to assume this property in some special frame since w is transforming under a real representation of the Lorentz group).

Then for a positive mass particle ($m > 0$) the spin

$$\underline{s} = \frac{1}{m}\left(\underline{w} - \frac{w_0 \underline{p}}{m+p_0}\right) = \frac{1}{m}\left(p_0 \underline{z} + i\, \underline{z} \wedge \underline{p} - \frac{\underline{z}\underline{p}}{m+p_0}\, \underline{p}\right) \qquad (1.11)$$

is also real. It is easily checked that $-\frac{1}{m^2} w^2 = \underline{s}^2 = \underline{z}^2 \, (= \rho^2)$. Under the above assumption ρ =0 is equivalent to $\underline{z}$ =0.

B. "Gauge" invariance with respect to the choice of the time parameter

According to the general prescription of Faddeev /10/ (see, in particular, the Appendix to that reference) the physical 1-particle phase space Γ^{+} is obtained from M by factoring out the trajectories of the constraint (1.7). In other words, we consider for a moment the function

$$\varphi = \frac{1}{2}(m^2 - p^2) \qquad (1.12)$$

as a Hamiltonian (φ/m is termed "relativistic Hamiltonian" in ref. /6/ or a "Lagrangian" in ref. /11/) . Then we can regard the variable

$$\tau = \frac{1}{m^2}\, xp \qquad (1.13)$$

as a conjugate ("proper") time (since $\{\tau, \varphi\} = 1$) and write for any (generalized) coordinate $q\,(x, p; z)$ the equation of motion

$$\frac{dq}{d\tau} = \{q, \varphi\}. \qquad (1.14)$$

(τ is actually the proper time divided by the rest mass; it has a non-zero limit for $m \to 0$-cf. /4/). We have, in particular,

$$\frac{dx}{d\tau} = p, \qquad \frac{dp}{d\tau} = 0 = \frac{dz}{d\tau} . \tag{1.14'}$$

Given a point $q_0 \in M$ Eq. (1.14) defines a unique trajectory through it. The phase space Γ^* is obtained from M by identifying all points on such a trajectory. We notice that the free particle world line $(x^\mu = p^\mu\tau + x_0^\mu, p_\mu; z_k)$ is thus identified with a point of Γ^*. Clearly, the special normalization of the function φ (i.e., the factor 1/2 in (1.12)) and the corresponding choice (1.13) of the time parameter are not important for this construction: for another choice we would have obtained a different parametrization of the <u>same</u> trajectory (and hence of the <u>same</u> factor space Γ^*).

An alternative way to look at the above construction is to consider Eq. (1.13) as a subsidiary condition, which fixes the "gauge" - in our case, the arbitrary parameter on the particle world line (cf. /12/). Such a choice amounts to picking up a representative point in each equivalence class of Γ^*. The constraint (1.13) defines the proper time which has (by definition) zero Poisson bracket with any physical quantity. (Note that the alternative non-covariant gauge $x^0 = t$ leads to the Newton-Wigner coordinates, see /17,12/).

Writing (1.13) in the form

$$\chi = \tau - \frac{xp}{m^2} = 0 \tag{1.13'}$$

we obtain pair of second class constraints $\varphi = 0 = \chi$ (in the terminology of Dirac [9]), since the Poisson bracket of φ and χ is not (a weak) zero:

$$\{\varphi, \chi\} = \frac{p^2}{m^2} = 1 . \tag{1.15}$$

This allows one to define a modified (Dirac) bracket $\{ , \}_*$ on Γ^* which respects the constraints

$$\{f, g\}_* = \{f, g\} + \{f, \varphi\}\{\chi, g\} - \{f, \chi\}\{\varphi, g\} . \tag{1.16}$$

(We have $\{f, \varphi\}_* = 0 = \{f, \chi\}_*$ for any function f on Γ .)
The modified bracket for the canonical coordinates and momenta are

$$\{p_\mu, p_\nu\}_* = 0, \qquad \{x_\mu, x_\nu\}_* = \frac{1}{m^2} L_{\mu\nu} = \frac{1}{m^2}(x_\mu p_\nu - x_\nu p_\mu) \tag{1.17a}$$

$$\{x^\mu, p_\nu\}_* = \frac{1}{m^2} p^\mu p_\nu - \delta^\mu_\nu . \tag{1.17b}$$

(The brackets of the spin variables remain unchanged).

