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G.I.Kolerov

# QUANTIZATION AND GLOBAL PROPERTIES OF MANYFOLDS

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# QUANTIZATION AND GLOBAL PROPERTIES OF MANYFOLDS

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Колеров Г.И. **E2 - 10119**

Квантование и глобальные свойства многообразий

Сформулировано условие квантования для систем с искривленным фазовым пространством $\mathfrak{M}$, рассмотрена связь этого условия с различными расслоениями над $\mathfrak{M}$ и интегралом Фейнмана по путям. Предложена топологическая модель "рождения" скалярной частицы как "приклеивание" трехмерной клетки.

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Kolerov G.I. **E2 - 10119**

Quantization and Global Properties of Manyfolds

The quantization conditions are formulated for the systems with curved phase space $\mathfrak{M}$ and relations of these conditions to different bundle spaces over $\mathfrak{M}$ and to the Feynman integrals over paths are analysed. A topological model of the scalar particle production is proposed as an attaching of the 3-dimensional cell.

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By quantization of a classical system is usually understood the construction of such a quantum system which describes the phenomena of microworld in more detail. The quantization means the introduction of a fundamental quantity h such that for $h \to 0$ the limit of the quantum system is just the corresponding classical system. Even intuitively, it is clear that this limiting transition is not unique as the quantum system describing microworld in more detail can be of different structure though having the same classical limit. Many different quantization schemes exist, however, it seems that all these can be splitted into two classes.

We call the first of them the local method. It employs the canonical variables $(q,p)$ and introduces differential operators describing the quantum-mechanical system. A mathematically rigorous formulation of the condition for quantization of this kind has been given by G.Weil in the form [1]

$$\exp\{i\alpha p\}\cdot\exp\{i\beta q\} = \exp\{ih\alpha\beta\}\cdot\exp\{i\beta q\}\cdot\exp\{i\alpha p\} \qquad (1)$$

different from the Heisenberg commutation relations

$$[p,q] = \frac{h}{i}, \qquad (2)$$

which are mathematically noncorrect due to unboundedness of operators $p$ and $q$. The Weil scheme of quantization, however, is applicable only to the classical systems with flat phase space since it is based on the use of canonical variables $(q,p)$.

We call the second class the global method because it is based on the global properties of manifolds. Widely known original Bohr quantization condition

$$\oint p\,dq = (2n + 1)\pi h \tag{3}$$

belongs to the second class, where the integral is taken over some plane in the phase space. However, the most beautiful and complete realization of the global method is the "Feynman method of summation over trajectories" /2/. The idea of the method is that the probability amplitude $\psi(x,t)$ is expressed via the probability amplitude $\psi(x',t')$ by using the propagation function which can be obtained calculating the classical function of action for all possible paths between points $(x,t)$ and $(x',t')$, more exactly

$$\psi(x,t) = \int \langle x,t|x't'\rangle \psi(x',t')dx',$$

where

$$\langle x,t|x',t'\rangle = \int \exp\{\frac{i}{h}\int_{x(t)}^{x'(t')} L[x(\tau),\dot{x}(\tau)]d\tau\}\mathfrak{D}[x(\tau)]\,. \tag{4}$$

An alternative way to calculate (4) is to define the skeleton history by indicating the sequence of intermediate moments of time $t<t_1<t_2<\ldots t'$, and giving, at these moments, the configurations $x, x_1, x_2, \ldots, x'$. The actual calculations are to be performed using finite differences of the type

$$\frac{x_{k+1} - x_k}{t_{k+1} - t_k} \tag{5}$$

in the function of action, instead of derivatives. There is, however, some arbitrariness in expressing derivative through these differences. Such an arbitrariness, connected with the ordering of cofactors in Lagrangian, results in the fact that different choices of the skeleton histories give different expressions for the propagation function. In other words, such a quantization method is not unique. What is the nature of this nonuniqueness? It is connected with an incorrect definition of quantization or from the very beginning underlied in its mathematical structure?



To answer this question, let us consider in more detail the global properties of quantum conditions using the methods of differential geometry and topology. The main idea of these methods is as follows: the quantization is in connection with introducing the linear connection in a bundle space, the Hilbert space of states being constructed of sections of the bundle space. These methods have been proposed by A.A.Kirillov /3/ and B.Kostant /4/ who used the idea of quantization in the theory of representations of Lie groups. Quantization of the systems with a curved phase space has been considered by P.Dirac /5/. Some models of quantization of the systems with phase space of a rather large class are analysed by F.A.Berezin /6/.

## *MATHEMATICAL APPARATUS FOR CLASSICAL SYSTEMS*

In the description of classical systems the concept of phase space is of primary importance. Points of this space are possible states of a system and functions on it specify the different physical quantities related to this system. Mathematically, the phase space is a symplectic manifold, that is a smooth manifold of even dimension on which is given a nondegenerated closed two-form.

Because in following we shall often use the concept of the bundle space /7/, let us define it:

A topological space $\mathfrak{L}$ is called a bundle space, if there exist

1. a topological space $\mathfrak{M}$ called the basic space,
2. a continuous map $\pi: \mathfrak{L} \to \mathfrak{M}$ called its projection,
3. a certain space $Y$, called the fiber, and the set $Y_x$, defined by the formula

$$Y_x = \pi^{-1}(x), \quad x \in \mathfrak{M}, \tag{1.1}$$

4. for each point $x \in \mathfrak{M}$ there should be such a neighbourhood U and such a map $\phi$ called the "coordinate system" that

$$\phi: U \times Y = \pi^{-1}(U);\ \pi \cdot \phi(x', y) = x',\ x' \in U,\ y \in Y. \tag{1.2}$$

Also a notion of the section is such a continuous map $f: \mathfrak{M} \to \mathfrak{L}$, that

$$\pi \cdot f(x) = x, \quad x \in \mathfrak{M}. \tag{1.3}$$

An example of a trivial bundle is the direct product $\mathfrak{M} \times Y$ with the natural projection $\pi$ onto the first factor. Every bundle can be "glued" out of the trivial ones in the following way:

