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FREE FERMIONS IN TWO-DIMENSIONAL ISING MODEL

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Свободные фермионы в двумерной модели Изинга

Исследуется двумерная модель Изинга как теория свободных фермионов на решетке. Рассмотрены метод фермионизации, основанный на зеркально-упорядоченной факторизации матрицы плотности, представление статсуммы в виде гауссовского фермионного интеграла, фермионная трактовка в импульсном пространстве, онсагеровский результат для свободной энергии, эффективные теории поля в континуальном пределе и сингулярности вблизи критической точки. Обсуждаются эффекты дальнодействующих фермионных корреляций для случая ненулевого магнитного поля и поведение теплоемкости на критической изотерме. Уделяется внимание выбору рациональных методов расчета.

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Free Fermions in Two-Dimensional Ising Model

The two-dimensional (2D) Ising model is reviewed as a theory of free fermions on a lattice. The discussion includes the fermionization procedure based on the mirror-ordered factorization of the density matrix, Gaussian fermionic integral representation for partition function, the momentum-space analysis and Onsager's result, the effective continuum-limit field theories and the critical-point singularities. The emergence of long-range fermionic correlations in a nonzero magnetic field and the behaviour of the specific heat along the critical isotherm are commented. Attention is given to the choice of rational computational devices.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

There are many remarkable analogues between the concepts and methods in statistical mechanics and quantum field theory. The two-dimensional Ising model (2DIM) may be a good example of this kind. In its original formulation, the 2D Ising model is a discrete-spin lattice model for the second-order phase transitions in magnets, for which the analytic results for the free energy and related functions are available over the whole temperature range. At the first stages, this model was analyzed rather as a complicated mathematical problem [1]-[8]. The canonical approaches to 2DIM are based on the transfer-matrix and combinatorial considerations [1]-[12]. The fermionic features in the 2DIM have been first recognized in [2,6,7]. Further developments in this direction provided new insight into the physical nature of problem and simplified the analysis in technical aspect [13]-[20]. The modern approaches to the 2DIM are based on the interpretation of the problem in terms of fermions [13]-[26]. In this article [1], we review some aspects of a simple noncombinatorial fermionic approach to the 2DIM based on the application of the anticommuting Grassmann-variable integrals and the mirror-ordered factorization ideas for the density matrix [18]-[20]. The method is simple, the transfer-matrices and combinatorics are not used. The appearance of fermions rather resembles the change of the basis in quantum mechanics [18]-[20]. The fermionization procedure is considered in Sect. 4. Grassmann variables are first introduced by factorization of the local bond Boltzmann weights in (11). The mirror-ordered factorized representation for the whole density matrix is then obtained in (16). This is a mixed spin-fermion representation, in which spin variables can be eliminated. Eliminating the spin variables from (16), the partition function, Q, appears as a Gaussian fermionic integral in (19). Equivalently, the 2D Ising model is reformulated as a theory of free fermions on a lattice. The above transformation of Q is performed for the most general inhomogeneous distribution of the bond coupling parameters over a lattice. In Sect. 5, the 2D Ising model on the standard homogeneous rectangular lattice is considered. By transformation to the momentum space for fermions, the partition function and free energy are obtained in a closed form (Onsager's result). In Sect. 6, coming back to previous discussion, we add few further remarks on the ordered products of Grassmannian factors, like those arising by factorization of the density matrix in (16). The nonlocal fermionic sums which appear in this context might be of interest for the 2D Ising model in a nonzero magnetic field, as is commented in Sect. 9. A refined version of the basic integral (19) for the partition function is obtained in Sect. 7. The resulting Gaussian integral for Q with two variables per site is given in (47). It is interesting to note that the Majorana-Dirac structures, somewhat mysteriously arising in the 2D Ising model, can be recognized in the action of (47) already at the lattice level. The effective continuum-limit field theories corresponding to the low-momentum sector of the exact lattice theory, which is responsible for the critical-point singularities near T_c, are then discussed in Sect. 8. The effective Majorana action of two-component massive fermions is obtained in (57) and (68). By doubling of the number of fermions in the Majorana representation, one can pass as well to the Dirac theory of charged fermions (69). In Sect. 9, the effects produced by a nonzero magnetic field in the fermionic system of the 2DIM Ising model near T_c are considered. It is argued that switching on of a nonzero magnetic field ($h \neq 0$) causes the long-range nonlocal interactions of the 2D Ising fermions. Within adopted approximation conjectures, the singularity in the specific heat at the critical isotherm is expected to be logarithmic. Finally, few concluding remarks are given in Sect. 10.

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2 Grassmann variables

Grassmann variables are the purely anticommuting fermionic symbols. Given a set of Grassmann variables, $a_1, a_2, a_3, \ldots, a_N$, we have: $a_i a_j + a_j a_i = 0$, $\quad a_j^2 = 0$. The linear superpositions of Grassmann variables are again purely anticommuting, their squares are zeros. The rules of the integration over Grassmann variables were first introduced by Berezin [27]. The elementary rules for one variable are:

$$\int da_j \cdot a_j = 1\,, \qquad \int da_j \cdot 1 = 0\,. \tag{1}$$

In multiple fermionic integrals, the differential symbols are again anticommuting with each other and with the variables [27]. Due to the nilpotent property of fermions, $a_j^2 = 0$, any natural (analytic) function definite a finite set of Grassmann variables can be represented, in principle, as a finite polynomial in these variables. The integration then reduces to the repeating use of the above elementary rules, keeping in mind that the fermionic symbols anticommute. The rules of change of variables under a linear substitution in the fermionic integrals readily follow from the basic rules of integration. As compared with the correspondent rules of commuting analysis, the only difference is that the Jacobian will now appear in the inverse power [27]. The Gaussian fermionic integral of the first kind is given as follows [27]:

$$\int \prod_{j=1}^{N} da_j^* da_j \, \exp\left(\sum_{i=1}^{N} \sum_{j=1}^{N} a_i A_{ij} a_j^* \right) = \det \hat{A}\,, \tag{2}$$

where all variables in the total set are purely anticommuting, the matrix $\hat{A}$ is arbitrary. By convention, the variables a_j and a_j^* can be considered as complex conjugated variables. In physical contexts this corresponds to charged fermions. Alternatively, one can consider a_j and a_j^* simply as independent variables. The Gaussian fermionic integral of the second kind, for real fermionic fields, is related to the Pfaffian [27]:

$$\int da_N \ldots da_2\, da_1 \, \exp\left(\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_i A_{ij} a_j \right) = \mathrm{Pfaff}\, \hat{A}\,, \tag{3}$$

where matrix $\hat{A}$ is now assumed to be skew-symmetric: $A + A^T = 0$, where A^T is the transposed matrix. In components: $A_{ij} + A_{ji} = 0$, $A_{ii} = 0$. This property is complimentary to fermionic anticommutativity. By formal definition, the Pfaffian is some combinatorial polynomial in elements A_{ij} [28, 29]. In fact, the combinatorics of the Pfaffian is identical to that of the fermionic version of Wick's theorem [28, 29]. The equation (3) can itself be assumed for an effective definition of the Pfaffian, wherefrom its basic properties readily follow. For any skew-symmetric matrix, the following algebraic identity holds true [28, 29]:

$$\det \hat{A} = (\,\mathrm{Pfaff}\, \hat{A}\,)^2\,. \tag{4}$$

The Pfaffian is thus the square root of the determinant of a skew-symmetric matrix. The above identity most easily can be proved just in terms of the integrals like (2) and (3). Let N be even. Assuming that the matrix in (2) is skew-symmetric, we make use of substitution:

$$a_k = \frac{1}{\sqrt{2}}(\xi_k + i\,\eta_k)\,, \qquad a_k^* = \frac{1}{\sqrt{2}}(\xi_k - i\,\eta_k)\,, \tag{5}$$

where ξ_k, η_k are the new variables of the integration. It is then easy to check that the integral (2) decouples into a product of two identical integrals like (3), which is equivalent to (4). For the normalized multifermionic averages associated with the Gaussian integrals like (2) and (3) one can apply fermionic Wick's theorem in a usual way. In the field-theoretical language, fermionic form in the exponentials like (2) and (3) is called action. In physical interpretations, the Gaussian fermionic integrals correspond to free-fermion field theories [21, 22].

3 Two-dimensional Ising model

In this section the 2D Ising model is formulated in terms of Ising spin variables. We start with a generalized formulation of the 2D Ising model, assuming arbitrary inhomogeneous distribution of the bond coupling parameters over a rectangular lattice net. The Ising spins, $\sigma_{mn} = \pm 1$, are disposed at the lattice sites, mn, labeled by pairs of integers, $m, n = 1, \ldots, L$, with m and n running in horizontal and vertical directions, respectively. L is the length of the lattice side. The total number of sites and spins on the lattice is $N = L^2$, at final stages we assume $N = L^2 \to \infty$. The hamiltonian is:

$$-\beta\, H(\sigma) = \sum_{m=1}^{L} \sum_{n=1}^{L} \left[\, b_{m+1n}^{(1)}\, \sigma_{mn}\sigma_{m+1n} + b_{mn+1}^{(2)}\, \sigma_{mn}\sigma_{mn+1} \,\right], \tag{6}$$

where $b_{mn}^{(\alpha)} = \beta J_{mn}^{(\alpha)}$ are the dimensionless bond coupling parameters, $J_{mn}^{(\alpha)}$ are the exchange energies, $\beta = 1/kT$ is the inverse temperature in the energy units. For L finite, to be definite, let us assume free boundary conditions for spin variables: $\sigma_{L+1n} = 0$, $\quad \sigma_{mL+1} = 0$. The partition function and the free energy per site are:

$$Z = \sum_{(\sigma)} \exp(\,-\beta H(\sigma))\,, \qquad -\beta\, f_Z = \lim_{N\to\infty} \frac{1}{N} \ln Z\,, \qquad \beta = 1/kT\,, \tag{7}$$

where the sum in Z is taken over all possible spin configurations provided by $\sigma_{mn} = \pm 1$ at each site. The internal (average) energy and specific heat follow by differentiating the free energy with respect to the temperature. The specific heat per site is:

$$C/k = \beta^2 \partial^2 (-\beta f_Z)/\partial \beta^2\,, \quad \beta = 1/kT\,, \tag{8}$$

where C/k is the dimensionless specific heat, k is Boltzmann's constant. Now, for a typical bond Boltzmann weight from (7), we note an identity: $\exp(b\,\sigma\sigma') = \cosh b + \sigma\sigma' \sinh b$, which readily follows from $\sigma\sigma' = \pm 1$. The partition function then appears in the form: $Z = R \times Q$, where R is a simple spin-independent prefactor and Q is the reduced partition function:

$$Q = \mathop{\mathrm{Sp}}_{(\sigma)} \left\{ \prod_{m=1}^{L} \prod_{n=1}^{L} (\,1 + t_{m+1n}^{(1)} \sigma_{mn}\sigma_{m+1n})\,(1 + t_{mn+1}^{(2)} \sigma_{mn}\sigma_{mn+1}) \right\}, \tag{9}$$

with $t_{mn}^{(1,2)} = \tanh b_{mn}^{(1,2)}$, and now we assume a properly normalized spin averaging:

$$\mathop{\mathrm{Sp}}_{(\sigma)} (\ldots) = \prod_{mn} \mathop{\mathrm{Sp}}_{(\sigma_{mn})} (\ldots)\,, \qquad \mathop{\mathrm{Sp}}_{(\sigma_{mn})} (\ldots) = \tfrac{1}{2} \sum_{\sigma_{mn}=\pm 1} (\ldots)\,, \tag{10}$$

so that at each site $\mathrm{Sp}\,(1) = 1$, $\mathrm{Sp}\,(\sigma_{mn}) = 0$. The reduced partition function Q will be the main subject of our interest in what follows.

