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ON PHASE SPACE REPRESENTATIONS. I

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О представлениях фазового пространства. I

Теория представлений Дирака обычно связана с формализмом амплитуд квантовой теории. В работе дано введение в теорию некоторых других представлений, которые применимы в формализме матрицы плотности и которые естественно назвать представлениями фазового пространства (ПФП). Они используют язык переменных фазового пространства (x и p одновременно) и дают описание, родственное описанию в фазовом пространстве в классике. Даны определения и рассмотрены алгебраические свойства таких ПФП как представления Вигнера, представление когерентных состояний и др. в квантовой механике. В качестве исходного пункта принято соотношение полноты матричного типа для соответствующих систем базисных операторов. Рассмотрен также случай квантовой теории поля.

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On Phase Space Representations. I

Dirac representation theory deals usually with the amplitude formalism of quantum theory. An introduction is given into a theory of some other representations, which are applicable in the density matrix formalism and can naturally be called phase space representations (PSR). They use terms of phase space variables (x and p simultaneously) and give a description, close to the classical phase space description. Definitions and algebraic properties are given in quantum mechanics for such PSR's as Wigner representation, coherent state representation and others. Completeness relations of a matrix type are used as a starting point. The case of quantum field theory is also outlined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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I. INRODUCTION

Quantum mechanics is formulated either in the wave function formalism or in the density operator ("matrix") one. With the latter there is connected a series of representations, which can naturally be called the phase space representations (PSR)/1-24/ (see other references therein). They use terms of phase space variables (x and p simultaneously) and phase space distribution functions, and such a description is close to the classical phase space description. However such a representation is not unique. We expose some of PSR's uniformly starting with the completeness relations:

$$|\mathbb{1}\|\otimes\|\mathbb{1}| = \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,|e^{-i\hbar^{-1}(\hat{x}p-\hat{p}x)}|\otimes\|e^{i\hbar^{-1}(\hat{x}p-\hat{p}x)}\| = \qquad (1.a)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,|:e^{-i\hbar^{-1}(\hat{x}p-\hat{p}x)}:|\otimes\|:e^{i\hbar^{-1}(\hat{x}p-\hat{p}x)}:\| = \qquad (1.b)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,|e^{i\hbar^{-1}\hat{p}x}\ e^{-i\hbar^{-1}\hat{x}p}|\otimes\|e^{i\hbar^{-1}\hat{x}p}\ e^{-i\hbar^{-1}\hat{p}x}\| = \qquad (1.c)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\left(\Lambda^{-1}|xp\rangle\langle xp|\right)\otimes\left(\Lambda^{-1}\|xp\rangle\langle xp\|\right) = \qquad (1.d)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,|xp\rangle\langle xp|\otimes\Lambda^{-2}\|xp\rangle\langle xp\| = \qquad (1.e)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\left(\Xi\,\Lambda^{-1}|xp\rangle\langle xp|\right)\otimes\left(\Xi^{-1}\Lambda^{-1}\|xp\rangle\langle xp\|\right). \qquad (1.f)$$

Here and in the following a many-dimensional case is implied: n is the number of degrees of freedom, $\hat{x}p-\hat{p}x \equiv \hat{x}_i p_i - \hat{p}_i x_i$, $dx \equiv dx_1 \ldots dx_n$ $dp = dp_1 \cdots dp_n$; $|xp\rangle \equiv |x_1 p_1 \cdots x_n p_n\rangle = e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}|0\rangle$ is a many-dimensional coherent state (see also Sec.2). $\hat{x}$ and $\hat{p}$

are coordinate and momentum operators. Single and double vertical lines are used as "matrix" notations. They indicate the disposition of "matrix indices". These completeness relations are similar to those for usual matrices (cf.ref. /25/).*) Λ is the symbol of Gauss transformation (see eqs. (110.b)-(110.e)), acting on x and p , and Λ^{-1} is the inverse operator. The operator Ξ is given by

$$\Xi = \exp\left(\frac{i}{2\hbar}\frac{\partial}{\partial x}\frac{\partial}{\partial p}\right). \tag{2}$$

The N -ordering (: :) and anti-N -ordering ($\vdots\ \vdots$) refer to the operators

$$\hat{a} = \sqrt{\frac{A}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2\hbar A}}\,\hat{p}\ , \quad \hat{a}^{+} = \sqrt{\frac{A}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2\hbar A}}\,\hat{p}\ , \tag{3}$$

where A is a symmetrical correlation matrix (see Sec.2). If the notations

$$a = \sqrt{\frac{A}{2\hbar}}\,x + \frac{i}{\sqrt{2\hbar A}}\,p\ , \quad a^{*} = \sqrt{\frac{A}{2\hbar}}\,x - \frac{i}{\sqrt{2\hbar A}}\,p \tag{4}$$

are introduced, then

$$:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}: = :e^{a\hat{a}^{+}-a^{*}\hat{a}}: = e^{a\hat{a}^{+}}\,e^{-a^{*}\hat{a}} \tag{5}$$

$$\vdots e^{i\hbar^{-1}(\hat{x}p-\hat{p}x)}\vdots = e^{-a^{*}\hat{a}}\,e^{a\hat{a}^{+}}. \tag{6}$$

Using the simplest case of the Baker-Campbell-Hausdorf formula

$$e^{A+B} = e^{A}\,e^{B}\,e^{-\frac{1}{2}[A,B]}\ , \quad \text{if} \quad [[A,B]A] = [[A,B]B] = 0 \tag{7}$$

we get

$$e^{i\hbar^{-1}(\hat{x}p-\hat{p}x)} = :e^{i\hbar^{-1}(\hat{x}p-\hat{p}x)}:\,e^{-\frac{1}{2}|a|^{2}} \quad \left(|a|^{2} = (2\hbar)^{-1}(Ax^{2}+A^{-1}p^{2})\ ^{*)}\right) \tag{8.a}$$

$$= \vdots e^{i\hbar^{-1}(\hat{x}p-\hat{p}x)}\vdots\,e^{\frac{1}{2}|a|^{2}}. \tag{8.b}$$

Hence the transition from (1.a) to (1.b) is clear. Equation (1.c) follows from eq.(1.a) analogously***). The derivation of relations (1) is illustrated in Appendix , where eq.(1d) is proved.

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*) For example, in similar notations the completeness relation for the γ -matrices takes the form $\sum_{A=1...16}|\gamma_A|\otimes\|\gamma_A\| = 4\,|\mathbb{1}|\otimes\|\mathbb{1}|$.

**) It is implied that $Ax^2 \equiv x_i A_{ij} x_j$, $A^{-1}p^2 \equiv p_i (A^{-1})_{ij} p_j$.

***) Using eq.(7), one can easily obtain the x,p-orderings (all x stand to the left (or right) of all p), which are adopted in eq. (1.c).

Each system of operators in eq.(1) forms a basis and can generate a corresponding phase space representation. (We do not pretend to be rigorous).

Many other bases are possible,e.g., $\Lambda^{\nu}|xp\rangle\langle xp|$ with $\nu > -1$.

In Sec. 2 a preliminary information concerning coherent states, ordered products, and bases is given.

In Sec. 3 we start with a coherent state representation (CSR) usual for the Dirac representation theory and then define phase space representations such as the Schwinger-Glauber (PSR-1, or CSR-1), Wigner (PSR-2, or CSR-2), Weyl representations and others. These PSR's are connected with each other by the Fourier transformations or by the Gauss ones. For example, a PSR-1 distribution function is a convolution of PSR-2 one with a given normal distribution (see eq. (110.b)).

In fact, all the consideration here (starting with eq.(1)) deals with various properties of the Weyl group with elements $e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}$.

In a finite-dimensional case similar representations can be constructed too, for some or other group. Group elements are also exponentials of linear forms of all the "canonical variables" (generators), e.g., for the spin $\frac{1}{2}$ $U(\vec{\omega}) = e^{i\vec{\omega}\vec{\sigma}}$ where $\vec{\sigma}$ are the Pauli operators (quaternions). See ref. /25/ for quaternion (group SO(3)) and octonion (exceptional group G_2) representations. Similar representations are possible for other classical groups.

We do not touch various more profound mathematical ideas concerning the representations of interest /18-23/.

2. BASES AND ORDERED PRODUCTS

Coherent states in quantum mechanics and in quantum field theory (we consider here both the cases parallel) can be defined in terms of interaction picture operators as follows

$$|xp\rangle = e^{i\hbar^{-1}(p(t)\hat{x}(t)-x(t)\hat{p}(t))}|0\rangle = e^{i\hbar^{-1}\int dt\, f(t)\hat{x}(t)} \tag{9}$$

$$|\varphi\rangle = e^{-\hbar^{-1}(\varphi,\hat{\varphi})}|0\rangle,\quad (\varphi,\hat{\varphi}) = \int d^3x\ \varphi(x)\overleftrightarrow{\partial_4}\hat{\varphi}(x) = -i\int d^4x\ J(x)\hat{\varphi}(x) \tag{10}$$

$$\left(\varphi\overleftrightarrow{\partial_4}\hat{\varphi} \equiv \partial_4\varphi\cdot\hat{\varphi} - \varphi\cdot\partial_4\hat{\varphi}\right)$$

Here $\hat{x}(t)$, $\hat{p}(t)=m\dot{\hat{x}}(t)$, $\hat{\varphi}(x)$ and $\dot{\hat{\varphi}}(x)$ are free (or interaction picture) operators of coordinate, momentum, scalar field and conjugated to the latter momentum. $x(t)$, $p(t)=m\dot{x}(t)$, $\varphi(x)$ and $\dot{\varphi}(x)$ are the corresponding classical counterparts of these operators.*) $|0\rangle$ are vacuum states, i.e., in fact, the special case of coherent states with $x(t)=p(t)=0$ or $\varphi(x)=\dot{\varphi}(x)=0$. The coordinates $\hat{x}(t)$ and $x(t)$ are implied to satisfy the same Newton equation for the free case or for any oscillator case

$$\ddot{\hat{x}}(t)=0\ , \qquad \ddot{x}(t)=0\ , \tag{11.a}$$

$$\ddot{\hat{x}}_k(t)=-\omega^2_{k\ell}\hat{x}_\ell(t)\ , \qquad \ddot{x}_k(t)=-\omega^2_{k\ell}x_\ell(t)\ . \tag{11.b}$$

Similarly the fields $\hat{\varphi}(x)$ and $\varphi(x)$ satisfy the free Klein-Gordon equation. Due to these equations of motion the scalar products, entering into eqs.(9) and (10) as exponents, and the coherent states themselves are conserved in time. Other consequences of these equations of motion are the evolution laws**)

$$\hat{x}(t)=mD(t-t')\overleftrightarrow{\frac{\partial}{\partial t'}}\hat{x}(t')\ , \quad x(t)=mD(t-t')\overleftrightarrow{\frac{\partial}{\partial t'}}x(t')=-m\int ds\,D(t-s)f(s) \tag{11.c}$$

$$\hat{\varphi}(x)=i\int d^3x'\,\Delta(x-x')\overleftrightarrow{\partial_4'}\hat{\varphi}(x'),\ \ \varphi(x)=i\int d^3x'\,\Delta(x-x')\overleftrightarrow{\partial_4'}\varphi(x')=-\int d^4y\,\Delta(x-y)J(y) \tag{12}$$

The "classical" quantities have the meaning of coherent state expectation values of the operator ones

$$x(t)=\langle xp|\hat{x}(t)|xp\rangle\ , \quad p(t)=m\dot{x}(t)=\langle xp|m\dot{\hat{x}}(t)|xp\rangle \tag{13}$$

$$\varphi(x)=\langle\varphi|\hat{\varphi}(x)|\varphi\rangle\ , \quad \dot{\varphi}(x)=\langle\varphi|\dot{\hat{\varphi}}(x)|\varphi\rangle\ . \tag{14}$$

These coordinates and momenta are independent quantities***), so that the initial values $x(t')$, $p(t')$ or $\varphi(\vec{x},t')$, $\dot{\varphi}(\vec{x},t')$ must be given simultaneously and may be chosen arbitrary at any initial time t' like in classics.