Cover the basic space $\mathfrak{M}$ by a set of neighbourhoods $\{U_i\}$, for every $U_i$ define the coordinate system $\phi_{ix}$ obeying (1.2). For any point $x$ on the overlapping of two neighbourhoods, i.e., $x \in U_i \cap U_j$, the map

$$\phi_{ix}^{-1} \cdot \phi_{jx} : Y \to Y \tag{1.4}$$

transforms the fiber into itself, i.e., it defines an element of a group G acting on the fiber

$$\phi_{ix}^{-1} \cdot \phi_{jx} : \mathcal{G}_{ij}(x): U_i \cap U_j \to G. \tag{1.5}$$

Functions $\mathcal{G}_{ij}(x)$ are called transition functions as they "link" fibers of a bundle space. These functions possess the following properties

$$\begin{aligned}
&\mathcal{G}_{kj}(x)\mathcal{G}_{ji}(x) = \mathcal{G}_{ki}(x), \quad \text{for } x \in U_i \cap U_j \cap U_k,\\
&\mathcal{G}_{ii}(x) = 1 \ \text{(identity element of } G\text{)},\ x \in U_i,\\
&\mathcal{G}_{ij}(x)\mathcal{G}_{ji}(x) = 1, \quad x \in U_i \cap U_j,\\
&\mathcal{G}_{ij}(x)\mathcal{G}_{jk}(x)\mathcal{G}_{ki}(x) = 1, \quad x \in U_i \cap U_j \cap U_k.
\end{aligned} \tag{1.6}$$



For the notion of "bundle" may be given another definition. Call points $(x_i, y_i) \in U_i \times Y$ and $(x_j, y_j) \in U_j \times Y$ equivalent, if

$$x_i = x_j, \quad y_i = \mathcal{G}_{ij} y_j, \tag{1.7}$$

then the bundle space $\mathfrak{L}$ is the coset space on this equivalence relation. The above defined section f over U is given by the set of functions

$$f_i : U \cap U_i \to Y, \qquad \text{where } U \subset \mathfrak{M} \tag{1.8}$$

called components of f and obeying the condition

$$f_i(x) = \mathcal{G}_{ij}(x) f_j(x), \quad x \in U \cap U_i \cap U_j. \tag{1.9}$$

Let $\mathcal{S}$ be an arbitrary smooth manifold of dimension n which we call configurational space. If $V_q(\mathcal{S})$ denotes a tangent to $\mathcal{S}$ space at point $q \in \mathcal{S}$ and $\{e_i = \frac{d}{dq^i}\}$ is its basis in a coordinate neighbourhood $\tilde{U}_i$ with coordinates $\{q^i\}$, then the basis in the dual to $V_q(\mathcal{S})$ space $V^*_q(\mathcal{S})$ (i.e., the space of linear functionals on $V_q(\mathcal{S})$) is defined by the following equations

$$dq^j \left\{\frac{d}{dq^i}\right\} = \delta^j_i \tag{1.10}$$

and any 1-form on $\tilde{U}_j$ can be represented in a unique way/8/ as

$$\sigma_{U_j} = \sum_k A_k dq^k, \tag{1.11}$$

where $A_i$ are called the vector coordinates on $V^*_q(\mathcal{S})$. The manifold $V^*(\mathcal{S}) = \bigcup_q V^*_q(\mathcal{S})$ in every neighbourhood $U_i$ can be represented as a set of pairs $\{q^i, p_i\}$, where $p_i \in V^*_q(\mathcal{S})$. The manifold $V^*(\mathcal{S})$ is the cotangent bundle space, $\{q^i, p_i\}$ being coordinates of that space in some neighbourhood $U_i \in V^*(\mathcal{S})$ which is an inverse image of $\tilde{U}_i$ relative to the natural projection $\pi: V^*(\mathcal{S}) \to \mathcal{S}$.

The conjugated map

$$d\pi^*_{(q,p)}: \quad V^*_q(\mathcal{S}) \to V^*_{(q,p)} \qquad (1.12)$$

allows a unique definition of a covariant vector field on $V^*(\mathcal{S})$

$$\sigma_0 = \sum_k p_k\, dq^k, \qquad (1.13)$$

called fundamental vector field of the manifold /9/. If on the space $V^*(\mathcal{S})$ one defines a nondegenerated and closed 2-form, which in the local coordinates $\{x^i\} \in U_i \subset V^*(\mathcal{S})$ has the form

$$\omega = \sum_{ij} \omega_{ij}\; dx^i \wedge dx^j, \qquad (1.14)$$

then it transforms into a symplectic one. That this form is nondegenerated means that $\|\omega_{ij}\| \neq 0$ and the condition that it is closed, $d\omega = 0$, where $d$ - denotes the exterior derivative, is written in the form

$$\partial_\alpha \omega_{\beta\gamma} + \partial_\beta \omega_{\gamma\alpha} + \partial_\gamma \omega_{\alpha\beta} = 0. \qquad (1.15)$$

In the geometry the following theorem takes place /10/: Let $\omega$ be a closed 2-form on a $2n$-dimensional manifold $\mathfrak{M}$ throughout having rank $2n$. Then, near each point in $\mathfrak{M}$ one may introduce such coordinates $\{q^i, p_i\}$ that

$$\omega = \sum_i dp_i \wedge dq^i, \qquad (1.16)$$

i.e., $\omega$ will be equal to the exterior derivative of (1.13). However, such a global separation of variables into $p$ and $q$ may not exist. In a geometrical sense this separation means to define the Lagrangian manifold, i.e., to define at each point $x \in U_i$ the manifold, the dimension of which is equal to that of the configurational space, and on which

$$\sum_\alpha \left\{ \frac{\partial p_\alpha}{\partial \beta_i} \frac{\partial q^\alpha}{\partial \beta_j} - \frac{\partial p_\alpha}{\partial \beta_j} \frac{\partial q^\alpha}{\partial \beta_i} \right\} = 0, \qquad (1.17)$$