4 Fermionization

In this section we transform Q into a fermionic Gaussian integral. The method is based on the mirror-ordered factorization procedure for the density matrix [18, 19, 20]. Let us start with a factorization of the local bond Boltzmann weights from (9). For the whole lattice, we introduce a set of the purely anticommuting Grassmann variables, a_{mn}, a_{mn}^*, b_{mn}, b_{mn}^*, a pair per bond, and write:

$$
\begin{aligned}
1+t^{(1)}_{m+1n}\sigma_{mn}\sigma_{m+1n} &= \\
&= \int da^{*}_{mn}da_{mn}\, e^{a_{mn}a^{*}_{mn}}\,(1+a_{mn}\sigma_{mn})\,(1+t^{(1)}_{m+1n}\,a^{*}_{mn}\sigma_{m+1n}) = \\
&= \underset{(a_{mn})}{\mathrm{Sp}}\ \{A_{mn}\,A^{*}_{m+1n}\}\,, \\
1+t^{(2)}_{mn+1}\sigma_{mn}\sigma_{mn+1} &= \\
&= \int db^{*}_{mn}db_{mn}\, e^{b_{mn}b^{*}_{mn}}\,(1+b_{mn}\sigma_{mn})\,(1+t^{(2)}_{mn+1}\,b^{*}_{mn}\sigma_{mn+1}) = \\
&= \underset{(b_{mn})}{\mathrm{Sp}}\ \{B_{mn}\,B^{*}_{m\,n+1}\}\,,
\end{aligned}
\tag{11}
$$

and $Sp(...)$ stand for the local Gaussian averaging like $da^{*}da\, e^{aa^{*}}(...)$ and $\int db^{*}db\, e^{bb^{*}}(...)$. These averaging symbols are totally commuting with any element of the algebra and can be gathered in one place, forming the symbol of the global Gaussian averaging. The identities (11) can be checked making use of the elementary rules of integration like (1), taking into account that $\exp(aa^{*}) = 1 + aa^{*}$ and $\exp(bb^{*}) = 1 + bb^{*}$. The mn indices in the above Grassmann factors are chosen to be equal to the indices of the spin variables involved in these factors. With this choice, it will be easy to control the position of any Grassmann factor with given spin variable among other such factors in their global products. The idea of the next step is to substitute (11) into (9) and to eliminate the spin variables in the resulting mixed representation for the density matrix. There are four Grassmann factors with the same spin variable, $A_{mn}, B_{mn}, A^{*}_{mn}, B^{*}_{mn}$, which come by factorization of the four different bonds attached to a given mn site. In the process of the spin averaging we have to keep these four factors nearby. Taking into account that the separable Grassmann factors are in general neither commuting nor anticommuting with each other, being the superpositions of commuting and anticommuting terms, we have to take care of a special ordering for the global products of such factors, in order the elimination of spin variables be really possible. In two dimensions, this problem is solvable [18, 19, 20]. What can really be used in the reordering arrangements of the products of Grassmann factors is the property that the doublets like $A_{mn}A^{*}_{m+1n}$ and $B_{mn}B^{*}_{mn+1}$, representing the local bond weights in (11), can be treated as totally commuting objects, if taken as a whole, under the sign of the Gaussian averaging arising by factorization.

In the subsequent transformations we shall apply as well two ordering principles illustrated below by tutorial examples. The first illustration (linear rearrangement) is:

$$
(\phi_0\,\bar{\phi}_1)\,(\phi_1\,\bar{\phi}_2)\,(\phi_2\,\bar{\phi}_3)\,(\phi_3\,\bar{\phi}_4) = \phi_0\,(\bar{\phi}_1\,\phi_1)\,(\bar{\phi}_2\,\phi_2)\,(\bar{\phi}_3\,\phi_3)\,\bar{\phi}_4\,, \tag{12}
$$

where we simply reread the product joining together the symbols with the same index. The second illustration (mirror rearrangement) is:

$$
(\phi_1\bar{\phi}_1)\,(\phi_2\bar{\phi}_2)\,(\phi_3\bar{\phi}_3) = (\,\phi_1\,(\phi_2\,(\phi_3\bar{\phi}_3)\,\bar{\phi}_2)\,\bar{\phi}_1) = \phi_1\,\phi_2\,\phi_3\cdot\bar{\phi}_3\,\bar{\phi}_2\,\bar{\phi}_1\,, \tag{13}
$$

where we assume that the doublets like $(\phi_j\bar{\phi}_j)$ are totally commuting with any individual factor from the common set, while the individual factors themselves may be noncommuting. Then we decouple proper and bar factors into separable products.