These initial values define the coherent state and therefore characterize the coherent state expectation values of all operators.

*) The more detailed notation of $|\varphi\rangle$ would be $|\varphi\dot{\varphi}\rangle$.

**) For D-functions in quantum mechanics see Appendix A of ref. /24f/

***) Unlike the mean squared coordinate and momentum, which are subjected to the uncertainty relation.

In quantum mechanics these quantities are functions of two variables $x(t')$ and $p(t')$ or, equivalently, functionals of one function $f(t)$.

In scalar field theory they are functionals of two functions $\varphi(\vec{x},t')$ and $\dot{\varphi}(\vec{x},t')$ of the 3-argument $\vec{x}$ or, equivalently, functionals of one scalar function $J(x)$ of 4-argument x_μ. The latter is more convenient due to covariance. For other fields analogously /24a,e/.

Coherent states in the Schrödinger picture have the same form (9) and (10): we should merely fix time t or omit it from these definitions.

Consider the correlation matrices

$$A=\|a_{\mu\nu}\|\ , \qquad \frac{\hbar}{2}a_{\mu\nu}\equiv\overline{\Delta p_\mu\Delta p_\nu}\equiv\langle xp|\Delta p_\mu\Delta p_\nu|xp\rangle\ , \tag{15.a}$$

$$C=\|c_{\mu\nu}\|\ , \qquad \frac{\hbar}{2}c_{\mu\nu}\equiv\overline{\Delta x_\mu\Delta x_\nu}\equiv\langle xp|\Delta x_\mu\Delta x_\nu|xp\rangle, \tag{15.b}$$

where $\Delta x_\mu\equiv\hat{x}_\mu-x_\mu$, $\Delta p_\mu\equiv\hat{p}_\mu-p_\mu$ all quantities being equal-time, and

$$\langle\varphi|\Delta\varphi(x)\Delta\varphi(y)|\varphi\rangle_{x_0=y_0}=\hbar\Delta^{(1)}(\vec{x}-\vec{y},0),\ \langle\varphi|\Delta\dot{\varphi}(x)\Delta\dot{\varphi}(y)|\varphi\rangle_{x_0=y_0}=\hbar\ddot{\Delta}^{(1)}(\vec{x}-\vec{y},0), \tag{16}$$

where $\Delta\varphi(x)\equiv\hat{\varphi}(x)-\varphi(x)$; $\Delta\dot{\varphi}(x)$ means the time derivative, and $\ddot{\Delta}^{(1)}\equiv\frac{\partial^2}{\partial x_0^2}\Delta^{(1)}$.

For the coherent states the following minimal uncertainty relations*)

$$4\,\overline{\Delta x_\mu\Delta x_\nu}\ \overline{\Delta p_\nu\Delta p_\lambda}=\hbar^2\delta_{\mu\lambda} \qquad (\text{i.e.,}\ AC=1\ ) \tag{17}$$

(the summation over ν is implied), and

$$-\int d^3y\ \Delta^{(1)}(\vec{x}-\vec{y},0)\ \ddot{\Delta}^{(1)}(\vec{y}-\vec{x},0)=\delta(\vec{x}-\vec{y}) \tag{18}$$

are valid.

Conversely, it is well-known, the coherent states can be found as those states for which the minimal uncertainty relations are valid. Namely, this requirement leads to the equations

$$[\hat{x}_\mu-x_\mu+ic_{\mu\nu}(\hat{p}_\nu-p_\nu)]|xp\rangle=0\ , \quad \text{or} \quad \hat{a}|xp\rangle=a|xp\rangle\ ,$$

$$\text{or} \quad \hat{x}^{(-)}(t)|xp\rangle=x^{(-)}(t)|xp\rangle; \tag{19}$$

$$\hat{\varphi}^{(-)}(x)|\varphi\rangle=\varphi^{(-)}(x)|\varphi\rangle\ , \tag{20}$$

where $\hat{x}^{(-)}(t)$, $x^{(-)}(t)$, $\hat{\varphi}^{(-)}(x)$ and $\varphi^{(-)}(x)$ are "positive"-frequency (annihilation) parts of the operators $\hat{x}(t)$ and $\hat{\varphi}(t)$

*) Note their covariant character.

and the functions $x(t)$ and $\varphi(x)$ (see Appendix A of ref. /24f/).

The coherent states are in fact characterized by the expectation values x_μ and p_μ (or φ and $\dot\varphi$) and a correlation matrix A (or $\Delta^{(1)}$). However the coherent states with all possible x_μ and p_μ and with fixed A form a complete set. Notation $|xp\rangle$ instead of $|xpA\rangle$ reflects this fact (similarly for $|\varphi\rangle$).

Let us write the coherent state $|qp\rangle$ in the x- and p-representations, in the oscillator (Fock) one and in the coherent state representation

$$\langle x|qp\rangle = N_1 e^{-\hbar^{-2}\overline{\Delta p_\mu \Delta p_\nu}(x_\mu - q_\mu)(x_\nu - q_\nu) + i\hbar^{-1}p_\nu x_\nu - i(2\hbar)^{-1}p_\nu q_\nu}, \qquad (21)$$

$$\langle k|qp\rangle = N_2 e^{-\hbar^{-2}\overline{\Delta x_\mu \Delta x_\nu}(k_\mu - p_\mu)(k_\nu - p_\nu) - i\hbar^{-1}k_\nu q_\nu + i(2\hbar)^{-1}p_\nu q_\nu}, \qquad (22)$$

$$\langle \mu_1 \cdots \mu_n|qp\rangle = N_3 \alpha_{\mu_1} \cdots \alpha_{\mu_n} e^{-\frac{1}{2}\alpha^*_\nu \alpha_\nu} \quad \left(\langle \mu_1 \cdots \mu_n| \equiv N_3 \langle 0|\hat\alpha_{\mu_1} \cdots \hat\alpha_{\mu_n}\right), \qquad (23)$$

$$\langle x_2 p_2|x_1 p_1\rangle = \exp\left(-(4\hbar)^{-1}\left(A(x_2 - x_1)^2 + A^{-1}(p_2 - p_1)^2\right) + i(2\hbar)^{-1}\left(p_{1\mu}x_{2\mu} - p_{2\mu}x_{1\mu}\right)\right), \qquad (24)$$

where N_1, N_2 and N_3 are normalization constants*). From eqs. (21),(22) and (24) it is seen that $|\langle x|qp\rangle|^2$, $|\langle k|qp\rangle|^2$ and $|\langle x_2 p_2|x_1 p_1\rangle|^2$ are the many-dimensional Gaussian distributions: coordinate one, momentum one, and simultaneously coordinate and momentum one. As to

$$|\langle \mu_1 \cdots \mu_n|qp\rangle|^2 = N_3^2 \, |\alpha_{\mu_1}|^2 \cdots |\alpha_{\mu_n}|^2 \, e^{-\alpha^*_\nu \alpha_\nu} \qquad (25)$$

it is the many-dimensional Poisson distribution.

In field theory analogs of coordinate and momentum representations are non-popular, and in Fock and coherent state representations we have

$$\langle 0|\hat\varphi^{(-)}(x_1) \cdots \hat\varphi^{(-)}(x_n)|\varphi\rangle = \varphi^{(-)}(x_1) \cdots \varphi^{(-)}(x_n)\langle 0|\varphi\rangle, \qquad (26)$$

$$\langle \varphi_2|\varphi_1\rangle = \exp\left(-(2\hbar)^{-1}(\varphi_1^{(-)}, \varphi_1^{(-)}) - (2\hbar)^{-1}(\varphi_2^{(-)}, \varphi_2^{(-)}) + \hbar^{-1}(\varphi_2^{(-)}, \varphi_1^{(-)})\right) =$$

$$= \exp\left(-(4\hbar)^{-1}\int d^4x\, d^4y \left(J_1(x) - J_2(x)\right)\Delta^{(1)}(x-y)\left(J_1(y) - J_2(y)\right) + i(2\hbar)^{-1}\int d^4x\, d^4y\, J_2(x)\Delta(x-y)J_1(y)\right). \qquad (27)$$

Being squared, they also give Poisson and Gaussian distributions.

*) For example, in the 3-dimensional and 4-dimensional (pseudo-Euclidean) cases

$$N_1 = (\pi\hbar)^{-\frac{3}{4}}(\det A)^{\frac{1}{4}}, \qquad N_2 = (\pi\hbar)^{-\frac{3}{4}}(\det A)^{-\frac{1}{4}},$$

$$N_1 = (\pi\hbar)^{-1}(-\det A)^{\frac{1}{4}}, \qquad N_2 = (\pi\hbar)^{-1}(-\det A)^{-\frac{1}{4}}.$$

$$|\langle 0|\varphi\rangle|^2 = e^{-\hbar^{-1}(\varphi^{(-)}, \varphi^{(-)})} = e^{-(2\hbar)^{-1}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y)}, \qquad (28.a)$$

$$|\langle 0|\sqrt{\omega_1}\hat\varphi^{(-)}(x_1) \cdots \sqrt{\omega_n}\hat\varphi^{(-)}(x_n)|\varphi\rangle|^2 = |\sqrt{\omega_1}\varphi^{(-)}(x_1)|^2 \cdots |\sqrt{\omega_n}\varphi^{(-)}(x_n)|^2 \, |\langle 0|\varphi\rangle|^2, \qquad (28.b)$$

$$|\langle \varphi_2|\varphi_1\rangle|^2 = e^{-\hbar^{-1}(\varphi_1^{(-)} - \varphi_2^{(-)}, \varphi_1^{(-)} - \varphi_2^{(-)})} = e^{-(2\hbar)^{-1}\int d^4x\, d^4y \left(J_1(x) - J_2(x)\right)\Delta^{(1)}(x-y)\left(J_1(y) - J_2(y)\right)}, \qquad (29)$$

$$\left(\sqrt{\omega}\,\varphi^{(-)}(x) \equiv \sqrt{-\partial_k \partial_k + m^2}\,\varphi^{(-)}(x)\right).$$

It is seen from the above Poisson distributions that

$$\alpha^+_\nu \alpha_\nu = (2\hbar)^{-1}\left(Ax^2 + A^{-1}p^2\right) \qquad (30)$$

and

$$\hbar^{-1}(\varphi^{(-)}, \varphi^{(-)}) = (2\hbar)^{-1}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y) \qquad (31)$$

are mean numbers of "quanta" in the coherent states, $\alpha^*_\nu \alpha_\nu$ with no summation being the mean number of a ν-th sort of "quanta". In fact, eqs. (30) and (31) are coherent state expectation values of the operators of the number of quanta

$$\hat N = \hat\alpha^+_\nu \hat\alpha_\nu = (2\hbar)^{-1} : Ax^2 + A^{-1}p^2 : , \qquad (32)$$

$$\hat N = \hbar^{-1}(\hat\varphi^{(-)}, \hat\varphi^{(-)}), \qquad (33)$$

respectively.