In order to rederive the equations of motion (1.14') in the $*$ - bracket formalism, we have to take into account that Eq. (1.13') gives an explicit τ - dependence to χ which cancels the implicit τ - dependence generated by the Poisson bracket of χ with the "Hamiltonian " φ :

$$\frac{d\chi}{d\tau} = \frac{\partial \chi}{\partial \tau} + \{\chi, \varphi\} = 1 - \frac{p^2}{m^2} = 0 . \tag{1.18}$$

Hence, for an arbitrary $f = f(x, p; \lambda)$, the τ-derivative is given by the (unmodified) Poisson bracket:

$$\frac{df}{d\tau} = \{f, \varphi\} + \{f, \chi\}\frac{d\chi}{d\tau} = \{f, \varphi\}. \tag{1.19}$$

For $m>0$, $\rho>0$ (positive mass and spin) the stability group of a point in Γ^* is 2-dimensional abelian: it consists of translations T_p along p and rotations in the 2-plane orthogonal to p and w. For $\rho=0$ ($m>0$). the stability group of a point on the orbit is 4-dimensional: It is $T_p \otimes SO(3)_p$, where $SO(3)_p$ is the (Wigner /27/) "little group" of p. For zero-mass particles the spin 4-vector w is proportional to the momentum

$$p^2 = 0 \Rightarrow w = \lambda p \tag{1.20}$$

($\lambda^2 = \rho^2$, λ need not be positive) Γ^* is the 6-dimensional space of points (x, p) satisfying (1.7), (1.13) (the "proper time" τ and the helicity λ being held fixed). The stability group of a point on Γ^* is in this case $T_p \otimes E(2)_p$ ($E(2)_p$ being the 2-dimensional Euclidean subgroup of the (proper) Lorentz group $SO^{\uparrow}(3,1)$, which leaves the vector p invariant).

2. A fully covariant formulation of the relativistic 2-body problem

A. Relativistic n-particle dynamics as a problem with constraints

The moral from our discussion of the phase space dynamics of a free relativistic particle can be stated as follows. The entire (invariant) information about the particle trajectory and equations of motion is contained in the constraint equation (1.7). There is no need to introduce a special Hamiltonian other than the function φ (1.12) in the left hand side of the constraints. The preferred Lorentz invariant gauge (1.13) is characterized by the property of the "proper time", defined by the right-hand side of (1.13), to be canonically conjugate to the "Hamiltonian" φ. We shall assume that for a system of n interacting particles the dynamics is given by n Poincaré invariant constraints on the points in $\Gamma = (\mathbb{R}^{8n} \times$ spin variables). These constraints should only reproduce the on-mass-shell conditions asymptotically, when the distance of a given particle to all others goes to infinity. Without such a relaxation of Eq. (1.7) it would leave no room for a potential energy in the non-relativistic limit of the theory.

We postulate the following set of constraints which should define the dynamics of n relativistic interacting particles

$$2\varphi_i \equiv m_i^2 - p_i^2 + \phi_i(p_1, w_1, \ldots, p_n, w_n;\, x_{jk}) \quad (x_{jk} = x_j - x_k) \tag{2.1}$$

$i, j, k = 1, \ldots, n.$

The functions ϕ_i are assumed to satisfy the conditions listed below.

(i) Lorentz invariant:

$$\phi_i(\Lambda p_j,\, \Lambda w_j;\, \Lambda x_{jk}) = \phi_i(p_j,\, w_j;\, x_{jk}); \tag{2.2}$$

here Λ may or may not involve reflections depending on the physical problem at hand. In case if it does one has to keep in mind that the w's are axial vectors. With our choice of variables in (2.1) translation invariance is automatic; therefore Eq. (2.2) actually implies the Poincaré invariance of the constraints.

(ii) Asymptotic on shell condition:

$$\phi_i(p_j,\, w_j;\, x_{jk}) \to 0 \quad \text{for} \quad \min_{\substack{k \neq i \\ k=1,\ldots,n}} (x_{ik}^2) \to \infty; \tag{2.3}$$

this condition reflects the physical requirement that if the i-th particle is far away from all the others, then it moves as a free particle of mass m_i ($\geqslant 0$).