i.e., the Lagrange brackets of $p(\beta), q(\beta)$ are zero. The equation for this manifold is as follows

$$p = p(q). \qquad (1.17')$$

Also it may be shown /11/ that if the Lagrangian manifold is uniquely projected onto $q$-space, it is given by a certain generating function $S$, so that

$$p_i = \partial S / \partial q^i. \qquad (1.18)$$

Let the system state at an instant $t_0$ be specified by a point $x$ of phase space $\mathfrak{M}$, then at instant $t > t_0$ the state of the given system will be defined by point $x = U_t(x_0)$, with the following equality

$$U_{t_1 + t_2} = U_{t_1} \cdot U_{t_2} \qquad (1.19)$$

being fulfilled, i.e., in other words, the totality of all $U_t$ composes a transformation semigroup, so-called dynamical semigroup. The manifold of all points $U_t(x_0)$ at fixed $x_0$ and varying $t$ makes up a trajectory in phase space. The infinitesimal generator of the dynamical group is the vector field defining the tangent vector at every point of the trajectory. This vector field $\xi$ is called Hamiltonian one if the induced by it a set of transformations conserves the form

$$L_\xi \omega = 0, \qquad (1.20)$$

where $L_\xi$ is the Lie derivative /12/ along the vector field $\xi$. In virtue of (1.20) the dynamical semigroup with such a field transforms a Lagrangian manifold again into a Lagrangian one. However, it may happen that at some $t'$ into one point $q$ of the configurational space $\mathcal{S}$ several points from the new Lagragian manifold can be projected, these points being called "critical" /13/. By introducing an operator $i_\xi$ lowering the degree of the differential form, the Lie derivative can be written in the form /14/

$$L_\xi = d\cdot i_\xi + i_\xi \cdot d \tag{1.21}$$

and equation (1.20) reads

$$d\omega_\xi = 0, \tag{1.22}$$

where

$$\omega_\xi = i_\xi\,\omega\,. \tag{1.23}$$

Relation (1.22) means that the form $\omega_\xi$ is the closed one. We call $\xi$ strictly Hamiltonian, if $\omega_\xi$ is the exact form, i.e., it takes place

$$\omega_\xi = dC \tag{1.24}$$

for some function $C$ on $\mathfrak{M}$. In general, the form $\omega_\xi$ only locally has the form $dC$ being continued onto the whole manifold it may be many-values. At the same time the set of all closed forms composes the vector space where the exact forms compose a subspace; the dimension of a coset space with respect to this subspace depends on the topology of a manifold only. This feature may be generalized for p-forms. If all p-forms on $\mathfrak{M}$ make up a linear space $\mathcal{G}^1_p(\mathfrak{M})$, and the closed p-forms compose its subspace $\mathcal{Z}_p(\mathfrak{M})$ (i.e., cocycles), and differentials $\mathcal{B}_p(\mathfrak{M})$ of the $(p+1)$ forms compose subspace of the space of closed forms, then the coset space

$$\mathcal{Z}_p / \mathcal{B}_p = H^p(\mathfrak{M}, R) \tag{1.25}$$

is called the p-dimensional cohomology group of de Rham on the manifold $\mathfrak{M}$ /15/. On the compact manifolds the group $H^p(\mathfrak{M},R)$ is finite-dimensional and its dimension is called the Betti p-number for manifold $\mathfrak{M}$. Definition (1.25) can be given another form. One says that two forms $\alpha_1$ and $\alpha_2$ are called cohomological, $\alpha_1 \sim \alpha_2$, if and only if they differ from each other by the total differential (the coboundary) $d\beta$, i.e.,

$$\alpha_1 - \alpha_2 = d\beta. \tag{1.26}$$



Hence, the group $\mathcal{B}_p(\mathfrak{M})$ consists of those forms $\alpha$ which are cohomological to zero, $\alpha \sim 0$, and the element $[\alpha] \in H^p$ is the equivalence class of the closed differentials $\alpha$, called the cohomological class. Then the dimension of $H^p(\mathfrak{M}, R)$ represents the number of linearly independent cohomological classes of closed differential forms.

If the vector $\xi$ in local coordinates $\{q^i, p_i\} \subset U_i$ has the form

$$\xi = \sum_i \{ a^i \frac{d}{dq^i} - b_i \frac{d}{dp_i} \}, \tag{1.27}$$

then from (1.23) and (1.16) it follows

$$\omega_\xi = \sum_k \{ b_k dq^k - a^k dp_k \}. \tag{1.28}$$

From (1.24) and (1.28) it is possible to express the Hamiltonian field components in terms of the generating function $C$:

$$b_k = \frac{\partial C}{\partial q^k}, \quad a^k = -\frac{\partial C}{\partial p_k}. \tag{1.29}$$

If in the phase space for some coordinate neighbourhood $U_i$ with the coordinates $\{q^i, p_i\}$ one defines the trajectory with the infinitesimal generator $\xi_C$, then the equations for it have the form

$$\frac{\partial q^i}{d\tau} = \frac{\partial C}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial C}{\partial q^i}. \tag{1.30}$$

Equations (1.30) give the one-parameter group of homogeneous tangent transformations in which $C$ is an arbitrary analytic function of variables $(q^i, p_i)$, homogeneous of first degree in $p_i$ /16/. If now in the configurational space $\mathcal{S}$ the Riemannian metrics is given

$$ds^2 = \sum_{ij} g_{ij}\, dq^i\, dq^j, \tag{1.31}$$

then as the generating function one may take

$$C = \sqrt{\sum_{ij} g^{ij} p_i p_j}\,. \tag{1.32}$$

In this case eqs. (1.30) are

$$\frac{dq^i}{ds} = \sum_j g^{ij} p_j , \quad \frac{dp_i}{ds} = -\frac{1}{2}\sum_{jk} \frac{\partial g^{jk}}{\partial q^i} p^j p^k \tag{1.33}$$

at $\tau = s$.