Further, note that the above equal-time correlation matrices are consequences of the following non-equal time ones

$$\langle xp|\Delta x_\mu(t)\Delta x_\nu(s)|xp\rangle = i\hbar\, D^{(-)}_{\mu\nu}(t,s), \qquad (34.a)$$

$$\langle xp|\{\Delta x_\mu(t)\Delta x_\nu(s)\}|xp\rangle = \hbar\, D^{(1)}_{\mu\nu}(t,s), \qquad (34.b)$$

$$\langle \varphi|\Delta\varphi(x)\Delta\varphi(y)|\varphi\rangle = i\hbar\,\Delta^{(-)}(x-y), \qquad (35.a)$$

$$\langle \varphi|\{\Delta\varphi(x)\Delta\varphi(y)\}|\varphi\rangle = \hbar\,\Delta^{(1)}(x-y). \qquad (35.b)$$

Hence it follows, in particular, that

$$\langle xp|\Delta x_\mu(t)\Delta p_\nu(t)|xp\rangle = \frac{i\hbar}{2}\delta_{\mu\nu}, \qquad \langle xp|\{\Delta x_\mu(t)\Delta p_\nu(t)\}|xp\rangle = 0, \qquad (36)$$

$$\langle \varphi|\Delta\varphi(x)\Delta\dot\varphi(y)|\varphi\rangle_{y_0 = x_0} = \frac{i\hbar}{2}\delta(\vec x - \vec y), \qquad \langle \varphi|\{\Delta\varphi(x)\Delta\dot\varphi(y)\}|\varphi\rangle_{y_0 = x_0} = 0. \qquad (37)$$

Equations (1), (9), (11),(13),(15),(17),(19),(21)-(25),(30), (32),(34) and (36) are valid not only for any number of degrees of freedom in the Euclidean case, but also in the pseudo- Euclidean relativistic case, when $\vec x$ and $x_4 = ict$ are treated alike (see ref. /26/ and Appendix Б in ref. /24c/). In particular, eq. (17) with $\mu,\nu,\lambda = 1,2,3,4$, $x_4 = ict$ is the relativistic minimal uncertainty relation for a relativistic particle. Non-minimal uncertainty relations (inequalities) in a covariant form were also obtained in ref.[26] both for Euclidean and pseudo-Euclidean cases. In the latter one-particle coherent states keep their form (9), (21)-(24). However, now conditions for the bilinear forms $\overline{\Delta x_\mu \Delta x_\nu}\, \eta_\mu \eta_\nu$ and $\overline{\Delta p_\mu \Delta p_\nu}\, \xi_\mu \xi_\nu$ to be positively definite are unusual ones (see ref.[26]). They exclude the possibility of infinitely growing Gaussian functions, like $e^{-(\vec{x}^2 - t^2)}$ ($\overline{\Delta x_\mu \Delta x_\nu}$ and $\overline{\Delta p_\mu \Delta p_\nu}$ cannot be a multiple of $\delta_{\mu\nu}$ unlike the Euclidean case).

Long ago M.A.Markov[27] has defined similar relativistic Gaussian packets from other considerations. Markov has proposed some special realizations of the correlation matrix $\overline{\Delta p_\mu \Delta p_\nu}$.

In the relativistic case a proper time-like parameter (instead of time) can serve as an evolution parameter.

Ordered products. In the representations of interest an important role is played by ordered products of coordinate and momentum operators, such as N-ordered, anti-N-ordered and symmetrized ones

$$:\hat{x} \cdots \hat{p} \cdots: \;, \qquad :\hat{x}(t_1) \cdots \hat{x}(t_n): \;, \tag{38}$$

$$\vdots\,\hat{x} \cdots \hat{p} \cdots \vdots \;, \qquad \vdots\,\hat{x}(t_1) \cdots \hat{x}(t_n)\,\vdots \;, \tag{39}$$

$$\mathrm{sym}(\hat{x} \cdots \hat{p} \cdots) = \frac{1}{n!}\{\hat{x} \cdots \hat{p} \cdots\}, \quad \mathrm{sym}(\hat{x}(t_1) \cdots \hat{x}(t_n)) = \frac{1}{n!}\{\hat{x}(t_1) \cdots \hat{x}(t_n)\}, \tag{40}$$

where $\{\cdots\}$ denotes the sum of all $n!$ permutations. The first expressions (38)-(40) follow from the second ones, if we differentiate the latter with respect to some t_i's, multiply by masses (to obtain $p(t_i) = m_i \dot{x}(t_i)$), and take the equal-time limit.

The normalization of the symmetrized products is clear from the example

$$\mathrm{sym}\, e^{i\hbar^{-1}(p\hat{x} - x\hat{p})} = e^{i\hbar^{-1}(p\hat{x} - x\hat{p})}. \tag{41}$$

This expression may be considered as a generating function for the symmetrized products.

All these products are commutative, but not associative ones, like the Jordan products. Moreover, the symmetrized products (40), in fact, coincide with the Jordan ones:

$$\frac{1}{n!}\{\hat{x}(t_1) \cdots \hat{x}(t_n)\} = \frac{1}{2^{n-1}}\{\hat{x}(t_n)\{\hat{x}(t_{n-1}) \cdots \{\hat{x}(t_2)\hat{x}(t_1)\} \cdots\}\}. \tag{42}$$

It is proved by induction, using the identity

$$\{a\{bc\}\} = \{b\{ac\}\} + [c[ba]] \tag{43}$$

and the fact, that commutators of the coordinates and momenta are c-numbers:

$$[\hat{x}_k, \hat{x}_n] = [\hat{p}_k, \hat{p}_n] = 0, \quad [\hat{x}_k, \hat{p}_n] = i\hbar\delta_{kn}, \quad [\hat{x}_k(t_1), \hat{x}_n(t_2)] = i\hbar\delta_{kn} D(t_1 - t_2). \tag{44}$$

It is true for both Schrödinger and interaction pictures.

Note that the Jordan product

$$\frac{1}{2^n}\{\hat{x}(t_n)\{\hat{x}(t_{n-1}) \cdots \{\hat{x}(t_2)\{\hat{x}(t_1)F\}\} \cdots\}\} \tag{45}$$

and the n-fold commutator

$$[\hat{x}(t_n)[\hat{x}(t_{n-1}) \cdots [\hat{x}(t_2)[\hat{x}(t_1), F]] \cdots]] \tag{46}$$

(where F is an arbitrary operator) are symmetric in $\hat{x}(t_i)$ due to the identity (43) and Jacobi one, respectively.

Simple products and T-products can be decomposed both over N-products (according to the well-known Dyson and Wick theorems), and over anti-N-products, and over symmetrized products (according to analogs of these theorems) as follows

$$\hat{x}(t_1) \cdots \hat{x}(t_n) = :\hat{x}(t_1) \cdots \hat{x}(t_n): + \sum_{\text{one pairing}} i\hbar D^{(-)}(t_1 - t_2) :\hat{x}(t_3) \cdots \hat{x}(t_n): + \sum_{\text{two pairings}} \cdots + \cdots \tag{47.a}$$

$$= \vdots\,\hat{x}(t_1) \cdots \hat{x}(t_n)\,\vdots + \sum_{\text{one pairing}} i\hbar D^{(+)}(t_1 - t_2)\, \vdots\,\hat{x}(t_3) \cdots \hat{x}(t_n)\,\vdots + \sum_{\text{two pairings}} \cdots + \cdots \tag{47.b}$$

$$= \frac{1}{n!}\{\hat{x}(t_1) \cdots \hat{x}(t_n)\} + \sum_{\text{one commutator}} \frac{1}{2}[\hat{x}(t_1), \hat{x}(t_2)] \frac{1}{(n-2)!}\{\hat{x}(t_3) \cdots \hat{x}(t_n)\} +$$

$$+ \sum_{\text{two commutators}} \frac{1}{2}[\hat{x}(t_1), \hat{x}(t_2)] \frac{1}{2}[\hat{x}(t_3), \hat{x}(t_4)] \frac{1}{(n-4)!}\{\hat{x}(t_5) \cdots \hat{x}(t_n)\} + \cdots \tag{47.c}$$

$$= \frac{1}{n!}\{\hat{x}(t_1) \cdots \hat{x}(t_n)\} + \sum \frac{i\hbar}{2} D(t_1 - t_2) \frac{1}{(n-2)!}\{\hat{x}(t_3) \cdots \hat{x}(t_n)\} + \sum \cdots + \cdots \tag{47.c'}$$

$$T\hat{x}(t_1) \cdots \hat{x}(t_n) = :\hat{x}(t_1) \cdots \hat{x}(t_n): + \sum (-i\hbar) D_+(t_1 - t_2) :\hat{x}(t_3) \cdots \hat{x}(t_n): + \sum \cdots + \cdots \tag{48.a}$$

$$= \vdots\,\hat{x}(t_1) \cdots \hat{x}(t_n)\,\vdots + \sum (-i\hbar) D_-(t_1 - t_2)\, \vdots\,\hat{x}(t_3) \cdots \hat{x}(t_n)\,\vdots + \sum \cdots + \cdots \tag{48.b}$$

$$= \frac{1}{n!}\{\hat{x}(t_1) \cdots \hat{x}(t_n)\} + \sum (-i\hbar) D_{sym}(t_1 - t_2) \frac{1}{(n-2)!}\{\hat{x}(t_3) \cdots \hat{x}(t_n)\} + \sum \cdots + \cdots \tag{48.c}$$

Let us give also decompositions of the symmetrized product over the N-and anti-N-products and inverse decompositions

$$\frac{1}{n!}\{\hat{x}(t_1)\cdots\hat{x}(t_n)\} = :\hat{x}(t_1)\cdots\hat{x}(t_n): + \sum \frac{\hbar}{2} D^{(1)}(t_1-t_2) :\hat{x}(t_3)\cdots\hat{x}(t_n): + \sum \cdots + \cdots \tag{49.a}$$

$$= \vdots\,\hat{x}(t_1)\cdots\hat{x}(t_n)\,\vdots + \sum \left(-\frac{\hbar}{2}\right) D^{(1)}(t_1-t_2)\,\vdots\,\hat{x}(t_3)\cdots\hat{x}(t_n)\,\vdots + \sum \cdots + \cdots \tag{49.b}$$

$$:\hat{x}(t_1)\cdots\hat{x}(t_n): = \frac{1}{n!}\{\hat{x}(t_1)\cdots\hat{x}(t_n)\} + \sum \left(-\frac{\hbar}{2}\right) D^{(1)}(t_1-t_2)\frac{1}{(n-2)!}\{\hat{x}(t_3)\cdots\hat{x}(t_n)\} + \sum \cdots + \cdots \tag{50}$$

$$\vdots\,\hat{x}(t_1)\cdots\hat{x}(t_n)\,\vdots = \frac{1}{n!}\{\hat{x}(t_1)\cdots\hat{x}(t_n)\} + \sum \frac{\hbar}{2} D^{(1)}(t_1-t_2)\frac{1}{(n-2)!}\{\hat{x}(t_3)\cdots\hat{x}(t_n)\} + \sum \cdots + \cdots \tag{51}$$

$$\vdots\,\hat{x}(t_1)\cdots\hat{x}(t_n)\,\vdots = :\hat{x}(t_1)\cdots\hat{x}(t_n): + \sum \hbar D^{(1)}(t_1-t_2) :\hat{x}(t_3)\cdots\hat{x}(t_n): + \sum \cdots + \cdots \tag{52}$$

$$:\hat{x}(t_1)\cdots\hat{x}(t_n): = \vdots\,\hat{x}(t_1)\cdots\hat{x}(t_n)\,\vdots + \sum (-\hbar) D^{(1)}(t_1-t_2)\,\vdots\,\hat{x}(t_3)\cdots\hat{x}(t_n)\,\vdots + \sum \cdots + \cdots \tag{53}$$

Equations (47)-(53) can be proved by induction.