(iii) Relativistic causality. The idea that the velocity of a particle cannot exceed the velocity of light is supposed to be valid in some form even in the interaction region, where the particle loses some of its identity (it has no, for instance, a fixed mass). We shall consider two inequivalent formulations of this property.

(a) Strict causality: the particle momentum p_i on the surface (2.1) should never become space-like. That implies the inequality

$$m_i^2 + \phi_i(p_j, w_j; x_{jk}) \geq 0 \qquad i = 1, \dots, n. \tag{2.4}$$

If we regard ϕ_i as a (generalized) potential then Eq. (2.4) tells us that very strong attractive potentials (in particular, singular negative potentials) are excluded. Such a requirement actually indicates the limitations of the classical theory with a fixed number of particles; in reality, if two high energy particles come close together they will create other particles (and particle-antiparticle pairs). On the other hand, it is technically too stringent, since it even excludes the attractive Coulomb potential. In order to allow for such (weakly singular) potentials we shall also consider an alternative, less restrictive, form of the causality assumption.

(b) Weak (or mean-value) causality. Starting with some equations of motion (to be specified below) we can regard the dynamical variables p_i, w_i, x_{jk} as functions of a time parameter τ. Then we shall require that a time average counterpart of (2.4) takes place:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left[m_i^2 + \phi_i(p_j, w_j; x_{jk}) \right] d\tau \geq 0. \tag{2.5}$$

In the non-relativistic limit that assumption should just exclude singular attractive potentials which would have led to falling on a centre.

Conditions (i)-(iii) should be supplemented by some regularity assumptions on the functions ϕ_i , which would guarantee, in particular, the existence of Poisson brackets.

B. A gauge invariant formulation of the two-body problem

Before trying to further develop and apply the above general scheme we shall proceed to the case of two interacting particles and shall make the following additional assumptions which will simplify our task.

We assume that the potentials ϕ_1 and ϕ_2 are equal x)

$$\phi_1 = \phi_2 \equiv \phi(\mathrm{P}, p; w_j; x) , \tag{2.6}$$

where

$$\mathrm{P} = p_1 + p_2 , \qquad x = x_1 - x_2 \tag{2.7a}$$

$$p = \mu_2 p_1 - \mu_1 p_2 = \frac{1}{2}(p_1 - p_2) - \frac{m_1^2 - m_2^2}{2s}\mathrm{P} \quad \left(\mu_1 + \mu_2 = 1,\ \mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{s},\ s = \mathrm{P}^2\right) \tag{2.7b}$$

and that ϕ satisfies the transversality relations

$$\mathrm{P}\nabla_x \Phi(\mathrm{P}, p; w_j, x) = 0, \tag{2.8}$$

$$\mathrm{P}\nabla_p \Phi(\mathrm{P}, p; w_j, x) = 0 \tag{2.9}$$

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x) An equivalent hypothesis is made in the quantum context in refs. /16,24/ .

on the surface (2.1) (often such equalities are termed "weak").

Conditions (2.6) and (2.8) imply that the constraints φ_1 and φ_2 (see Eq. (2.1)) are in involution, that is, their Poisson bracket $\{\varphi_1, \varphi_2\}$ is weakly zero. The transversality relations (2.8) (2.9) would be automatically satisfied if ϕ depends on x and p through the (pseudo) scalars

$$x^2 - \frac{1}{5}(Px)^2, \quad p^2, \quad xp, \quad x\wedge p \wedge P \wedge w_i, \ (i = 1, 2) \qquad (2.10)$$
$$x \wedge P \wedge w_1 \wedge w_2, \qquad p \wedge P \wedge w_1 \wedge w_2.$$