In general, every real function $f$ given on a manifold $\mathfrak{M}$ may be treated as a generating function of the strict Hamiltonian field $\xi_f$ defined in the local coordinate system $\{q^i, p_i\}$ by the formula

$$\xi_f = \sum_i \left\{ \frac{\partial f}{\partial q^i}\frac{d}{dp_i} - \frac{\partial f}{\partial p_i}\frac{d}{dq^i} \right\}. \tag{1.34}$$

Any two smooth functions $f$ and $g$ given on the manifold $\mathfrak{M}$ obey the equality

$$\{f, g\} = \omega(\xi_f, \xi_g) = \xi_f(g) = -\xi_g(f). \tag{1.35}$$

The value of expression (1.35) is called the Poisson brackets. They have the following properties

$$\begin{aligned} &\{f, g\} = -\{g, f\}, \\ &\{f\{g, h\}\} + \{g\{h, f\}\} + \{h\{f, g\}\} = 0, \end{aligned} \tag{1.36}$$

and therefore the space $C^{\infty}(\mathfrak{M}, R)$ of smooth real functions on $\mathfrak{M}$ composes a Lie algebra infinite-dimensional relative to these brackets

*QUANTIZATION*

The procedure of quantization is as follows: among a great number of physical quantities describing the behaviour of a classical system there is separated such a subset $\{f_i\}$ which produces a Lie algebra with respect to the Poisson bracket (1.35). Then to each $f_i$ one makes correspond its quantum analog



$$\begin{aligned} \hat{A}(f_i) &= \hat{f}_i , \\ \hat{A}(I) &= \hat{e} , \end{aligned} \qquad (\hat{e} \text{ - identity operator}) \tag{2.1}$$

so that the relation

$$\exp i\hat{A}(f) \exp i\hat{A}(g) = \exp ih\,\omega(\xi_f, \xi_g) \exp i\hat{A}(f+g) \tag{2.2}$$

is fulfilled, or in the infinitesimal form:

$$[\hat{A}(f), \hat{A}(g)] = ih\hat{A}(\{f, g\}) = ih\,\omega(\xi_f, \xi_g). \tag{2.3}$$

Let $\mathfrak{M}$ be the symplectic $2n$-dimensional manifold with 1-form $\sigma = \sum_i \sigma_i dx^i$, then for any two vectors $X$ and $Y$ on $\mathfrak{M}$ the relation [8]

$$2d\sigma(X, Y) = X\sigma(Y) - Y\sigma(X) - \sigma([X, Y]) \tag{2.4}$$

is fulfilled.

Now let us construct the bundle space $\mathfrak{L}$ over $\mathfrak{M}$ with the fiber $C$ i.e., the complex plane)

$$\begin{array}{ccc} C & \rightarrow & \mathfrak{L} \\ & & \downarrow \pi \\ & & \mathfrak{M} \end{array}$$

by introducing the linear connection

$$\nabla_x f = \frac{\sigma(x)}{ih} f, \tag{2.5}$$

where $X$ is a vector on $\mathfrak{M}$ and $f$ is a section. In this case for every function $\phi \in C$ one has [4]

$$\begin{aligned} &\nabla_{\phi X} = \phi \cdot \nabla_X , \\ &\nabla_X(\phi f) = (X\phi) f + \phi \;{}_X f . \end{aligned} \tag{2.6}$$

The Hilbert space of states of a physical system is constructed from sections f of this bundle space.

Using (2.5) and (2.6), relation (2.4) can be rewritten in the following form

$$2d\sigma(X,Y) = ih\{[\nabla_X \nabla_Y] - \nabla_{[X,Y]}\} = ihR(X,Y), \qquad (2.7)$$

where $R(X,Y)$ is the curvature tensor. Substituting (2.7) into (2.3), one obtains

$$[\hat{A}(f), \hat{A}(g)] = -\frac{h^2}{2} R(\xi_f, \xi_g). \qquad (2.8)$$

Consider now the curve $c(r)$ in $\mathfrak{M}$ given by eqs. (1.30). Using (2.5) one may define the covariant derivative

$$\frac{\mathfrak{D} f}{\partial r} = \xi_C(f) + \frac{1}{ih}\sigma(\xi_C) f \qquad (2.9)$$

of section $f \in \Gamma(\mathfrak{L}, \mathfrak{M})$ along this curve. Or, it also reads

$$\frac{\mathfrak{D} f}{\partial r} = \frac{df}{dr} + \frac{1}{ih}\sum_i \sigma_i \frac{dx^i}{dr} f. \qquad (2.10)$$

The equality of the covariant derivative to zero gives the condition for translation of a differentiable path from the basic space $\mathfrak{M}$ to the bundle space $\mathfrak{L}$ and thus defines the connection in the latter:

$$df + \frac{1}{ih}\sum_i \sigma_i dx^i f = 0. \qquad (2.11)$$

Solving this equation, we obtain

$$f = \exp\{\frac{i}{h}\int_{c_i}\sum_k \sigma_k dx^k\}, \qquad (2.12)$$

where the integral is taken along the path $c_i(r)$ in the neighbourhood $U_i \subset \mathfrak{M}$. Provided that as a 1-form the fundamental vector field (1.13) is taken, one has

$$f = \exp\{\frac{i}{h}\int_{c_i}\sum_k p_k dq^k\}. \qquad (2.13)$$

If $dq^k$ is taken from the first equations in (1.33) and substituted into (2.13), then

$$f = \exp\{\frac{i}{h}\int_{c_i} \mathcal{H}\, dr\}, \qquad (2.14)$$

where $\mathcal{H} = \sum_{ij} g^{ij} p_i p_j$.

Taking different paths $c_i \in \Omega_i$ one obtains different f. Thus, expression (2.12) defines the representations of the manifold of paths $\Omega_i$ of neighbourhood $U_i$ in a set of the unit modulus complex numbers $Z$ which composes the group $T_1$ with respect to multiplication of these numbers, i.e.,

$$f: \Omega_i \to T_1. \qquad (2.15)$$

In other words, we have found the main bundle space $\mathfrak{L}$ with the basis $\mathfrak{M}$ and structure group $T_1$ (see (1.5)). Hence, the problem of uniqueness of quantization reduces to the algebraic topology problem on classification of bundle spaces with a given basis and fiber. This classification is performed by connecting a system of topological invariants to each equivalence class of bundle spaces.