In the following we need the relations

$$\Lambda^{-2}\langle xp|\,\vdots\,\hat{x}\cdots\hat{p}\cdots\vdots\,|xp\rangle = \Lambda^{-1}\langle xp|\mathrm{sym}(\hat{x}\cdots\hat{p}\cdots)|xp\rangle$$
$$= \langle xp| :\hat{x}\cdots\hat{p}\cdots: |xp\rangle = x\cdots p\cdots \tag{54}$$

$$\Lambda^{-2}\langle xp|\,\vdots\,\hat{x}(t_1)\cdots\hat{x}(t_n)\,\vdots\,|xp\rangle = \Lambda^{-1}\langle xp|\mathrm{sym}(\hat{x}(t_1)\cdots\hat{x}(t_n))|xp\rangle =$$
$$= \langle xp| :\hat{x}(t_1)\cdots\hat{x}(t_n): |xp\rangle = x(t_1)\cdots x(t_n) \tag{55}$$

For example, we are interested in

$$\Lambda^{-2}\langle xp|\,\vdots\, e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}\,\vdots\,|xp\rangle = \Lambda^{-1}\langle xp| e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}|xp\rangle =$$
$$= \langle xp| : e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} : |xp\rangle = e^{i\hbar^{-1}(p'x-x'p)}. \tag{56}$$

It follows from eqs.(54) and (55), that

$$\langle xp|\,\vdots\,\hat{x}\cdots\hat{p}\cdots\vdots\,|xp\rangle = \Lambda^2\, x\cdots p\cdots\Lambda^{-2}\cdot 1 \tag{57}$$

$$\langle xp|\mathrm{sym}(\hat{x}\cdots\hat{p}\cdots)|xp\rangle = \Lambda\, x\cdots p\cdots\Lambda^{-1}\cdot 1 \tag{58}$$

$$\langle xp|\,\vdots\,\hat{x}(t_1)\cdots\hat{x}(t_n)\,\vdots\,|xp\rangle = \Lambda^2\, x(t_1)\cdots x(t_n)\Lambda^{-2}\cdot 1 \tag{59}$$

$$\langle xp|\mathrm{sym}(\hat{x}(t_1)\cdots\hat{x}(t_n))|xp\rangle = \Lambda\, x(t_1)\cdots x(t_n)\Lambda^{-1}\cdot 1 \tag{60}$$

$$\frac{1}{2^n}\langle xp|\{\hat{x}(t_n)\{\hat{x}(t_{n-1})\cdots\{\hat{x}(t_1)F\}\cdots\}\}|xp\rangle = \Lambda x(t_1)\cdots x(t_n)\Lambda^{-1}\langle xp|F|xp\rangle \tag{61}$$

$$\langle xp|[\hat{x}(t_n)[\hat{x}(t_{n-1})\cdots[\hat{x}(t_1),F]\cdots]]|xp\rangle = (i\hbar)^n\frac{\delta}{\delta f(t_1)}\cdots\frac{\delta}{\delta f(t_n)}\langle xp|F|xp\rangle. \tag{62}$$

The latter expressions of eqs. (57)-(61) may be treated using

$$\Lambda x_k \Lambda^{-1} = x_k + \frac{\hbar}{2} A^{-1}_{k\ell}\frac{\partial}{\partial x_\ell}, \quad \Lambda p_k \Lambda^{-1} = p_k + \frac{\hbar}{2} A_{k\ell}\frac{\partial}{\partial p_\ell}$$

$$\Lambda x_k(t)\Lambda^{-1} = x_k(t) + \frac{\hbar}{2} D^{(1)}_{k\ell}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_\ell(s)} \tag{63}$$

$$\Lambda^2 x_k \Lambda^{-2} = x_k + \hbar A^{-1}_{k\ell}\frac{\partial}{\partial x_\ell}, \quad \Lambda^2 p_k \Lambda^{-2} = p_k + \hbar A_{k\ell}\frac{\partial}{\partial p_\ell}$$

$$\Lambda^2 x_k(t)\Lambda^{-2} = x_k(t) + \hbar D^{(1)}_{k\ell}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_\ell(s)} \tag{64}$$

Note that for non-diagonal matrix elements we have

$$\langle x_2 p_2| :\hat{x}(t_1)\cdots\hat{x}(t_n): |x_1 p_1\rangle = x_{21}(t_1)\cdots x_{21}(t_n)\langle x_2 p_2|x_1 p_1\rangle$$
$$x_{21}(t) = x_1^{(-)}(t) + x_2^{(+)}(t). \tag{65}$$

If some operator is ordered in one of the above manners then, using eqs.(54) or (55), we can convert it into a function of the phase space variables x and p or into a functional of $f(t)$, thus obtaining c-number (classical) equivalent (representation) of this operator.[14] Examples of such "classical" representations of field operators, observables and S-matrix (in quantum field theory) see, e.g., in refs.[24c,e] Note that symmetrized product decompositions lead to causal coefficient functions.

<u>Basis sets of operators</u>. The coherent states will be used mainly, for evaluation of expectation values in these states, what, of course, may be done in terms of the corresponding density matrix

$$\hat{\rho}(x,p) = |xp\rangle\langle xp| = \tag{66.a}$$

$$= : e^{-(2\hbar)^{-1}(A(\hat{x}-x)^2 + A^{-1}(\hat{p}-p)^2)} : = : e^{-(\hat{a}^+ - \alpha^*)(\hat{a}-\alpha)} : = \tag{66.b}$$

$$= : e^{-(2\hbar)^{-1}(\hat{x}^{(1)}(t) - x^{(1)}(t))\overleftrightarrow{\frac{\partial}{\partial t}}(\hat{x}(t) - x(t))} : = \tag{66.c}$$

$$= e^{-(2\hbar)^{-1}\int dt\, ds\, f(t) D^{(1)}(t-s) f(s)} : e^{-(2\hbar)^{-1}\hat{x}^{(1)}(t)\frac{\partial}{\partial t}\hat{x}(t) - \hbar^{-1}\int dt\, \hat{x}^{(1)}(t) f(t)} : \tag{66.d}$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx' dp'\, e^{-i\hbar^{-1}(xp'-px')}\, e^{-(2\hbar)^{-1}(Ax'^2 + A^{-1}p'^2)} : e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} : \tag{66.e}$$

$$= \Lambda^2 \frac{1}{(2\pi\hbar)^n}\int dx' dp'\, e^{-i\hbar^{-1}(xp'-px')}\, \vdots\, e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}\, \vdots = \tag{66.f}$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx' dp'\, e^{-i\hbar^{-1}(xp'-px')}\, e^{-(4\hbar)^{-1}(Ax'^2 + A^{-1}p'^2)}\, e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} = \tag{66.g}$$

$$= 2^n \mathrm{sym}\left(e^{-(4\hbar)^{-1}(A(\hat{x}-x)^2 + A^{-1}(\hat{p}-p)^2)}\right) = 2^n \mathrm{sym}\left(e^{-\frac{1}{2}(\hat{a}^+ - \alpha^*)(\hat{a}-\alpha)}\right) = \tag{66.h}$$

$$= \Lambda \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')}\; e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} = \qquad (66.i)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')} \vdots\, e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} \vdots \qquad (66.j)$$

where possible expressions for $|xp\rangle\langle xp|$ are given. Other operators can be decomposed over the operators $\hat{\rho}(xp)$.

One can use, instead of $\hat{\rho}(xp)$, other sets of operators

$$\hat{\sigma}(xp) = \Lambda^{-1}|xp\rangle\langle xp| = \qquad (67.a)$$

$$= \hbar^n\int d\alpha\, e^{-ip\alpha}|x-\tfrac{\hbar}{2}\alpha\rangle\langle x+\tfrac{\hbar}{2}\alpha| = \hbar^n\int d\beta\, e^{ix\beta}|p-\tfrac{\hbar}{2}\beta\rangle\langle p+\tfrac{\hbar}{2}\beta| = \qquad (67.b)$$

$$= 2^n : e^{-(4\hbar)^{-1}(A(\hat{x}-x)^2+A^{-1}(\hat{p}-p)^2)} : = 2^n : e^{-\frac{1}{2}(\hat{a}^+-\alpha^*)(\hat{a}-\alpha)} := \qquad (67.c)$$

$$= 2^n : e^{-(4\hbar)^{-1}(\hat{x}^{(1)}(t)-x^{(1)}(t))\frac{\overleftrightarrow{\partial}}{\partial t}(\hat{x}(t)-x(t))} := \qquad (67.d)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')}\, e^{-(4\hbar)^{-1}(Ax'^2+A^{-1}p'^2)} : e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} : \qquad (67.e)$$

$$= \Lambda\frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')} : e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} := \qquad (67.f)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')}\; e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} \qquad (67.g)$$

$$\hat{\tau}(xp) = \Lambda^{-2}|xp\rangle\langle xp| = \qquad (68.a)$$

$$= \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')} \vdots\, e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})} \vdots \qquad (68.b)$$

$$\chi(xp) = e^{i\hbar^{-1}(x\hat{p}-p\hat{x})} \qquad (69)$$

$$U(xp) = : e^{i\hbar^{-1}(x\hat{p}-p\hat{x})} : \qquad (70)$$

$$\omega(xp) = \vdots\, e^{i\hbar^{-1}(x\hat{p}-p\hat{x})} \vdots \qquad (71)$$

$$\Xi\Lambda^{-1}|xp\rangle\langle xp| = (2\pi\hbar)^{\frac{n}{2}} e^{i\hbar^{-1}xp}|x\rangle\langle p| = \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')} e^{i\hbar^{-1}p'\hat{x}}\, e^{-i\hbar^{-1}x'\hat{p}} \qquad (72)$$

$$\Xi^{-1}\Lambda^{-1}|xp\rangle\langle xp| = (2\pi\hbar)^{\frac{n}{2}} e^{-i\hbar^{-1}xp}|p\rangle\langle x| = \frac{1}{(2\pi\hbar)^n}\int dx'dp'\, e^{-i\hbar^{-1}(xp'-px')} e^{-i\hbar^{-1}x'\hat{p}}\, e^{i\hbar^{-1}p'\hat{x}} \qquad (73)$$

They are not density matrices in the usual sense, e.g., have no the property to be positively definite. However, each of these sets of operators forms a basis (labelled by the phase space variables x and p) for decomposing other operators (density operators, observables).