Following again Faddeev's prescription /10/ we can define the physical phase space Γ^* in a gauge invariant manner (cf. Sec.1b). For the reader's convenience we give here a pedestrian (non-rigorous) summary of the Appendix to ref. /10/. One starts with the one parameter family of "Hamiltonians"

$$H_\alpha = \alpha\varphi_1 + (1-\alpha)\varphi_2 \qquad 0 \le \alpha \le 1,$$

where $\varphi_{1,2}$ are the functions defined by the left-hand side of the constraints (2.1). Such a *Hamiltonian* gives rise to a trajectory through each point q of the phase space Γ, defined as the solution of the system of differential equations

$$\frac{dq}{dt_\alpha} = \{q, H_\alpha\}$$

(where q is in general a 20-component quantity: $q \in \Gamma \approx R^{16} \times S^2 \times S^2$). We then define Γ^* by identifying the points on all such trajectories (when t and α vary) on the manifold

$$M = \{ q \in \Gamma \ ; \ \varphi_1 = 0 = \varphi_2 \} . \tag{2.11}$$

(The assumption that $\{ \varphi_1 , \varphi_2 \} = 0$ is a prerequisite for the consistency of the above construction.) In such a way M can be regarded as a fibre bundle with base Γ^* and a two dimensional fibre in each point of Γ^* (generated by the "Hamiltonians" H_α).

C. Description of the relative motion in a Lorentz invariant gauge

Our task in this subsection is to separate the centre-of--mass motion of the two particles and to give an explicit Lorentz invariant description of the non-trivial relative motion.

To do that, we start by rewriting the constraints (2.1) in the form

$$\varphi (\equiv \varphi_2 - \varphi_1) = \tfrac{1}{2} (m_2^2 - m_1^2 + p_1^2 - p_2^2) = P p = 0 \tag{2.12}$$

$$H \left(\equiv \mu_2 \varphi_1 + \mu_1 \varphi_2 \right) = \tfrac{1}{2} \left[\Phi (P, p, w_1, w_2, x) - b^2 (s) - p^2 \right] \tag{2.13}$$

where P, p and x are given by (2.7) and

$$b^2 (s) = \frac{1}{4s} \left[s^2 - 2 (m_1^2 + m_2^2) s + (m_1^2 - m_2^2)^2 \right] \qquad (s = P^2 > 0) \tag{2.14}$$

is the on-shell cenre-of-mass 3-momentum squared.

We shall impose one Lorentz invariant gauge condition which fixes the relative time variable to be zero in the centre of-mass frame:

$$\chi = x P = 0. \tag{2.15}$$

It is conjugate to φ in the sense that

$$\{\varphi, \chi\} = s \,(>0) \qquad \{H, \chi\} \left(= \tfrac{1}{2} P \nabla_p \Phi - p P\right) = 0 \tag{2.16}$$

(In deriving the last equality we used (2.9) and (2.12) .)

Further, we identify the "Hamiltonian" with the function H (2.13) (which defines the second constraints). The total momentum P has zero Poisson brackets with H (as well as with φ and χ), and hence does not change in time

$$\frac{dP}{d\tau} = 0 \;\left(= \frac{ds}{d\tau}\right). \tag{2.17}$$

On the other hand, the centre-of-mass variable does not enter either of the functions (2.12)-(2.15). Therefore, we can study (for fixed P_μ) the relative motion of the two particles in the phase space

$$\Gamma_{rel} = \{ (p,x) \in \mathbb{R}^8 ; \; \varphi = 0 = \chi \} \times S^2 \times S^2. \tag{2.18}$$

The Dirac brackets of the relative variables are given by

$$\{ p_\mu, p_\nu \}_* = 0 = \{ x^\mu, x^\nu \}_* \tag{2.19a}$$

(since $\{p_\mu, \varphi\} = 0 = \{x^\mu, \chi\}$) and

$$\{p_\mu, x^\nu\}_* = \{p_\mu, x^\nu\} - \frac{1}{s} \{p_\mu, Px\} \{Pp, x^\nu\} = \tag{2.19b}$$

$$= \delta_\mu^\nu - \frac{1}{s} P_\mu P^\nu \equiv \Pi_\mu^\nu .$$

In the centre of mass frame we recover the conventional canonical commutation relations for the 3-vectors $\underline{p}$ and $\underline{x}$.