Let us connect to each differentiable representation f the main bundle space $(\mathfrak{L}, \pi, \mathfrak{M}, T_1)$ with basis $\mathfrak{M}$ and structure group $T_1$, then $\mathfrak{L}$ is a space of representation without fixed points the orbits of which being fibers. Cover manifold $\mathfrak{M}$ by coordinate neighbourhoods $\{U_i\}$, then to each bundle space there corresponds the system of transition functions

$$\mathcal{G}_{ij}: U_i \cap U_j \to T_1 \qquad (2.16)$$

with properties (1.6). Using these, features it may be shown that the system of maps $\mathcal{G}_{ij}$ composes a cocycle of dimension 1 and there is fulfilled the *theorem A* /17/.

*The equivalence classes of the main bundle spaces with basis* $\mathfrak{M}$ *and group* $T_1$ *make up a set* $\mathcal{E}(\mathfrak{M}, T_1)$

*which is in one-to-one correspondence with the set* $H^2(\mathfrak{M},\mathbb{Z})$, where $H^2(\mathfrak{M},\mathbb{Z})$ is the two-dimensional cohomology group of de Rham on the manifold $\mathfrak{M}$ with integer coefficients (see (1.25)).

Let two cocycles $\{\mathcal{G}_{ij}\}$ and $\{\mathcal{G}'_{ij}\}$ are given, then the following functions

$$\mathcal{G}''_{ij} = \mathcal{G}_{ij} \cdot \mathcal{G}'_{ij} \tag{2.17}$$

compose new cocycles. If the given cocycles correspond to the equivalence classes $\xi$, $\xi' \in \mathcal{E}(T_1, \mathfrak{M})$, then the class of equivalence containing the bundle space defined by cocycle $\{g''_{ij}\}$ will be $\xi \otimes \xi'$ and it is called the tensor product of classes $\xi$ and $\xi'$. The set $\mathcal{E}(\mathfrak{M}, T_1)$ in which such an operation is given is the Abelian group isomorphic to group $H^2(\mathfrak{M},\mathbb{Z})$, i.e.,

$$ch: \mathcal{E}(\mathfrak{M}, T_1) \to H^2(\mathfrak{M}, Z). \tag{2.18}$$

This map makes correspond to each class of the bundle space $\xi$ with basis $\mathfrak{M}$ the class of integer cohomologies $ch(\xi)$ of dimension 2. This class is called the characteristic class of bundle space. From exp.(1.6) and the theorem A it follows that for the bundle space $(\mathfrak{L}, \pi, \mathfrak{M}, T_1) \in \xi$ and covering $\{U_i\}$ the class $ch(\xi)$ is represented by the cocycle $\{\mathcal{G}_{ijk}\}$ defined by the following formula

$$\ln \mathcal{G}_{ijk} = \ln \mathcal{G}_{ik} - \ln \mathcal{G}_{jk} + \ln \mathcal{G}_{ij}, \tag{2.19}$$

where $x \in U_i \cap U_j \cap U_k$, the function $\ln \mathcal{G}_{ij}(x)$ being assumed to be determined for each pair $(U_i, U_j)$. This is not a constraint if one supposes that the intersections of sets $(U_i \cap U_j)$ are small enough so that

$$T_1 \neq \mathcal{G}_{ij}(U_i \cap U_j) \tag{2.19'}$$

Let the covering $\{U_i\}$ be chosen so that the intersections $(U_i \cap U_j)$ are simply connected and $(U_i \cap U_j \cap U_k)$ connected. Since $\{\mathcal{G}_{ij}\}$ is a cocycle, from (1.6) it follows that $\ln \mathcal{G}_{ijk}$ are integer functions independent of $x \in U_i \cap U_j \cap U_k$ and composing a two-dimensional cocycle.



By the de Rham theorem/15/, to this cocycle the cohomological class of external forms of the second order corresponds. It can be shown that form $\omega$ being the form of curvature of infinitesimal connection (2.11) belongs to this class, i.e., the image of the characteristic class of the bundle space $(\mathfrak{L}, \pi, \mathfrak{M}, T_1)$ under the de Rham isomorphism contains the form of curvature of infinitesimal connection. These forms may be introduced into the bundle space by means of 1-forms $\sigma$ defined on $\mathfrak{M}$ and invariant relative the group $T_1$. In a neighbourhood of the bundle space $\pi^{-1}(U_i)$ with coordinates $\{x^i, z\}$ (see (1.7)) these 1-forms are

$$\sigma = \sum_i a_i\, dx^i + \omega_0 (d\phi - \sum_i \sigma_i\, dx^i), \quad (z = e^{2\pi i \phi}), \tag{2.20}$$

where $a_i$, $\omega_0$, $\sigma_i$ do not depend on $\phi$.

Then $\omega$ is given by the formula

$$\omega = \frac{1}{2} \sum_{ij} \left\{ \frac{\partial \sigma_i}{\partial x^j} - \frac{\partial \sigma_j}{\partial x^i} \right\} dx^i \wedge dx^j . \tag{2.21}$$

Let assume that there exists an invariant under the dynamical semigroup Lagrangian manifold $\mathfrak{L}$ lying at the energy $\mathcal{H} = \text{const}$. Then the condition for form $\omega$ to be integer means that

$$\int_{\mathfrak{L}} \omega = n \cdot h \tag{2.22}$$

whence, by the Stokes theorem, one obtains that for every closed contour $\gamma$ on $\mathfrak{L}$ the relation

$$\oint_\gamma \sum_i p_i\, dq^i = nh \tag{2.23}$$

holds which is equivalent to the Bohr quantization condition (3). If the characteristic class of bundle spaces $(\mathfrak{L}, \pi, \mathfrak{M}, T_1)$ for covering $\{U_i\}$ equals zero, then the cocycle $\{\mathcal{G}_{ijk}\}$ is a coboundary. In this case the bundle space is trivial, i.e., equivalent to the one $(\mathfrak{L}', \pi', \mathfrak{M})$ in which $\mathfrak{L}' = \mathfrak{M} \times T_1$ and $\pi'$ is the projection of $\mathfrak{L}'$ on the first factor. All trivial bundle spaces belong to one equivalence class.