Let us give properties of these bases. The coherent states are not orthogonal vectors (see eq.(24)). However, they are normalized and form a complete (in fact, overcomplete) set of states

$$\mathrm{Tr}\,\hat{\rho} = \langle xp|xp\rangle = 1\,, \qquad (74)$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,|xp\rangle\langle xp| = \mathbb{1}\,. \qquad (75)$$

Further,

$$\frac{1}{(2\pi\hbar)^n}\int dp\,|xp\rangle\langle xp| = N_1^2\int dx'\,|x'\rangle\, e^{-\hbar^{-1}A(x'-x)^2}\langle x'|\,, \qquad (76)$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,|xp\rangle\langle xp| = N_2^2\int dp'\,|p'\rangle\, e^{-\hbar^{-1}A^{-1}(p'-p)^2}\langle p'|\,, \qquad (77)$$

$$|x_1p_1\rangle\langle x_1p_1|x_2p_2\rangle\langle x_2p_2| = 4^n\Lambda_1\Lambda_2\, e^{2i\hbar^{-1}(x_1p_2-x_2p_1)}\, e^{2i\hbar^{-1}((p_1-p_2)\hat{x}-(x_1-x_2)\hat{p})} =$$

$$= 4^n\Lambda_1\Lambda_2\, e^{2i\hbar^{-1}(x_1p_2-x_2p_1)}\frac{1}{(2\pi\hbar)^n}\int dx\,dp\, e^{2i\hbar^{-1}(p_1-p_2)x-2i\hbar^{-1}(x_1-x_2)p}\Lambda^{-1}|xp\rangle\langle xp|. \qquad (78)$$

The latter equation is the multiplication table for $\hat{\rho}(xp)$. The operators $\hat{\rho}(xp)$ together with $\hat{\tau}(xp)$ satisfy the equations

$$\mathrm{Tr}\left(|xp\rangle\langle xp|\Lambda'^{-2}|x'p'\rangle\langle x'p'|\right) = \Lambda'^{-2}\,|\langle xp|x'p'\rangle|^2 = (2\pi\hbar)^{-1}\delta(x-x')\delta(p-p') \qquad (79)$$

and (1.e), which are orthogonality and completeness relations, similar to matrix ones. For $\hat{\tau}(xp)$ analogs of equations (74) and (75) are

$$\mathrm{Tr}\,\hat{\tau}(xp) = \Lambda^{-2}\langle xp|x'p'\rangle = 1\,, \qquad (80)$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\Lambda^{-2}\,|xp\rangle\langle xp| = \mathbb{1}\,. \qquad (81)$$

For the basis operators $\hat{\sigma}(xp)$ we have

$$\mathrm{Tr}\,\hat{\sigma}(xp) = \Lambda^{-1}\langle xp|xp\rangle = 1\,, \qquad (82)$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\Lambda^{-1}\,|xp\rangle\langle xp| = \mathbb{1}\,, \qquad (83)$$

$$\frac{1}{(2\pi\hbar)^n}\int dp\,\Lambda^{-1}|xp\rangle\langle xp| = |x\rangle\langle x|\,, \qquad (84)$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,\Lambda^{-1}|xp\rangle\langle xp| = |p\rangle\langle p|\,, \qquad (85)$$

$$\frac{1}{(2\pi\hbar)^n}\int dq\, e^{-iq\beta}\,\Lambda^{-1}|qp\rangle\langle qp| = |p+\tfrac{\hbar}{2}\beta\rangle\langle p-\tfrac{\hbar}{2}\beta| \equiv |k'\rangle\langle k|\,, \qquad (86)$$

$$\frac{1}{(2\pi\hbar)^n}\int dp\, e^{ip\alpha}\,\Lambda^{-1}|qp\rangle\langle qp| = |q+\tfrac{\hbar}{2}\alpha\rangle\langle q-\tfrac{\hbar}{2}\alpha| \equiv |x'\rangle\langle x|\,, \qquad (87)$$

$$(\Lambda_1^{-1}|x_1p_1\rangle\langle x_1p_1|)(\Lambda_2^{-1}|x_2p_2\rangle\langle x_2p_2|)=4^n e^{2i\hbar^{-1}(x_1p_2-x_2p_1)}e^{2i\hbar^{-1}((p_1-p_2)\hat{x}-(x_1-x_2)\hat{p})}$$
$$=4^n e^{2i\hbar^{-1}(x_1p_2-x_2p_1)}\frac{1}{(2\pi\hbar)^n}\int dx\,dp\, e^{2i\hbar^{-1}(p_1-p_2)x-2i\hbar^{-1}(x_1-x_2)p}\Lambda^{-1}|xp\rangle\langle xp|, \quad (88)$$
$$\mathrm{Tr}\big((\Lambda^{-1}|xp\rangle\langle xp|)(\Lambda'^{-1}|x'p'\rangle\langle x'p'|)\big)=\Lambda^{-1}\Lambda'^{-1}|\langle xp|x'p'\rangle|^2=(2\pi\hbar)^n\delta(x-x')\delta(p-p'). \quad (89)$$

Equations (89) and (1.d) are orthogonality and completeness relations, again similar to matrix ones. Equation (88) is the multiplication table.

For the basis $\hat{\chi}(xp)$

$$\mathrm{Tr}\left(e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}\right)=(2\pi\hbar)^n\delta(x)\delta(p), \quad (90)$$
$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}=2^n\int dx'|-x'\rangle\langle x'|, \quad (91)$$
$$e^{i\hbar^{-1}(p_1\hat{x}-x_1\hat{p})}e^{i\hbar^{-1}(p_2\hat{x}-x_2\hat{p})}=e^{i(2\hbar)^{-1}(p_1x_2-x_1p_2)}e^{i\hbar^{-1}((p_1+p_2)\hat{x}-(x_1+x_2)\hat{p})}, \quad (92)$$
$$\mathrm{Tr}\left(e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}\right)=(2\pi\hbar)^n\delta(x-x')\delta(p-p'). \quad (93)$$

Thus, for $\hat{\chi}(xp)$ orthogonality and completeness relations in the matrix sence are valid, too (eqs.(93) and (1.a)). Multiplication table (92) for elements of the Weyl group is simplest. Using it, one can obtain multiplication tables for other bases, such as eqs. (78) and (88).

Note that $\hat{\upsilon}(xp)$ and $\hat{\omega}(xp)$ satisfy the relations

$$\mathrm{Tr}\left(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:\right)=\mathrm{Tr}\left(\vdots e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}\vdots\right)=(2\pi\hbar)^n\delta(x)\delta(p), \quad (94)$$
$$\mathrm{Tr}\left(:e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}:\vdots e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}\vdots\right)=(2\pi\hbar)^n\delta(x-x')\delta(p-p'). \quad (95)$$

Again, eqs. (95) and (1.b) are orthogonality and completeness relations, similar to matrix ones.

Interrelations between the bases are clear from the above formulas.

3. COHERENT STATE REPRESENTATION AND PHASE SPACE REPRESENTATIONS

<u>CSR</u> is such a representation, where each operator F is represented by all its matrix elements between coherent states

$$\langle x_2p_2|F|x_1p_1\rangle=\mathcal{F}(\alpha_2^*,\alpha_1)\langle x_2p_2|x_1p_1\rangle. \quad (96)$$

The operator F can be easily reconstructed in the form

$$F=\frac{1}{(2\pi\hbar)^{2n}}\int dx_1dp_1dx_2dp_2|x_2p_2\rangle\langle x_2p_2|F|x_1p_1\rangle\langle x_1p_1| \quad (97)$$

(the reconstruction theorem). Equation (97) permits us to obtain the operator in any other representation, e.g., in the x-representation $(\langle x|F|x'\rangle)$, the p-representation $(\langle p|F|p'\rangle)$ etc.

Multiplications of F by $\hat{x}$ or $\hat{p}$ on the left or right may be written as operators, acting on the representative (96), i.e., on the representation variables x_1, p_1, x_2, p_2 :*)

$$\langle x_2p_2|\hat{x}F|x_1p_1\rangle=x^l\langle x_2p_2|F|x_1p_1\rangle,\ \langle x_2p_2|F\hat{x}|x_1p_1\rangle=x^r\langle x_2p_2|F|x_1p_1\rangle,\ \dots \quad (98)$$
$$x^l=\frac{1}{\sqrt{2\hbar A}}\left(\frac{1}{2}\alpha_2+\alpha_2^*+\frac{\partial}{\partial\alpha_2^*}\right),\quad p^l=-i\sqrt{\frac{A}{2\hbar}}\left(\frac{1}{2}\alpha_2-\alpha_2^*+\frac{\partial}{\partial\alpha_2^*}\right) \quad (99)$$
$$x_\mu^l(t)=\frac{1}{2}x_{2\mu}^{(-)}(t)+x_{2\mu}^{(+)}(t)+iD_{\mu\nu}^{(-)}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_{2\nu}(s)}. \quad (100)$$

Right operator representatives are obtained from the left ones by complex conjugation and substitution $2\to 1$. The representative (96) may be used, e.g., for a density matrix $(F=\rho)$ and the left and right operator representatives for operators (observables) acting on it (e.g., for a Hamiltonian). As general rules for all representations, 1) the left representatives commute with the righ ones (due to the associativity**)), 2) the left representatives are multiplied in the same order as original operators while right ones in the inverse order.

Note, however, that the set of coherent states $|xp\rangle$ is overcomplete, and majority of matrix elements (96) are redundant. Keeping among matrix elements (96) only diagonal ones for each operator F, one obtains

A) PSR-1 (or CSR-1)***), the Schwinger-Glauber representation,

$$\mathcal{F}_1(x,p)=\langle xp|F|xp\rangle=\mathrm{Tr}(|xp\rangle\langle xp|F)\qquad(\mathcal{F}_1(xp)=\mathcal{F}(\alpha^*,\alpha)) \quad (101)$$

---

*) For derivation of eqs. (99) and (100) and alternative forms see, e.g., ref. /24c/ , Appendix Б;

$$\frac{\partial}{\partial\alpha}=\sqrt{\frac{\hbar}{2A}}\frac{\partial}{\partial x}-i\sqrt{\frac{\hbar A}{2}}\frac{\partial}{\partial p},\quad \frac{\partial}{\partial\alpha^*}=\sqrt{\frac{\hbar}{2A}}\frac{\partial}{\partial x}+i\sqrt{\frac{\hbar A}{2}}\frac{\partial}{\partial p}.$$

**) For a non-associative case see ref. /25/

***) It is also called "coherent state representation" (as we did in Abstract).

with the reconstruction theorem (due to eq.(1.e))

$$F=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\left(\Lambda^{-2}|xp\rangle\langle xp|\right)\,\langle xp|F|xp\rangle\,. \tag{102}$$

Thus, any operator is completely determined by its coherent state expectation values.