The equations of motion read

$$\dot{p}\left(\equiv \frac{dp}{d\tau}\right) = \{p, H\}_* = \{p, H\} = \tfrac{1}{2}\nabla_x \Phi . \tag{2.20}$$

$$\dot{x} = \{x, H\}_* = \{x, H\} = p - \tfrac{1}{2}\nabla_p \Phi . \tag{2.21}$$

We should remember that τ has dimension of $(\text{mass})^{-2}$ in these equations. In the non-relativistic limit Eqs. (2.20) (2.21) (in the centre of mass frame) go into the familiar Newton equations, if we set

$$\tau = \frac{m_1 m_2}{m_1 + m_2} t , \qquad \Phi = 2\frac{m_1 m_2}{m_1 + m_2} V \tag{2.22}$$

and demand that V is independent of p.

3. Quantization. Relation to the quasipotential approach

Given the phase space formulation of classical mechanics in terms of Poisson (or Dirac) brackets the problem of quantization becomes trivial (at least in a practical sense): we have just to replace the classical brackets by the commutator (devided by i). We shall discuss here the Schrödinger representation of the quantized relative motion of two relativistic particles.

We consider the Hilbert space $\mathcal{H} = \mathcal{H}_{s_1 s_2}$ of vector valued wave functions $\psi(x)$ defined in a neighbourhood of the hyperplane [x] (2.15), taking values in the $(2s_1+1)(2s_2+1)$ dimensional complex space $\mathcal{V}_{s_1}\otimes\mathcal{V}_{s_2}$ (where s_1 and s_2 are the spin values of each of the two particles), and having finite norm:

$$\|\psi\|^2 = \langle\psi,\psi\rangle = \int (\psi(x),\psi(x))\,\delta(Px)\,d^4x < \infty, \tag{3.1}$$

where

$$(\psi,\psi) = \sum_{\zeta_1=-s_1}^{s_1}\sum_{\zeta_2=-s_2}^{s_2} \overline{\psi}_{\zeta_1\zeta_2}\psi_{\zeta_1\zeta_2}$$

is the scalar product in $\mathcal{V}_{s_1}\otimes\mathcal{V}_{s_2}$.

Then x^μ is defined (as usual) as a multiplication operator with domain $D_{[x^\mu]} = \{\psi\in\mathcal{H} : \|x^\mu\psi\|<\infty\}$, while p_μ is given by

$$p_\mu = i\,\Pi_\mu^{\ \nu}\frac{\partial}{\partial x^\nu} = i\left[\frac{\partial}{\partial x^\mu} - \frac{1}{s}\,P_\mu(P\nabla_x)\right] \tag{3.2}$$

(where Π is the projection operator (2.19b)). In the centre of mass frame, setting $\underline{x} = (x^k)$, $\underline{p} = (p^k)$, $k=1,2,3$ we come to the conventional expression

$$\underline{p} = -i\frac{\partial}{\partial\underline{x}} \qquad \left(\text{for} \quad P = (\sqrt{s},\underline{0}),\ p^0 = 0\right). \tag{3.2'}$$

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[x] More precisely $\mathcal{H}$ should be defined as the completion of the set of equivalence classes of continuous functions $\psi(x)$ (continuity is necessary if we wish to give unambiguous meaning of the integral in (3.1)). Two functions should be ascribed to the same class if they coincide on the hyperplane (2.15).

In order to give a quantum theoretical meaning to the potential Φ, we assume that it is a polynomial in $\underline{p}$ and present each term of this polynomial in a symmetric form (so that real Φ go into Hermitian operators). The spin projections are given by the infinitesimal operators of the representations (s_1) and (s_2) of SU(2).

We notice that the constraint (2.12) is automatically satisfied by the expression (3.2) for the relative momentum operator. The gauge condition (2.15) is taken care of by the δ -function in the definition of the scalar product (3.1). The constraint (2.13) on the other hand should be imposed as a subsidiary condition to the wave function. In the centre of mass frame it assumes the form of a stationary Schrödinger equation:

$$\left[\Delta + b^2(s) - \Phi(s; \underline{x}, -i\underline{\nabla}) \right] \psi(s, x) = 0 \qquad (3.3)$$

where Φ is, in general, a matrix in the spin indices.

This is the type of equation encountered in the local version of the quasipotential approach, introduced (and successfully applied to problems of quantum electrodynamics) in refs. /24,25,20,21/ (in that case the potential is extracted from the Feynman perturbation expansion of the elastic scattering amplitude). An investigation of the classical counterpart of this equation for specific choices of Φ is under way.

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