It is easy to guess that the linear ordering principle (12) is by itself enough to solve the 1D Ising chain via fermionization. This is not the case, however, in two dimensions, where there is a contradiction between preferable m-ordering for the horizontal weights and preferable n-ordering for the vertical weights with respect to the linear-ordering rule (12). Therefore, we

shall apply first the mirror-ordering principle (13) to factorize a horizontal ladder of the vertical weights, $B_{mn}B^{*}_{mn+1}$, in a horizontal-like fashion with respect to index m, with n fixed. This will provide us with an opportunity to introduce properly the remaining horizontal weights at the next stage, so that the spin variables can be finally completely eliminated from the density matrix. In transformations from (14) to (15) we omit, for brevity, the signs of the Gaussian averaging introduced by factorization of the local weights. The totally commuting bond Boltzmann weights are now given by $A_{mn}A^{*}_{m+1n}$ and $B_{mn}B^{*}_{mn+1}$.

For the first step, we multiply a subset of vertical weights over m, with n fixed. Making use of the mirror-ordering rule (13), we write:

$$
\prod_{m=1}^{L}(1+t^{(2)}_{mn+1}\sigma_{mn}\sigma_{mn+1}) = \prod_{m=1}^{L} B_{mn}B^{*}_{mn+1} = \prod_{m=1}^{L}\overset{m}{\overleftarrow{B_{mn}}}\cdot\prod_{m=1}^{L}\overset{m}{\overrightarrow{B^{*}_{mn+1}}}\,. \tag{14}
$$

In the final expression, there are two m-ordered products with $m = 1,\dots,L$ going in the opposite directions (mirror ordering). Already at this stage the ordering is favourable for introducing the horizontal weights, $A_{mn}A^{*}_{m+1n}$, which possibility will be used below. Multiplying the above partial products (14) taken as a whole over $n = 1,\dots,L$, with n increasing from left to right, and making use of the linear-ordering rule (12) with respect to index n, we write:

$$
\begin{aligned}
&\prod_{n=1}^{L}\prod_{m=1}^{L}(1+t_{mn+1}\sigma_{mn}\sigma_{mn+1}) = \\
&=\prod_{n=1}^{\overset{\xrightarrow{n}}{L}}\Big[\prod_{m=1}^{L}\overset{m}{\overleftarrow{B_{mn}}}\cdot\prod_{m=1}^{L}\overset{m}{\overrightarrow{B^{*}_{mn+1}}}\Big] = \\
&=\prod_{n=1}^{\overset{\xrightarrow{n}}{L}}\Big[\prod_{m=1}^{L}\overset{m}{\overrightarrow{B^{*}_{mn}}}\cdot\prod_{m=1}^{L}\overset{m}{\overleftarrow{B_{mn}}}\Big].
\end{aligned}
\tag{15}
$$

In the last line, it was taken into account that $B^{*}_{mL+1} = 1$, since $\sigma_{mL+1} = 0$. Respectively, we have corrected the same expression at the left end introducing, formally, the lacking product of factors $B^{*}_{m1} = 1 + t_{m1}\sigma_{m1}b^{*}_{m0}$ with $b^{*}_{m0} = 0$, so that, actually, $B^{*}_{m1} = 1$. In this way, the free boundary conditions for spins are now elaborated into the analogous conditions for fermions. All vertical weights are already involved in (15). It remains only to introduce properly the commuting horizontal weights. For the complete density matrix we thus find:

$$
\begin{aligned}
Q(\sigma) &= \prod_{n=1}^{L}\prod_{m=1}^{L}(1+t_{mn+1}\sigma_{mn}\sigma_{mn+1})\,(1+t_{m+1n}\sigma_{mn}\sigma_{m+1n}) = \\
&= \underset{(a,b)}{\mathrm{Sp}}\ \prod_{n=1}^{\overset{\xrightarrow{n}}{L}}\Big[\prod_{m=1}^{L} B^{*}_{mn}\overset{m}{\overrightarrow{A_{mn}}}A^{*}_{m+1n}\cdot\prod_{m=1}^{L}\overset{m}{\overleftarrow{B_{mn}}}\Big] = \\
&= \underset{(a,b)}{\mathrm{Sp}}\ \prod_{n=1}^{\overset{\xrightarrow{n}}{L}}\Big[\prod_{m=1}^{L} A^{*}_{mn}\overset{m}{\overrightarrow{B^{*}_{mn}}}A_{mn}\cdot\prod_{m=1}^{L}\overset{m}{\overleftarrow{B_{mn}}}\Big],
\end{aligned}
\tag{16}
$$

where use was made of the linear ordering rule (13) with respect to m. By analogy with the boundary transformations in (15), we eliminate in the final line the extra factors $A^{*}_{L+1n} = 1$, with $\sigma_{L+1n} = 0$, and insert, formally, the lacking factors $A^{*}_{1n} = 1$, assuming $a^{*}_{0n} = 0$. In (16) we also restore the symbol of the diagonal Gaussian averaging arising by factorization of the local weights:

$$\mathop{\mathrm{Sp}}_{(a,b)} \{...\} = \int \prod_{m=1}^{L} \prod_{n=1}^{L} da^{*}_{mn} da_{mn} db^{*}_{mn} db_{mn} \; \exp \sum_{m=1}^{L} \sum_{n=1}^{L} (\, a_{mn} a^{*}_{mn} + b_{mn} b^{*}_{mn}\,) \{ ... \} \,. \tag{17}$$

The expression in the final line of (16) is what we call the mirror-ordered factorized representation for the density matrix. This representation is exact, assuming the free-boundary conditions for fermions: $a^{*}_{0n} = 0$, $b^{*}_{m0} = 0$. The density matrix is now completely prepared for the elimination of the spin variables. The partition function arises by summing over $\sigma_{mn} = \pm 1$ at each site in (16).