The following reasoning may be also given. In classics a system with n degrees of freedom is characterized by 2n variables such as x_i and p_i (i=1...n). In quantum mechanics each operator F for a similar system is also characterized by 2n variables, e.g., $\langle x''_1\cdots x''_n|F|x'_1\cdots x'_n\rangle$ in x-representation. The same is valid for the coherent state expectation values $\langle x_1p_1\cdots x_np_n|F|x_1p_1\cdots x_np_n\rangle$, too, in contrast to non-diagonal matrix elements, depending on 4n-variables.

Besides the representative (101), we find left and right operator representatives:*)

$$\langle xp|\hat{x}F|xp\rangle=x^{l}\langle xp|F|xp\rangle\,,\quad \langle xp|F\hat{x}|xp\rangle=x^{r}\langle xp|F|xp\rangle\,,\ \cdots \tag{103}$$

$$x^{\substack{l\\r}}_k=x_k+\frac{\hbar}{2}A^{-1}{}_{km}\frac{\partial}{\partial x_m}\pm\frac{i\hbar}{2}\frac{\partial}{\partial p_k}=\Lambda\left(x_k\pm\frac{i\hbar}{2}\frac{\partial}{\partial p_k}\right)\Lambda^{-1}\,, \tag{104}$$

$$p^{\substack{l\\r}}_k=p_k+\frac{\hbar}{2}A_{km}\frac{\partial}{\partial p_m}\mp\frac{i\hbar}{2}\frac{\partial}{\partial x_k}=\Lambda\left(p_k\mp\frac{i\hbar}{2}\frac{\partial}{\partial x_k}\right)\Lambda^{-1}\,, \tag{105}$$

$$x^{\substack{l\\r}}_k(t)=x_k(t)\pm i\hbar D^{(\mp)}{}_{km}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_m(s)}=\Lambda\left(x_k(t)\pm\frac{i\hbar}{2}\frac{\delta}{\delta f_k(t)}\right)\Lambda^{-1}. \tag{106}$$

The left operators commute with the right ones:

$$[x^{l}_k,x^{r}_m]=[x^{l}_k,p^{r}_m]=[p^{l}_k,x^{r}_m]=[x^{l}_k(t),x^{r}_m(s)]=0\,. \tag{107}$$

The operators of one kind satisfy usual commutation relations (but sign changes for the right ones because of the above rule)

$$[x^{\substack{l\\r}}_k,x^{\substack{l\\r}}_m]=0\,,\quad [x^{\substack{l\\r}}_k,p^{\substack{l\\r}}_m]=\pm i\hbar\delta_{km}\,,\quad [p^{\substack{l\\r}}_k,p^{\substack{l\\r}}_m]=0 \tag{108}$$

$$[x^{\substack{l\\r}}_k(t),x^{\substack{l\\r}}_m(s)]=\pm i\hbar\delta_{km}D(t-s)\,. \tag{109}$$

Let us state some relations with other representations.

a) Forestalling what follows, we give connections with subsequent representations:

*) Either directly using the definition (9) of the coherent states (see, e.g., ref. /24c/) or exploiting connection (117) and transforming operators (115) and (116). For derivation of the latter see Appendix.

$$\langle xp|F|xp\rangle\equiv\mathcal{F}_1(x,p)=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{i\hbar^{-1}(px'-xp')}\tilde{\mathcal{F}}_1(x',p')= \tag{110.a}$$

$$=\Lambda\mathcal{F}_2(x,p)\equiv\pi^{-n}\int dx'dp'\,e^{-\hbar^{-1}(A(x-x')^2+A^{-1}(p-p')^2)}\mathcal{F}_2(x',p')= \tag{110.b}$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{i\hbar^{-1}(px'-xp')}\,e^{-(4\hbar)^{-1}(Ax'^2+A^{-1}p'^2)}\tilde{\mathcal{F}}_2(x',p')= \tag{110.c}$$

$$=\exp\left(\frac{\hbar}{4}\left(A\frac{\partial}{\partial p}\frac{\partial}{\partial p}+A^{-1}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right)\right)\mathcal{F}_2(x,p)=\exp\left(\frac{1}{2}\frac{\partial}{\partial\alpha^*}\frac{\partial}{\partial\alpha}\right)\mathcal{F}_2(x,p)= \tag{110.d}$$

$$=\exp\left(-\frac{\hbar}{4}\frac{\delta}{\delta f(\tau)}\overleftrightarrow{\frac{\partial}{\partial\tau}}D^{(1)}(\tau,\tau')\overleftrightarrow{\frac{\partial}{\partial\tau'}}\frac{\delta}{\delta f(\tau')}\right)\mathcal{F}_2(x,p)=\quad(\tau,\tau'\ \text{are arbitrary}) \tag{110.e}$$

$$=\exp\left(\frac{\hbar}{4}D^{(1)}(\tau,\tau')\left(\frac{\partial}{\partial x(t')}+\overleftarrow{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial\dot{x}(t')}\right)\left(\frac{\partial}{\partial x(t')}+\overleftarrow{\frac{\partial}{\partial\tau'}}\frac{\partial}{\partial\dot{x}(t')}\right)\right)_{\tau=\tau'=t'}\mathcal{F}_2(x,p)\,, \tag{110.f}$$

where $\mathcal{F}_2$ is the Gauss transform of $\mathcal{F}_1$, and $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ are the Fourier transforms of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. In fact, eqs. (110.b)-(110.f) give different definitions of the operator Λ.

b) The following representations of an operator F

$$F=:\mathcal{F}_1(\hat{x},\hat{p}):= \tag{111.a}$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,:e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}:\tilde{\mathcal{F}}_1(x,p)= \tag{111.b}$$

$$=:e^{\hat{x}\frac{\partial}{\partial x}+\hat{p}\frac{\partial}{\partial p}}:\mathcal{F}_1(x,p)_{|x=p=0}= \tag{111.c}$$

$$=:\exp\left(\hat{x}(\tau)\overleftrightarrow{\frac{\partial}{\partial\tau}}\frac{\delta}{\delta f(\tau)}\right):\mathcal{F}_1(x,p)_{|f=0}= \tag{111.d}$$

$$=:\Lambda\mathcal{F}_2(\hat{x},\hat{p}):=\pi^{-n}\int dx\,dp\,:e^{-\hbar^{-1}(A(\hat{x}-x)^2+A^{-1}(\hat{p}-p)^2)}:\mathcal{F}_2(x,p)= \tag{111.e}$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,:e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}:e^{-(4\hbar)^{-1}(Ax^2+A^{-1}p^2)}\tilde{\mathcal{F}}_2(x,p)= \tag{111.f}$$

$$=\mathrm{sym}(\mathcal{F}_2(\hat{x},\hat{p}))=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}\tilde{\mathcal{F}}_2(x,p)= \tag{111.g}$$

$$=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,e^{-i\hbar^{-1}p\hat{x}}\,e^{i\hbar^{-1}x\hat{p}}\,e^{-i(2\hbar)^{-1}xp}\tilde{\mathcal{F}}_2(x,p)= \tag{111.h}$$

$$=\vdots\mathcal{F}_3(\hat{x},\hat{p})\vdots=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\vdots e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}\vdots\,\tilde{\mathcal{F}}_3(x,p) \tag{111.i}$$

are more practicable, than eq.(102). For example eq.(111.a) directly reconstructs the operator F in the N-ordered form, if the function $\mathcal{F}_1(x,p)$ is known.

c) Conversely, from eq. (111.a) one can immediately obtain F in CSR (96) (and, in particular in PSR-1 (101)), and further in any other representation (x-, p-, etc.) due to eq. (97).

d) There is a more direct way to the occupation number representation, i.e.,to matrix elements between oscillator (Fock) states:

$$\langle m|F|n\rangle=\frac{1}{\sqrt{m!n!}}\left(\frac{\partial}{\partial a^*}\right)^m\left(\frac{\partial}{\partial a}\right)^n\frac{\langle xp|F|xp\rangle}{|\langle xp|0\rangle|^2}\bigg|_{x=p=0}\quad\left(|\langle xp|0\rangle|^2=e^{-a^*a}\right). \tag{112}$$

That means that PSR-1 (101) for an operator F serves as a generating function for obtaining matrix elements between oscillator states. This approach in quantum mechanics and field theory is due to J.Schwinger.

B) $$\widetilde{\mathcal{F}}_1(x,p)=\mathrm{Tr}\left(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F\right) \tag{113}$$

$$F=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,:e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}:\,\mathrm{Tr}\left(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F\right) \tag{114}$$

Here the left and right operator representatives are (see Appendix)

$$x_k^{\substack{\ell\\ r}}=-i\hbar\frac{\partial}{\partial p_k}-\frac{i}{2}A^{-1}_{km}p_m\mp\frac{1}{2}x_k=e^{-\frac{1}{2}|a|^2}\left(-i\hbar\frac{\partial}{\partial p_k}\mp\frac{1}{2}x_k\right)e^{\frac{1}{2}|a|^2}, \tag{115}$$

$$p_k^{\substack{\ell\\ r}}=i\hbar\frac{\partial}{\partial x_k}+\frac{i}{2}A_{km}x_m\mp\frac{1}{2}p_k=e^{-\frac{1}{2}|a|^2}\left(i\hbar\frac{\partial}{\partial x_k}\mp\frac{1}{2}p_k\right)e^{\frac{1}{2}|a|^2}. \tag{116}$$

The representations A and B are connected by the Fourier transformation

$$\langle xp|F|xp\rangle=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\,\mathrm{Tr}\left(:e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}:F\right) \tag{117}$$

$$\mathrm{Tr}\left(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F\right)=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\langle x'p'|F|x'p'\rangle. \tag{118}$$

If $F=\rho$ is a density matrix, then $\mathcal{F}_1(x,p)$ is a phase space distribution function and $\widetilde{\mathcal{F}}_1(x,p)$ is its characteristic function.