Consequently, the bundle space $(\mathfrak{L}, \pi, \mathfrak{M}, T_1)$ with the integrable connection, i.e., with that one the curvature form of which is zero, is, generally, trivial. It is nontrivial only if the group $H^2(\mathfrak{M}, Z)$ contains nonzero periodic elements. The bundle spaces, for which these elements are characteristic classes, are spaces associated with the universal covering of manifold $\mathfrak{M}$ and representations of the homotropic group $\pi_1(\mathfrak{M})$ in group $T_1$ /19/.

Assume that the manifold $\mathfrak{M}$ is covered by system of the neighbourhoods $\{U_i\}$.

Let in a neighbourhood $U_i$ at point $x_0$ at instant $t_0$ a section $f_i \in \Gamma(\mathfrak{L}, \mathfrak{M})$ be given. Since the dynamical system is defined on the Lagrangian manifold (1.17), then, in virtue of (1.17 ), the sections depend only on the projection $\pi : \mathfrak{M} \to \mathfrak{S}$ of points of phase space onto the configurational one, i.e., sections $f_i$ are functions of coordinates only. Hence, according to (1.9), (2.13) and (2.14) the section at point $x' \in \pi(U_i)$ and instant $t'$ is expressed through the section at point $x \in \pi(U_i)$ and instant $t_0$ by the formula

$$f_i(x',t') = [\exp \frac{i}{h} \int_{x_0}^{x} \sum_k p_k dq^k \cdot \exp \frac{i}{h} \int_{t_0}^{t'} \mathcal{H} d\tau] f_i(x_0, t_0) =$$

$$= [\exp \frac{i}{h} \int_{x_0(t_0)}^{x(t)} L_{U_i} d\tau] f_i(x_0, t_0), \qquad (2.24)$$

where integrals are taken along some path $c_i \subset \Omega_i$ on $U_i$ and should obey condition (2.19). Continuing the 1-form $\sigma_{U_i}$ defined on $U_i$ onto the whole space covered by neighbourhoods one finds the transition function for sections from one point $x_0$ to another $x$ of manifold $\pi(\mathfrak{M})$ as the limit /20/

$$G(x,t; x_0, t_0) = \lim_{k \to \infty} \prod_k \exp\{\frac{i}{h} \int_{x_k}^{x_{k+1}} L_{U_k} d\tau\} =$$

$$= \int_{x_0(t_0)}^{x(t)} \exp[\frac{i}{h} \int_{t_0}^{t} L d\tau] \mathfrak{D}[c(\tau)], \qquad (2.25)$$



where the latter is the continual integral taken along all possible paths $c \in \Omega_{x_0 x}$ on $\mathfrak{M}$ joining two given points. Then the difference between two nearly paths going through different neighbourhoods (see *Fig. 1*) is

$$\exp \frac{i}{h} \int_{x_0}^{x} \sigma + \exp \frac{i}{h} \int_{\Delta} d\sigma = e^{-\frac{i}{h} \int_{t_0}^{t} \mathcal{H} d\tau} \, e^{\frac{i}{h} \int_{x_0}^{x} \sigma} \, e^{\frac{i}{h} \int_{t_0}^{t} \mathcal{H} d\tau} . \qquad (2.26)$$

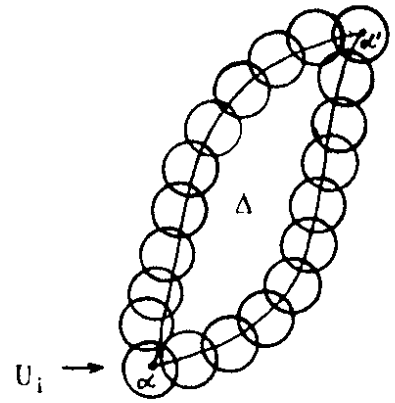


*Fig. 1*

As is clear from the above, for the bundle space $\mathfrak{L}$ to be not trivial, it is necessary that its characteristic class be not zero, i.e., the form of the connection should not be integrable. Consequently, only those paths contribute into the continual integral for which

$$\int_{\Delta} d\sigma \neq 0. \qquad (2.27)$$

If $\sigma = \sum_k p_k dq^k$, then taking account of (2.7), from the integrality condition we get

$$\int_{\Delta} dp \wedge dq \sim \frac{nh}{2} \qquad (2.28)$$

whence

$$\Delta p \cdot \Delta q \sim \frac{h}{2}. \qquad (2.29)$$

Let all the paths joining two given points in phase space be projected on a configurational space $\pi : \mathfrak{M} \to \mathcal{S}$ which is assumed to be smooth, and let $a, a'$ be projections of these points. Then points $a$ and $a'$ are joined by a totality of piecewise smooth paths lying on $\mathcal{S}$. This totality will be denoted by $\Omega(a, a', \mathcal{S})$.

Let us assume that in $\mathcal{S}$ the metric (1.31) is defined. For the path $c \in \Omega$ let us define the action of $c$ from $a$ to $a'$:

$$S = \int_a^{a'} \left| \frac{dc}{d\tau} \right|^2 d\tau. \qquad (2.30)$$

If $\bar{\beta} : U \to \Omega$ is the two-parameter variation of a geodesic $\gamma$ to which there correspond the variation vector fields

$$w_i = \frac{\partial \beta}{\partial u_i}(0,0) \in V\Omega_\gamma, \quad i = 1,2$$

then the second derivative of the function of action $\frac{1}{2} \frac{\partial^2 S}{\partial u_1 \partial u_2}$ equals [21]

$$- \Sigma < w_2, \Delta_\tau \frac{\mathfrak{D} w_1}{d\tau} > - \int_a^{a'} < w_2, \frac{\mathfrak{D}^2 w_1}{d\tau^2} + R(v, w_1) v > d\tau, \qquad (2.31)$$

where

$$v = \frac{d\gamma}{d\tau},$$

and

$$\Delta_\tau \frac{\mathfrak{D} w_1}{d\tau} = \frac{\mathfrak{D} w_1}{d\tau}(\tau^+) - \frac{\mathfrak{D} w_1}{d\tau}(\tau^-)$$

refers to the discontinuity at these points, R is the tensor of curvature.