The averaging over $\sigma_{mn} = \pm 1$ is to be performed at the junction of the two m-ordered products in (16), with fixed n. This is a step by step procedure. The local averaging at the junction is given by:

$$\begin{aligned}
&\mathop{\mathrm{Sp}}_{(\sigma_{mn})} \{ A^{*}_{mn} B^{*}_{mn} A_{mn} B_{mn} \} = \\
&= \frac{1}{2} \sum_{\sigma_{mn} = \pm 1} (1 + t^{(1)}_{mn} \sigma_{mn} a^{*}_{m-1n})(1 + t^{(2)}_{mn} \sigma_{mn} b^{*}_{mn-1})(1 + \sigma_{mn} a_{mn})(1 + \sigma_{mn} b_{mn}) = \\
&= 1 + a_{mn} b_{mn} + t^{(1)}_{mn} t^{(2)}_{mn} a^{*}_{m-1n} b^{*}_{mn-1} + (t^{(1)}_{mn} a^{*}_{m-1n} + t^{(2)}_{mn} b^{*}_{mn-1})(a_{mn} + b_{mn}) + \\
&+ t^{(1)}_{mn} t^{(2)}_{mn} a^{*}_{m-1n} b^{*}_{mn-1} a_{mn} b_{mn} = \\
&= \exp [\, a_{mn} b_{mn} + t^{(1)}_{mn} t^{(2)}_{mn} a^{*}_{m-1n} b^{*}_{mn-1} + (t^{(1)}_{mn} a^{*}_{m-1n} + t^{(2)}_{mn} b^{*}_{mn-1})(a_{mn} + b_{mn}) \,] \,.
\end{aligned} \tag{18}$$

The result of the averaging is a purely fermionic polynomial, even in the variables, which is equivalent to the Gaussian exponential factor given in the last line. This equivalence can be checked, for instance, by the series expansion of the exponential, taking into account the nilpotent property of fermions. Another way to see this equivalence is explained in Sect. 6. Let n be fixed, at the junction of the two m-ordered products in (16) we just find the four relevant Grassmann factors (18) with the same index mn placed nearby, with $m = L$, given n. The local averaging (18) results the Gaussian exponential factor from the last line, which is even in fermions and is thus totally commuting with any element of the algebra. We then remove this commuting Gaussian factor from the junction somewhere to the very left end of the remaining ordered product, and find again, at the junction, a new set of four neighboring factors like (18) with the same index mn and the same spin variable, $m = L - 1$, given n. We then repeat the same averaging procedure at the junction for $m = L-1$, and then for $m = L-2, \ldots, 1$, for given n, and all over again for other values of $n = 1, \ldots, L$. The spin variables being completely eliminated, over the whole lattice, the partition appears to be given by the product of partial Gaussian exponential factors from (18) taken under the sign of the global Gaussian averaging (17). Thus we come to the result:

$$\begin{aligned}
Q \;=\; &\int \prod_{m=1}^{L} \prod_{n=1}^{L} db^{*}_{mn} db_{mn} da^{*}_{mn} da_{mn} \; \exp \Big\{ \sum_{m=1}^{L} \sum_{n=1}^{L} \Big[a_{mn} a^{*}_{mn} + b_{mn} b^{*}_{mn} + \\
&+ a_{mn} b_{mn} + t^{(1)}_{mn} t^{(2)}_{mn} a^{*}_{m-1n} b^{*}_{mn-1} + (t^{(1)}_{mn} a^{*}_{m-1n} + t^{(2)}_{mn} b^{*}_{mn-1})(a_{mn} + b_{mn}) \Big] \Big\} \,,
\end{aligned} \tag{19}$$

where $a^{*}_{0n} = 0$, $b^{*}_{m0} = 0$. The partition function is now presented as a fermionic Gaussian integral. This representation is exact. The fermionic integral (19) is completely equivalent to the original expression (9) assuming the free boundary conditions both for spins and fermions.

5 The momentum-space fermions

In this section we consider the 2D Ising model settled on the standard homogeneous lattice. The partition function Q can be explicitly evaluated in this case by the transformation to the momentum space for fermions. This results the Onsager expressions for partition function and free energy of the standard 2D Ising model. For the homogeneous (though yet anisotropic) lattice, in the hamiltonian (6) we put: $b^{(1)}_{mn}, b^{(2)}_{mn} \to b_1, b_2$, where $b_{1,2} = \beta J_{1,2}$ are the dimensionless coupling constants in the horizontal and vertical directions, respectively. The partition function becomes: $Z = (2 \cosh b_1 \cosh b_2)^{L^2} Q$, with the reduced partition function:

$$Q = \mathop{\mathrm{Sp}}_{(\sigma)} \Big\{ \prod_{m=1}^{L} \prod_{n=1}^{L} (1 + t_1 \sigma_{mn} \sigma_{m+1n})(1 + t_2 \sigma_{mn} \sigma_{mn+1}) \Big\} \,, \tag{20}$$

where $t_{1,2} = \tanh b_{1,2}$. From (19), the same partition function is given by the Gaussian integral:

$$\begin{aligned}
Q \;=\; &\int \prod_{m=1}^{L} \prod_{n=1}^{L} db^{*}_{mn} db_{mn} da^{*}_{mn} da_{mn} \; \exp \Big\{ \sum_{m=1}^{L} \sum_{n=1}^{L} \Big[a_{mn} a^{*}_{mn} + b_{mn} b^{*}_{mn} + \\
&+ \; a_{mn} b_{mn} + t_1 t_2\, a^{*}_{m-1n} b^{*}_{mn-1} + (t_1\, a^{*}_{m-1n} + t_2\, b^{*}_{mn-1})(a_{mn} + b_{mn}) \Big] \Big\} \,,
\end{aligned} \tag{21}$$