C) PSR-2 (CSR-2), the Wigner representation,

$$\mathcal{F}_2(x,p)=\Lambda^{-1}\langle xp|F|xp\rangle=\mathrm{Tr}\left(\Lambda^{-1}|xp\rangle\langle xp|F\right)= \tag{119.a}$$

$$=\hbar^n\int da\,e^{-ipa}\langle x+\tfrac{\hbar}{2}a|F|x-\tfrac{\hbar}{2}a\rangle= \tag{119.b}$$

$$=\hbar^n\int db\,e^{ixb}\langle p+\tfrac{\hbar}{2}b|F|p-\tfrac{\hbar}{2}b\rangle= \tag{119.c}$$

$$=(2\pi\hbar)^{\frac{n}{2}}\,\Xi\,e^{-i\hbar^{-1}xp}\langle x|F|p\rangle \tag{119.d}$$

$$=(2\pi\hbar)^{\frac{n}{2}}\,\Xi^{-1}e^{i\hbar^{-1}xp}\langle p|F|x\rangle. \tag{119.e}$$

According to (1.d) the reconstruction theorem is written as

$$F=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\left(\Lambda^{-1}|xp\rangle\langle xp|\right)\Lambda^{-1}\langle xp|F|xp\rangle, \tag{120}$$

and the left and right operator representatives have the form

$$x_k^{\substack{\ell\\ r}}=x_k\pm\frac{i\hbar}{2}\frac{\partial}{\partial p_k}, \tag{121}$$

$$p_k^{\substack{\ell\\ r}}=p_k\mp\frac{i\hbar}{2}\frac{\partial}{\partial x_k}, \tag{122}$$

$$x_k^{\substack{\ell\\ r}}(t)=x_k(t)\pm\frac{i\hbar}{2}\frac{\delta}{\delta f_k(t)}\qquad\left(\frac{\delta}{\delta f_k(t)}\equiv D_{km}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_m(s)}\right). \tag{123}$$

D) The Weyl representation:

$$\widetilde{\mathcal{F}}_2(x,p)=\mathrm{Tr}\left(e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}F\right), \tag{124}$$

$$F=\frac{1}{(2\pi\hbar)^n}\int dx\,dp\;e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}\,\mathrm{Tr}\left(e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}F\right), \tag{125}$$

$$x_k^{\substack{\ell\\ r}}=-i\hbar\frac{\partial}{\partial p_k}\mp\frac{1}{2}x_k, \tag{126}$$

$$p_k^{\substack{\ell\\ r}}=i\hbar\frac{\partial}{\partial x_k}\mp\frac{1}{2}p_k. \tag{127}$$

The representations $\mathcal{F}_2$ and $\widetilde{\mathcal{F}}_2$ are connected by the Fourier transformation

$$\Lambda^{-1}\langle xp|F|xp\rangle=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\,\mathrm{Tr}\left(e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}F\right), \tag{128}$$

$$\mathrm{Tr}\left(e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}F\right)=\frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\Lambda^{-1}\langle x'p'|F|x'p'\rangle. \tag{129}$$

Hence, if $F=\rho$ (a density matrix) and $W(x,p)=\mathcal{F}_2(x,p)$ is considered as a phase space distribution function (the Wigner distribution function), then $\widetilde{\mathcal{F}}_2(x,p)$ is its characteristic function /5/. Recall, however, that $W(x,p)$ is not guarantied to be positively definite, and therefore it is not the true distribution function. However, it contains the whole information and is connected with positively definite distribution $\mathcal{F}_1(x,p)$ (for $F=\rho$) by the Gauss transformation (110.b). It is easy to obtain the coordinate or momentum distributions separately, by integrating $W(x,p)$ over superfluous canonical variables (due to eqs. (84) and (85)), like in classics. Hence the rule: a transition probability (cross section) is the integral of $W(x,p)$ over all the initial and final coordinates":

$$w(p_f,p_i)\sim\frac{1}{(2\pi\hbar)^{m+n}}\int dx_{i1}\cdots dx_{im}dx_{f1}\cdots dx_{fn}W(x_fp_ft_f,x_ip_it_i)$$

$$W(x_fp_ft_f,x_ip_it_i)=\Lambda_f^{-1}\Lambda_i^{-1}|\langle x_fp_f|e^{-i\hbar^{-1}\hat{H}(t_f-t_i)}|x_ip_i\rangle|^2 \tag{130}$$

E) $\mathcal{F}_3(x,p) = \Lambda^{-2}\langle xp|F|xp\rangle = \mathrm{Tr}\,(\Lambda^{-2}|xp\rangle\langle xp|F)$, (131)

$$F = \frac{1}{(2\pi\hbar)^n}\int dx\,dp\; |xp\rangle\langle xp|\;\Lambda^{-2}\langle xp|F|xp\rangle\,, \tag{132}$$

$$x_k^{\substack{\ell\\ r}} = x_k - \frac{\hbar}{2}A^{-1}_{km}\frac{\partial}{\partial x_m} \pm \frac{i\hbar}{2}\frac{\partial}{\partial p_k} = \Lambda^{-1}\Big(x_k \pm \frac{i\hbar}{2}\frac{\partial}{\partial p_k}\Big)\Lambda\,, \tag{133}$$

$$p_k^{\substack{\ell\\ r}} = p_k - \frac{\hbar}{2}A_{km}\frac{\partial}{\partial p_m} \mp \frac{i\hbar}{2}\frac{\partial}{\partial x_k} = \Lambda^{-1}\Big(p_k \mp \frac{i\hbar}{2}\frac{\partial}{\partial x_k}\Big)\Lambda, \tag{134}$$

$$x_k^{\substack{\ell\\ r}}(t) = x_k(t) \pm i\hbar D^{(\pm)}_{km}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_m(s)} = \Lambda^{-1}\Big(x_k(t) \pm \frac{i\hbar}{2}\frac{\delta}{\delta f_k(t)}\Big)\Lambda. \tag{135}$$

F) $\widetilde{\mathcal{F}}_3(x,p) = \mathrm{Tr}\,(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F)$, (136)

$$F = \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,:e^{-i\hbar^{-1}(p\hat{x}-x\hat{p})}:\;\mathrm{Tr}\,(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F)\,, \tag{137}$$

$$x_k^{\substack{\ell\\ r}} = -i\hbar\frac{\partial}{\partial p_k} + \frac{i}{2}A^{-1}_{km}p_m \mp \frac{1}{2}x_k = e^{\frac{1}{2}|\alpha|^2}\Big(-i\hbar\frac{\partial}{\partial p_k} \mp \frac{1}{2}x_k\Big)e^{-\frac{1}{2}|\alpha|^2}, \tag{138}$$

$$p_k^{\substack{\ell\\ r}} = i\hbar\frac{\partial}{\partial x_k} - \frac{i}{2}A_{km}x_m \mp \frac{1}{2}p_k = e^{\frac{1}{2}|\alpha|^2}\Big(i\hbar\frac{\partial}{\partial x_k} \mp \frac{1}{2}p_k\Big)e^{-\frac{1}{2}|\alpha|^2}. \tag{139}$$

Representations E and F are connected by the Fourier transformation:

$$\Lambda^{-2}\langle xp|F|xp\rangle = \frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\mathrm{Tr}\big(:e^{i\hbar^{-1}(p'\hat{x}-x'\hat{p})}:F\big) \tag{140}$$

Note, that eqs. (1.c) and (1.f) give four more representations, pairs of which are connected by the Fourier transformations

$$\Xi\Lambda^{-1}\langle xp|F|xp\rangle = \frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\mathrm{Tr}\big(e^{i\hbar^{-1}p'\hat{x}}\,e^{-i\hbar^{-1}x'\hat{p}}F\big) \tag{141}$$

$$\Xi^{-1}\Lambda^{-1}\langle xp|F|xp\rangle = \frac{1}{(2\pi\hbar)^n}\int dx'dp'\,e^{-i\hbar^{-1}(xp'-px')}\mathrm{Tr}\big(e^{-i\hbar^{-1}x'\hat{p}}\,e^{i\hbar^{-1}p'\hat{x}}F\big) \tag{142}$$

Left and right operator representatives can be easily obtained, using eqs. (119.d) and (119.e), which demonstrate the connection with the representations $\langle p|F|x\rangle$ and $\langle x|F|p\rangle$. The mixed x,p-representation $\langle x|F|p\rangle$ has been used by D.I.Blokhintsev[4] as a phase space representation.

In quantum field theory we can follow the above patterns, given for quantum mechanics. So, we can formulate the completeness relations:

$$\|\mathbb{1}\|\otimes\|\mathbb{1}\| = \int\delta^2\varphi\,|e^{\hbar^{-1}(\varphi,\hat{\varphi})}|0\rangle\!\rangle\langle\!\langle 0|e^{-\hbar^{-1}(\varphi,\hat{\varphi})}\| = \qquad \Big(\delta^2\varphi \equiv \prod_{\vec{x}}\frac{d\varphi(\vec{x},t')\,d\dot{\varphi}(\vec{x},t')}{2\pi\hbar}\Big)$$

$$= \int\delta^2\varphi\,|:e^{\hbar^{-1}(\varphi,\hat{\varphi})}:|0\rangle\!\rangle\langle\!\langle 0|:e^{-\hbar^{-1}(\varphi,\hat{\varphi})}:\| =$$

$$= \int\delta^2\varphi\,(\Lambda^{-1}|\varphi\rangle\langle\varphi|)\otimes(\Lambda^{-1}\|\varphi\rangle\langle\varphi\|) = \int\delta^2\varphi\,|\varphi\rangle\langle\varphi|\otimes\Lambda^{-2}\|\varphi\rangle\langle\varphi\|\,, \tag{143}$$

$$\int\delta^2\varphi\,|\varphi\rangle\langle\varphi| = \int\delta^2\varphi\,\Lambda^{-1}|\varphi\rangle\langle\varphi| = \int\delta^2\varphi\,\Lambda^{-2}|\varphi\rangle\langle\varphi| = \mathbb{1}\,, \tag{144}$$

other relevant relations, such as multiplication tables, e.g.,

$$e^{\hbar^{-1}(\varphi_1,\hat{\varphi})}\,e^{\hbar^{-1}(\varphi_2,\hat{\varphi})} = e^{-(2\hbar)^{-1}(\varphi_1,\varphi_2)+\hbar^{-1}(\varphi_1+\varphi_2,\hat{\varphi})} \tag{145}$$

and define the phase space representations:

A) $\mathcal{F}_1(\varphi) = \langle\varphi|F|\varphi\rangle = \mathrm{Tr}\,(|\varphi\rangle\langle\varphi|F) =$ (146)

$$= \Lambda\mathcal{F}_2(\varphi) = \exp\Big(\frac{\hbar}{4}\int d^3x\,d^3y\,\frac{\delta}{\delta J(x)}\overleftrightarrow{\partial}{}^x_4\,\Delta^{(1)}(x-y)\overleftrightarrow{\partial}{}^y_4\frac{\delta}{\delta J(y)}\Big)\mathcal{F}_2(\varphi) = \tag{147.a}$$

$$= c\int\delta^2\varphi'\exp\Big(-\hbar^{-1}\int d^3x\,d^3y\,\big[-\ddot{\Delta}^{(1)}(\vec{x}-\vec{y},0)(\varphi(\vec{x})-\varphi'(\vec{x}))(\varphi(\vec{y})-\varphi'(\vec{y}))+$$
$$+\Delta^{(1)}(\vec{x}-\vec{y},0)(\dot{\varphi}(\vec{x})-\dot{\varphi}'(\vec{x}))(\dot{\varphi}(\vec{y})-\dot{\varphi}'(\vec{y}))\big]\Big)\mathcal{F}_2(\varphi') \tag{147.b}$$

$$F = \int\delta^2\varphi\,(\Lambda^{-2}|\varphi\rangle\langle\varphi|)\langle\varphi|F|\varphi\rangle = :\mathcal{F}_1(\hat{\varphi}): = \mathrm{sym}\,\mathcal{F}_2(\hat{\varphi}) = \vdots\,\mathcal{F}_3(\hat{\varphi})\,\vdots \tag{148}$$

$$\varphi^{\substack{\ell\\ r}}(x) = \varphi(x) \mp \hbar\int d^3y\,\Delta^{(\mp)}(x-y)\overleftrightarrow{\partial}{}^y_4\frac{\delta}{\delta J(y)} = \Lambda\Big(\varphi(x) \pm \frac{i\hbar}{2}\frac{\delta}{\delta J(x)}\Big)\Lambda^{-1} \tag{149}$$