Assume that a point $a'$ is conjugate to $a$ along a geodesic $\gamma$, the variation vector field $w$ obeys the Jacobi equation



$$\frac{\mathfrak{D}^2 w}{d\tau^2} - R(v, w) v = 0. \qquad (2.32)$$

If along $\gamma$ one chooses the orthonormal parallel vector fields $\{\hat{e}_i\}$ then at $w = \sum_i \phi^i e_i$ eq. (2.32) takes the form

$$\frac{d^2 \phi^i}{d\tau^2} + \sum_j a^i_j \phi^j = 0, \quad i = 1, 2, \ldots, \qquad (2.33)$$

where $a^i_j = < R(v, e_j) v, e_i >$. With notations

$$K_v(w) = R(v, w) v \qquad (2.34)$$

properties of the tensor of curvature give that the operator $K_v$ is self-conjugated, i.e.,

$$< K_v(w), w' > = < w, K_v(w') >. \qquad (2.35)$$

Hence, the orthonormalized basis $\{\hat{e}_i\}$ can be chosen so that

$$K_v \hat{e}_i = \omega_i^2 \hat{e}_i \qquad (2.36)$$

Now eq. (2.33) reads

$$\frac{d^2 \phi^i}{d\tau^2} + \omega_i^2 \phi^i = 0, \quad i = 1, 2, \ldots . \qquad (2.37)$$

Let us show that to a conjugate point $a'$ there corresponds a stationary point $(q, p)$ of phase space of the dynamical system, i.e., a point such that

$$U_t(q, p) = (q, p). \qquad (2.38)$$

Relation (2.38) holds only when

$$p = 0, \quad (dV)_q = 0 \qquad (2.39)$$

(where V is a potential).

The Lagrangian in a neighbourhood of such a point has the form

$$L = \sum_{ij} [a_{ij} \dot{q}^i \dot{q}^j - b_{ij} q^i q^j] \qquad (2.40)$$

where $a_{ij}$, $b_{ij}$ are constants. By the known theorem of the linear algebra, there exists such a linear transformation which diagonalizes both the quadratic forms, and, consequently, the function L in new coordinates reads

$$L = \sum_i [(\dot{\phi}^i)^2 - \omega_i^2 (\phi^i)^2] \qquad (2.41)$$

and the Lagrange equation is of the form

$$\ddot{\phi}^i + \omega_i^2 \phi^i = 0, \quad i = 1,2,\ldots \qquad (2.42)$$

and thus coincides with eq. (2.37).

For the dynamical system with infinite number of degrees of freedom eq. (2.42) takes the form

$$\ddot{\phi}(\vec{k}) + \omega^2(\vec{k})\, \phi(\vec{k}) = 0. \qquad (2.43)$$

The basis in this case is provided by the functions

$$\{ \hat{e}_k = e^{i\vec{k}\vec{x}} \}. \qquad (2.44)$$

Multiplying both sides of eq. (2.43) by these functions and integrating over $\vec{k}$, we get

$$\int \ddot{\phi}(\vec{k}) e^{i\vec{k}\vec{x}}\, d\vec{k} + \int \omega^2(\vec{k}) \phi(\vec{k}) e^{i\vec{k}\vec{x}}\, d\vec{k} = 0. \qquad (2.45)$$

Keeping the idea that $\omega(\vec{k})$ is a frequence and that it should be invariant under rotations in the three-dimensional space, one can assume that $\omega^2(\vec{k}) = \vec{k}^2$. Then eq. (2.42) may be rewritten as follows

$$\frac{1}{c^2}\frac{\partial^2 \phi(x)}{\partial t^2} - \frac{\partial^2 \phi(x)}{\partial x_1^2} - \frac{\partial^2 \phi(x)}{\partial x_2^2} - \frac{\partial^2 \phi(x)}{\partial x_3^2} = 0, \qquad (2.46)$$

i.e., one arrives at the wave equation. This equation is correspondent to the Jacobi equation for the variation vector field, also infinite-dimensional.

Since the dynamical process is realized in space and time, to each point of configurational space $\mathcal{S}$ of the dynamical system there corresponds a point of space-time continuum $\nu: \mathcal{S} \to N$. If now the action (2.30) is treated as a function of the second point $a'$, then it, in its turn, becomes a function of space-time coordinates $(\vec{x}, x_0) \in N$; in other words,



$$S: N \to R. \qquad (2.47)$$

Denote $N^a = S^{-1}(-\infty, a), \{p \in N, S(p) < a\}$.

If different hypersurfaces $S(x) = \text{const.}$ are considered, then equations of the orthogonal trajectories having in the local coordinate system $\{x^i\}$ the form

$$\frac{\partial x^i}{d\tau} = \sum_i g^{ij} \frac{\partial S}{\partial x^j}. \qquad (2.48)$$

define the shift $N^a \to N^b$ without changing the homotopy type of manifold $N^a$, if the set $S^{-1}[a,b]$ does not contain the critical points (these points correspond to the conjugate ones). If the level $S(x) = \delta$ is critical, i.e., contains the critical point p, the orthogonal trajectories in neighbourhoods of noncritical points of this level behave like in other points, since

$$\frac{\partial S}{\partial \tau} = |\,\text{grad } S|^2 > \epsilon > 0. \qquad (2.49)$$

for all points lying outside small enough cylindrical neighbourhood of the critical point (see *Fig. 2*) /21/ which is left fixed by those trajectories. In a neighbourhood of the critical point there holds the Theorem B /22/.