with $a^{*}_{0n} = b^{*}_{m0} = 0$. The integral (21) is equivalent to (20) for any finite lattice size L under the free boundary conditions. In what follows, however, it will be more suitable to impose in (21) the periodic boundary conditions for fermions, $a^{*}_{0n} = a^{*}_{Ln}$, $b^{*}_{m0} = b^{*}_{mL}$. This change can be viewed as a boundary approximation inessential for infinite lattice, as $N = L^2 \to \infty$. Finally, we are interesting in the free energy per site for infinite lattice. Assuming now periodic boundary conditions for fermions, let us pass in (21) to the momentum space by means of the standard Fourier substitution:

$$\begin{aligned}
a_{mn} &= \frac{1}{L} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} a_{pq}\, \mathrm{e}^{\, i \frac{2\pi}{L} mp + i \frac{2\pi}{L} nq} \,, & a^{*}_{mn} &= \frac{1}{L} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} a^{*}_{pq}\, \mathrm{e}^{- i \frac{2\pi}{L} mp - i \frac{2\pi}{L} nq} \,, \\
b_{mn} &= \frac{1}{L} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} b_{pq}\, \mathrm{e}^{\, i \frac{2\pi}{L} mp + i \frac{2\pi}{L} nq} \,, & b^{*}_{mn} &= \frac{1}{L} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} b^{*}_{pq}\, \mathrm{e}^{- i \frac{2\pi}{L} mp - i \frac{2\pi}{L} nq} \,.
\end{aligned} \tag{22}$$

In the momentum space, the integral (21) becomes:

$$\begin{aligned}
Q \;=\; &\int \prod_{p=0}^{L-1} \prod_{q=0}^{L-1} da^{*}_{pq} da_{pq} db^{*}_{pq} db_{pq} \; \exp \; \Big\{ \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} \Big[a_{pq} a^{*}_{pq} + b_{pq} b^{*}_{pq} + a_{pq} b_{L-pL-q} + \\
&+ t_1 t_2\, \mathrm{e}^{\, i \frac{2\pi p}{L} - i \frac{2\pi q}{L}}\, a^{*}_{pq} b^{*}_{L-pL-q} + (t_1\, \mathrm{e}^{\, i \frac{2\pi p}{L}}\, a^{*}_{pq} + t_2\, \mathrm{e}^{\, i \frac{2\pi q}{L}}\, b^{*}_{pq})(a_{pq} + b_{pq}) \Big] \Big\} \,.
\end{aligned} \tag{23}$$

where $a_{pq}, a^{*}_{pq}, b_{pq}, b^{*}_{pq}$ are the new variables of the integration. In the above transformation, the orthogonality relations for the Fourier exponentials were also taken into account:

$$\frac{1}{L^2} \sum_{m=1}^{L} \sum_{n=1}^{L} \exp \Big[\, i \frac{2\pi}{L} m(p \pm p') + i \frac{2\pi}{L} (q \pm q') n \Big] = \delta\, (p \pm p' \,|\, q \pm q')_{\,\mathrm{mod}\, L} \,, \tag{24}$$

where $\delta\,(p\,|\,q)_{\,\mathrm{mod}\, L}$ is the Kronecker symbol modulo L in both directions. The fermionic measure in (23) transforms in a trivial way (Jacobian equals to unity) due to the unitarity of the combined Fourier substitution (22), which property follows from (24). Thus we have to evaluate explicitly the momentum-space integral (23).

The fermionic action in the momentum space admits a block-diagonal structure and the integral decouples into a product of low-dimensional integrals over the groups of the variables with momenta p, q and $L-p, L-q$. Since the variables with conjugated momenta pq and $L-pL-q$ are interacting, in order to single out explicitly the true independent subsets of the variables in the action, we have to combine together in the pq-sum in (23) the terms with conjugated momenta p, q and $L-p$, $L-q$. Equivalently, the pq-sum is to be symmetrized with respect to conjugation $p, q \leftrightarrow L-p, L-q$. After such a symmetrization, the integral (23) factorizes into a product of independent integral factors of the following kind:

$$
\begin{aligned}
Q^2_{pq} = \int & da^*_{pq} da_{pq} db^*_{pq} db_{pq} da^*_{L-pL-q} da_{L-pL-q} db^*_{L-pL-q} db_{L-pL-q} \exp \left[(a_{pq} a^*_{pq} + \right. \\
& + b_{pq} b^*_{pq} + a_{L-pL-q} a^*_{L-pL-q} + b_{L-pL-q} b^*_{L-pL-q}) + (a_{pq} b_{L-pL-q} + a_{L-pL-q} b_{pq}) + \\
& + (\hat{t}_1 \hat{t}^*_2 \, a^*_{pq} b^*_{L-pL-q} + \hat{t}^*_1 \hat{t}_2 \, a^*_{L-pL-q} b^*_{pq}) + (\hat{t}_1 \, a^*_{pq} + \hat{t}_2 \, b^*_{pq}) (a_{pq} + b_{pq}) + \\
& \left. + (\hat{t}^*_1 \, a^*_{L-pL-q} + \hat{t}^*_2 \, b^*_{L-pL-q}) (a_{L-pL-q} + b_{L-pL-q}) \right] ,
\end{aligned} \tag{25}
$$

where we assume abbreviations:

$$
\hat{t}_1 = t_1 \, e^{i \frac{2\pi p}{L}} , \quad \hat{t}_2 = t_2 \, e^{i \frac{2\pi q}{L}} , \quad \hat{t}^*_1 = t_1 \, e^{-i \frac{2\pi p}{L}} , \quad \hat{t}^*_2 = t_2 \, e^{-i \frac{2\pi q}{L}} . \tag{26}
$$