B) $\widetilde{\mathcal{F}}_1(\varphi) = \mathrm{Tr}\,(:e^{-\hbar^{-1}(\varphi,\hat{\varphi})}:F)$ (150)

$$F = \int\delta^2\varphi\,:e^{\hbar^{-1}(\hat{\varphi},\varphi)}:\,\mathrm{Tr}\,(:e^{-\hbar^{-1}(\varphi,\hat{\varphi})}:F) \tag{151}$$

$$\langle\varphi|F|\varphi\rangle = \int\delta^2\varphi'\,e^{-\hbar^{-1}(\varphi,\varphi')}\,\mathrm{Tr}\,(:e^{-\hbar^{-1}(\varphi',\hat{\varphi})}:F) \tag{152}$$

$$\mathrm{Tr}(:e^{-\hbar^{-1}(\varphi,\hat{\varphi})}:F) = \int\delta^2\varphi'\,e^{\hbar^{-1}(\varphi,\varphi')}\langle\varphi'|F|\varphi'\rangle \tag{153}$$

C) $\mathcal{F}_2(\varphi) = \Lambda^{-1}\langle\varphi|F|\varphi\rangle = \mathrm{Tr}\,(\Lambda^{-1}|\varphi\rangle\langle\varphi|F)$ (154)

$$F = \int\delta^2\varphi\,(\Lambda^{-1}|\varphi\rangle\langle\varphi|)\,\Lambda^{-1}\langle\varphi|F|\varphi\rangle \tag{155}$$

$$\varphi^{\substack{\ell\\ r}}(x) = \varphi(x) \pm \frac{i\hbar}{2}\frac{\delta}{\delta J(x)} \tag{156}$$

D) $\widetilde{\mathcal{F}}_2(\varphi) = \mathrm{Tr}\big(e^{-\hbar^{-1}(\varphi,\hat{\varphi})}F\big)$ (157)

$$F = \int\delta^2\varphi\,e^{\hbar^{-1}(\hat{\varphi},\varphi)}\,\mathrm{Tr}\big(e^{-\hbar^{-1}(\varphi,\hat{\varphi})}F\big) \tag{158}$$

$$\Lambda^{-1}\langle\varphi|F|\varphi\rangle = \int\delta^2\varphi'\,e^{-\hbar^{-1}(\varphi,\varphi')}\,\mathrm{Tr}\big(e^{-\hbar^{-1}(\varphi',\hat{\varphi})}F\big) \tag{159}$$

$$\mathrm{Tr}\big(e^{-\hbar^{-1}(\varphi,\hat{\varphi})}F\big) = \int\delta^2\varphi'\,e^{\hbar^{-1}(\varphi,\varphi')}\,\Lambda^{-1}\langle\varphi'|F|\varphi'\rangle\,, \tag{160}$$

etc. Other left and right operator representatives are also similar to the above quantum-mechanical ones, and may be obtained by the substitutions

$x_k \equiv x_k(t') \to \varphi(\vec{x},t')$, $p_k \equiv p_k(t') \to \dot{\varphi}(\vec{x},t')$, $x(t) \to \varphi(x)$, $f(t) \to J(x)$

(cf. refs.[24c,e,f] for the cases A and C).

Other fields may be treated analogously.

APPENDIX

It is easy to check the completeness relation (75) for the coherent states in the x-representation (using eq. (21)):

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\langle x''|xp\rangle\langle xp|x'\rangle = \delta(x''-x'). \tag{A.1}$$

In this manner one can derive, e.g., formulas (1),(67.b),(76) and (77). With eq. (21) we have

$$\langle x''|\left(\Lambda^{-1}|xp\rangle\langle xp|\right)|x'\rangle = \Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle =$$
$$= N_1^2\Lambda^{-1}e^{-(2\hbar)^{-1}A(x''-x)^2-(2\hbar)^{-1}A(x'-x)^2+i\hbar^{-1}p(x''-x')} . \tag{A.2}$$

Further, we obtain

$$e^{-\frac{\hbar}{4}A\frac{\partial}{\partial p}\frac{\partial}{\partial p}}\; e^{i\hbar^{-1}p(x''-x')} = e^{(4\hbar)^{-1}A(x''-x')^2+i\hbar^{-1}p(x''-x')}, \tag{A.3}$$

$$e^{-\frac{\hbar}{4}A^{-1}\frac{\partial}{\partial x}\frac{\partial}{\partial x}}\; e^{-(2\hbar)^{-1}A(x''-x)^2-(2\hbar)^{-1}A(x'-x)^2} =$$
$$= e^{-\frac{\hbar}{4}A^{-1}\frac{\partial}{\partial x}\frac{\partial}{\partial x}}\;\mathcal{N}^2\int du\,dv\; e^{i(x''-x)u+i(x'-x)v-\frac{\hbar}{2}A^{-1}(u^2+v^2)} =$$
$$= \mathcal{N}^2\int du\,dv\; e^{i(x''-x)u+i(x'-x)v-\frac{\hbar}{4}A^{-1}(u-v)^2} =$$
$$= \mathcal{N}^2\mathcal{N}'(2\pi)^n\,\delta\left(x-\frac{x'+x''}{2}\right)e^{-(4\hbar)^{-1}A(x''-x')^2}, \tag{A.4}$$

where $N_1^2\mathcal{N}^2\mathcal{N}'(2\pi)^n = 1$, e.g., in the 3-dimensional case: $n = 3$,

$$N_1^2 = (\pi\hbar)^{-\frac{3}{2}}(\det A)^{\frac{1}{2}},\quad \mathcal{N} = \left(\frac{\hbar}{2\pi}\right)^{\frac{3}{2}}(\det A)^{-\frac{1}{2}},\quad \mathcal{N}' = \left(\frac{\pi}{\hbar}\right)^{\frac{3}{2}}(\det A)^{\frac{1}{2}}. \tag{A.5}$$

Therefore,

$$\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle = e^{i\hbar^{-1}p(x''-x')}\,\delta\left(x-\frac{x'+x''}{2}\right). \tag{A.6}$$

Similarly, using eq. (22), we get

$$\Lambda^{-1}\langle p''|xp\rangle\langle xp|p'\rangle = e^{-i\hbar^{-1}x(p''-p')}\,\delta\left(p-\frac{p'+p''}{2}\right). \tag{A.7}$$

Hence

$$\frac{1}{(2\pi\hbar)^n}\int dp\,\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle = \delta(x-x')\,\delta(x-x''), \tag{A.8}$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,\Lambda^{-1}\langle p''|xp\rangle\langle xp|p'\rangle = \delta(p-p')\,\delta(p-p''), \tag{A.9}$$

$$\frac{1}{(2\pi\hbar)^n}\int dx\,dp\left(\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle\right)\left(\Lambda^{-1}\langle y''|xp\rangle\langle xp|y'\rangle\right) =$$
$$= \frac{1}{(2\pi\hbar)^n}\int dx\,dp\,\delta\left(x-\frac{x'+x''}{2}\right)e^{i\hbar^{-1}p(x''-x')}\,\delta\left(x-\frac{y'+y''}{2}\right)e^{i\hbar^{-1}p(y''-y')} =$$
$$= \delta(x''-y')\,\delta(y''-x') . \tag{A.10}$$

Using eqs. (A.6)-(A.10) one can obtain formulas (1),(67.b), as follows

$$\frac{1}{(2\pi\hbar)^n}\int dp\,\Lambda^{-1}|xp\rangle\langle xp| = \frac{1}{(2\pi\hbar)^n}\int dp\,dx''dx'\,|x''\rangle\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle\langle x'| = |x\rangle\langle x|, \tag{A.11}$$

$$\Lambda^{-1}|xp\rangle\langle xp| = \int dx''dx'\,|x''\rangle\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle\langle x'| =$$
$$= \hbar^n\int dy\,d\alpha\; e^{-ip\alpha}\,\delta(x-y)\,|y-\tfrac{\hbar}{2}\alpha\rangle\langle y+\tfrac{\hbar}{2}\alpha| =$$
$$= \hbar^n\int d\alpha\; e^{-ip\alpha}\,|x-\tfrac{\hbar}{2}\alpha\rangle\langle y+\tfrac{\hbar}{2}\alpha|, \tag{A.12}$$

and so on.

The above calculations may be generalized for the bases $\Lambda^{\nu}|xp\rangle\langle xp|$ with $\nu > -1$ /24f/, but not for $\Lambda^{-2}|xp\rangle\langle xp|$. Taking apart the analysis of existence of $\Lambda'^{-2}\langle x'''|x'p'\rangle\langle x'p'|x''\rangle$, note that $\Lambda'^{-2}|\langle xp|x'p'\rangle|^2 = (2\pi\hbar)^n\delta(x-x')\delta(p-p')$, and $\Lambda'^{-2}\Lambda|\langle xp|x'p'\rangle|^2$ is a function.

Let us derive operators (115) and (116). To find x^l and p^r we represent the exponential in $\mathrm{Tr}(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:\hat{x}F)$ and $\mathrm{Tr}(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F\hat{p})$ as follows

$$:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}: = e^{-\frac{1}{2}|\alpha|^2}e^{i\hbar^{-1}(p\hat{x}-x\hat{p})} = e^{-\frac{1}{2}|\alpha|^2+\frac{i}{2}\hbar^{-1}xp}\,e^{-i\hbar^{-1}x\hat{p}}\,e^{i\hbar^{-1}p\hat{x}} \tag{A.13}$$

Here we used twofold the Baker-Campbell-Hausdorf formula (7) . Further,

$$\mathrm{Tr}(e^{-i\hbar^{-1}x\hat{p}}\,e^{i\hbar^{-1}p\hat{x}}\,\hat{x}\,F) = -i\hbar\frac{\partial}{\partial p}\mathrm{Tr}(e^{-i\hbar^{-1}x\hat{p}}\,e^{i\hbar^{-1}p\hat{x}}\,F), \tag{A.14}$$

$$\mathrm{Tr}(e^{-i\hbar^{-1}x\hat{p}}\,e^{i\hbar^{-1}p\hat{x}}\,F\,\hat{p}) = i\hbar\frac{\partial}{\partial x}\mathrm{Tr}(e^{-i\hbar^{-1}x\hat{p}}\,e^{i\hbar^{-1}p\hat{x}}\,F), \tag{A.15}$$

and after returning to the anti-N-ordered exponential we get x^l

$$\mathrm{Tr}(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:\hat{x}F) =$$
$$= e^{-\frac{1}{2}|\alpha|^2+\frac{i}{2}\hbar^{-1}xp}\,\frac{\hbar}{i}\frac{\partial}{\partial p}\,e^{\frac{1}{2}|\alpha|^2-\frac{i}{2}\hbar^{-1}xp}\,\mathrm{Tr}(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F) =$$
$$= \left(-i\hbar\frac{\partial}{\partial p}-i(2\hbar)^{-1}A^{-1}p-\frac{1}{2}x\right)\mathrm{Tr}\left(:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}:F\right), \tag{A.16}$$

and similarly p^r. One can obtain x^r and p^l analogously, using, instead of eq. (A.13), the formula

$$:e^{i\hbar^{-1}(p\hat{x}-x\hat{p})}: = e^{-\frac{1}{2}|\alpha|^2}e^{i\hbar^{-1}(p\hat{x}-x\hat{p})} = e^{-\frac{1}{2}|\alpha|^2-\frac{i}{2}\hbar^{-1}xp}\,e^{i\hbar^{-1}p\hat{x}}\,e^{-i\hbar^{-1}x\hat{p}} . \tag{A.17}$$

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