*Let* $S: N \to R$ *be a smooth function and* p *its nondegenerated critical point with index* $\lambda$. *Assume that the set* $S^{-1}[\delta-\epsilon, \delta+\epsilon]$ *where* $\delta = S(p)$ *is compact and does not contain the critical points of* S *other than* p *for some* $\epsilon > 0$. *Then for all sufficiently small* $\epsilon > 0$ *the set* $N^{\delta+\epsilon}$ *has the homotopy type* $N^{\delta-\epsilon}$ *with an attached cell of dimension* $\lambda$.

Now let us make the assumption (*) that *"production of an elementary particle with mass* m *topologically is equivalent to attaching the three-dimensional cell* $e^3$

$$x_1^2 + x_2^2 + x_3^2 \leq \epsilon < (\frac{h}{mc})^2 \qquad (2.50)$$

*in the neighbourhood of the critical point of the function of action.*

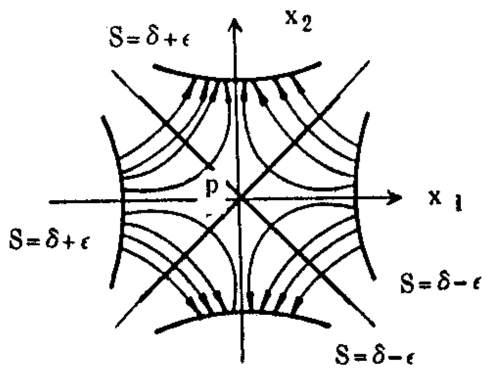


*Fig. 2. The cylindrical neighbourhood of point:* $S^2 = -x_1^2 + x_2^2$.

Then, by Theorem B, the particle production is adequate to transition through the conjugate point defined by the Jacobi equation (2.32). Indeed, physically, the elementary particle is realized at the point of the interferring wave phases. Since the action (2.30) plays the role of phase, then the above mentioned point is just the point at which geodesics intersect with different $\omega(\vec{k})$ but equal $S$ (i.e., the conjugate point).

By a Morse lemma[22], in a neighbourhood $U$ of the critical point $p$ there exists such a local system of coordinates $\{x_i\}$ in $U$ that the identity

$$S^2(x) = S^2(p) + a^2\{x_0^2 - x_1^2 - x_2^2 - \cdots - x_\lambda\} \qquad (2.51)$$

holds, where $\lambda$ is the index of $S$ at point $p$, $S(p)$ the critical value of $S$. By an appropriate choice of a constant one can obtain $S(p)=0$. Keeping in mind that $S$ plays the role of phase and comparing (2.51) with the action for free particle one finds that $a^2 = (mc/h)^2$. Since, in our case also $\lambda = 3$ the final expression for $S^2(x)$ is as follows



$$S^2(x) = (\frac{mc}{h})^2\{x_0^2 - x_1^2 - x_2^2 - x_3^2\}. \qquad (2.52)$$

Let us take $\epsilon > 0$ small enough so that

1. the neighbourhood of the critical level does not contain critical points other than $p$;
2. the image of $U$ under the following imbedding

$$\{x^i\}: U \to R^4 \qquad (2.53)$$

does contain the closed sphere

$$\{(x_0, x_1, x_2, x_3): \Sigma(x_i)^2 \leq 2\epsilon. \qquad (2.54)$$

Define now $e^3$ as a set of points from $U$, where

$$x_1^2 + x_2^2 + x_3^2 \leq \epsilon, \quad x_0 = 0. \qquad (2.55)$$

The configuration obtained is schematically drawn in *Fig. 3.* Note that $e^3 \cap N^{\delta-\epsilon}$ is exactly the boundary $e^3$ and thus the cell $e^3$ is attached to $N^{\delta-\epsilon}$ in the topological sense[22]. From the assumption (∗) it follows that the equation for a produced elementary particle can be obtained from the condition that the eigenvalue $\mu$ of the quadratic functional of the second variation of the function of action (2.31) with the normalization

$$\int_a^{a'} |w|^2 dr = 1 \qquad (2.56)$$

be nonzero. In other words, the Jacobi equation (2.32) a system with infinite number of degrees of freedom which reads

$$\ddot{\phi}(\vec{k}) + \omega^2(\vec{k})\phi(\vec{k}) = -\mu\phi(\vec{k}). \qquad (2.57)$$

Here $\mu$ should be negative as the interval $(a, a')$ includes a conjugate point. On the other hand, since $\mu$ is an eigenvalue of the functional of the second variation of $S$, then

$$\mu = \langle \mathrm{grad}\, S \rangle^2 \tag{2.58}$$

whence, by using (2.52) we find

$$\mu = \left(\frac{mc}{h}\right)^2. \tag{2.59}$$

Multiplying both sides of eq. (2.57) by $e^{i\vec{k}\vec{x}}$ and integrating over $\vec{k}$ we arrive at the Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x_1^2} - \frac{\partial^2\phi}{\partial x_2^2} - \frac{\partial^2\phi}{\partial x_3^2} + \left(\frac{mc}{h}\right)^2 \phi = 0. \tag{2.60}$$

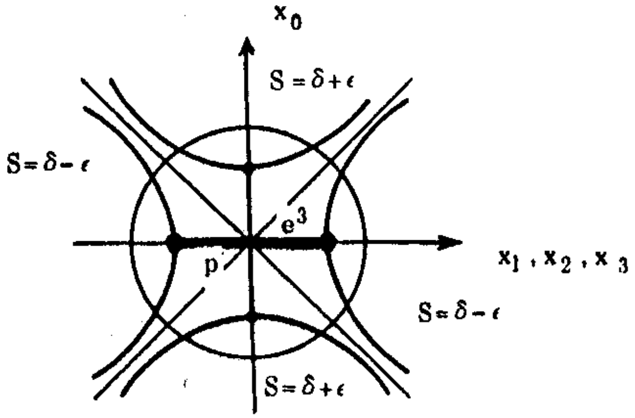


*Fig. 3.*

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