The elementary Gaussian integral (25) can be evaluated in different ways. The straightforward method is to expand the nondiagonal part of the exponential into a series and to integrate step by step over the subsets of the conjugated variables by means of elementary rules like (1). In the advanced version of this method, one makes use of the selection rules for the diagonal Gaussian averages that can be observed in the relations like (11) and (46). Another method is to interpret (25) as determinantal Gaussian integral like (2) with $N = 4$. In such representation, one assumes the Gaussian action in the form: $S = a \hat{A} a^*$. This is possible, for instance, with the following choice of the conjugated fields:

$$
\begin{aligned}
& a_1 \, , a_2 \, , a_3 \, , a_4 \quad \leftrightarrow \quad a_{pq} \, , b_{pq} \, , a^*_{L-pL-q} \, , b^*_{L-pL-q} \, , \\
& a^*_1 \, , a^*_2 \, , a^*_3 \, , a^*_4 \quad \leftrightarrow \quad a^*_{pq} \, , b^*_{pq} \, , a_{L-pL-q} \, , b_{L-pL-q} \, .
\end{aligned} \tag{27}
$$

The integral factor (25) then equals to the determinant of matrix A given explicitly in (28). Thus, we find:

$$
Q^2_{pq} = \det \begin{vmatrix} 1-\hat{t}_1 & -\hat{t}_2 & 0 & 1 \\ -\hat{t}_1 & 1-\hat{t}_2 & -1 & 0 \\ 0 & \hat{t}^*_1 \hat{t}_2 & -1+\hat{t}^*_1 & \hat{t}^*_1 \\ -\hat{t}_1 \hat{t}^*_2 & 0 & \hat{t}^*_2 & -1+\hat{t}^*_2 \end{vmatrix} . \tag{28}
$$

By a straightforward though somewhat lengthy calculation of the above determinant, we arrive to the following expressions:

$$
\begin{aligned}
Q^2_{pq} & = (1 + |\hat{t}_1|^2)(1 + |\hat{t}_2|^2) - (\hat{t}_1 + \hat{t}^*_1)(1 - |\hat{t}_2|^2) - (\hat{t}_2 + \hat{t}^*_2)(1 - |\hat{t}_1|^2) = \\
& = (1 + t_1^2)(1 + t_2^2) - 2t_1 (1 - t_2^2) \cos \frac{2\pi p}{L} - 2t_2 (1 - t_1^2) \cos \frac{2\pi q}{L} .
\end{aligned} \tag{29}
$$

To obtain the partition function, Q, we have to multiply the factors (29) over all distinct pairs of the conjugated momentum-lattice points $(p, q \,|\, L-p, L-q)$. That is, if the factor Q^2_{pq} with

given pq is already included into the product, then the factor Q^2_{L-pL-q} is not to be included, and vice versa (notice by the way that $Q^2_{pq} = Q^2_{L-pL-q}$). The above prescription can be seen also comparing fermionic measures in (23) and (25). Respectively, if we multiply the factors Q^2_{pq} over all the points of the momentum lattice with no restrictions, this will yield squared partition function, Q^2. Thus, we find:

$$
Q^2 = \prod_{p=0}^{L-1} \prod_{q=0}^{L-1} \left[(1 + t_1^2)(1 + t_2^2) - 2t_1(1 - t_2^2) \cos \frac{2\pi p}{L} - 2t_2(1 - t_1^2) \cos \frac{2\pi q}{L} \right] . \tag{30}
$$

The trigonometric product (30) is the exact solution for Q^2 in the limit $L^2 \to \infty$. The correspondent free energy per site readily follows:

$$
\begin{aligned}
-\beta f_Q & = \frac{1}{L^2} \ln Q \Big|_{L\to\infty} = \\
& = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \frac{dp}{2\pi} \frac{dq}{2\pi} \ln \left[(1 + t_1^2)(1 + t_2^2) - 2t_1(1 - t_2^2) \cos p - 2t_2(1 - t_1^2) \cos q \right] .
\end{aligned} \tag{31}
$$

This is the free energy for the reduced partition function, Q, while the true free energy per site, for Z, is to be recalculated from $Z = (2 \cosh b_1 \cosh b_2)^{L^2} Q$, and we find:

$$
-\beta f_Z = \ln 2 + \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \frac{dp}{2\pi} \frac{dq}{2\pi} \ln \left[\cosh 2b_1 \cosh 2b_2 - \sinh 2b_1 \cos p - \sinh 2b_2 \cos q \right] , \tag{32}
$$

which is Eq. (108) in [1]. It is not necessary to say that the method we have applied above to obtain (32) significantly differs from the original approach [1]. An interesting comment by Lars Onsager on the history of his remarkable solution can be seen in [30].

In conclusion to this section, let us add few remarks on the properties of the 2D Ising model that follow from the exact solution. In what follows, we assume ferromagnetic case, $b_{1,2} > 0$. As regards the critical behaviour near T_c, there is no essential difference between (31) and (32) since the factor between Q and Z is nonsingular at all temperatures. From (31), it can be then deduced that the point of phase transition is given by the condition:

$$
1 - t_1 - t_2 - t_1 t_2 = 0 , \tag{33}
$$

where $t_1 = \tanh b_1$, $t_2 = \tanh b_2$, with $b_1 = J_1/kT$, $b_2 = J_2/kT$. Equivalently, this condition can be written in the form: $\sinh 2b_1 \cdot \sinh 2b_2 = 1$, which rather corresponds to the free energy in the form (32). The specific heat exhibits the logarithmic singularity as $T \to T_c$:

$$
C/k \simeq A_c |\log \tau| \to \infty , \qquad \tau = \left| \frac{T - T_c}{T_c} \right| \to 0 , \tag{34}
$$

where C/k is the dimensionless specific heat, k is Boltzmann's constant. The parameter A_c is called the specific-heat critical amplitude. The value of A_c is the same by approaching T_c from above and from below even for the anisotropic lattice, this is a particular feature of the 2D Ising model. In the isotropic case ($t_1 = t_2 = \tanh b$) the specific-heat amplitude is a fixed number: $A_c = \frac{8}{\pi} b_c^2 \simeq 0.495$, where $b_c = \frac{1}{2} \ln(1 + \sqrt{2}) \simeq 0.441$ is the inverse critical temperature, $b_c = J/kT_c$. The analytic expression is also known for the spontaneous magnetization in the ferromagnetic 2D Ising model [3, 5]. This expression is simple, though its derivation by any known method is very complicated. This is a yet unsolved puzzle in the two-dimensional Ising model [3]. For related comments also see [20, 25]. The analysis of correlation functions in 2DIM also have been performed within different approaches [8, 11, 31].

6 Ordered products of grassmann factors and gaussian exponentials

In this section we add few more remarks about the ordered products of Grassmann factors typically arising, as we have seen, by the fermionic interpretation of the 2D Ising model within the factorization method. Let L_1 and L_2 be arbitrary linear forms in Grassmann variables. Then we have:

$$(1+L_1)(1+L_2) = e^{L_1L_2}(1+L_1+L_2), \tag{35}$$

where the nilpotent properties of fermions where taken into account. In the above equation the two Grassmann factors are combined into a one Grassmann factor accompanied by a Gaussian exponential. The resulting identity can be iterated further on, and we find:

$$(1+L_1)(1+L_2)(1+L_3)\dots(1+L_N) = \left(1+\sum_{i=1}^{N} L_i\right)\exp\left(\sum_{1\le i<j\le N} L_iL_j\right), \tag{36}$$

where $L_1,\dots,L_N$ are arbitrary linear forms in Grassmann variables. Let $\sigma_0 = \pm 1$ be Ising spin, notice that $\sigma_0^2 = 1$. Making substitution $L_j \to L_j\,\sigma_0$ in (36), we obtain the identity:

$$(1+L_1\sigma_0)(1+L_2\sigma_0)(1+L_3\sigma_0)\dots(1+L_N\sigma_0) = \left(1+\sigma_0\sum_{i=1}^{N} L_i\right)\exp\left(\sum_{1\le i<j\le N} L_iL_j\right). \tag{37}$$

The averaging over the spin states then results:

$$\underset{(\sigma_0)}{\mathrm{Sp}}\left\{(1+L_1\sigma_0)(1+L_2\sigma_0)\dots(1+L_N\sigma_0)\right\} =$$
$$= \exp\left(\sum_{1\le i<j\le N} L_iL_j\right), \qquad \underset{(\sigma_0)}{\mathrm{Sp}}(\dots) = \frac{1}{2}\sum_{\sigma_0=\pm1}(\dots). \tag{38}$$

We see that the averaging of a product of any number of the Grassmannian factors like (38) over spin states, $\sigma_0 = \pm 1$, always results Gaussian fermionic exponential, assuming the Ising spin being the same in all the factors. This property have been used already in the analysis of the 2D Ising models on irregular (in the geometrical sense) planar lattices [24]. The appearance of the Gaussian exponential when we average at the junction in (18) is also evident from (38). In the same manner, we can elaborate the products of Grassmann factors with different spins, like those appearing in (16):

$$(1+L_1\sigma_1)(1+L_2\sigma_2)\dots(1+L_N\sigma_N) = \left(1+\sum_{i=1}^{N} L_i\sigma_i\right)\exp\left(\sum_{1\le i<j\le N} \sigma_i\sigma_j\,L_iL_j\right). \tag{39}$$

This identity is a generalization (or a particular case) of (36). In Eqs. (35)-(39), it is only important that L_1, ..., L_N are the purely anticommuting symbols, satisfying also the nilpotent property. In principle, in the most general case, we may assume in the above identities L_1, ..., L_N to be arbitrary odd polynomials in Grassmann variables.

The identities like (36) and (39) and related may be of interest also with respect to the 2D Ising model in a nonzero magnetic field. The inclusion of the nonzero magnetic field corresponds to the additional terms ... $+\,\beta h\,\sigma_{mn}$ in the hamiltonian (6), which results in the appearance of the additional Boltzmann factors $1 + t_0\,\sigma_{mn}$ in the partition function (9) and (16), which are linear in spin variables. Here $t_0 = \tanh(h)$, and h is conventional magnetic field

in the energy units, $h = \beta\mathcal{H}$, for small field, $t_0 \simeq h$. The appearance of such factors prevents the exact solution since the spin variables can not be easily eliminated from the density matrix (16) in this case. Within approximations, however, it can be expected that $h \neq 0$ will make the spins in the ordered products of Grassmann factors like in (16) and (39) to be "frozen", which will induce the nonlocal terms in the action like in (36). With respect to the problem of a non-zero magnetic field in 2D Ising model, and in view of some other potential applications, it may be therefore of interest to consider the nonlocal fermionic action like the one arising in (36) in the momentum space representation.

For visual convenience, let us change the index in the nonlocal fermionic sum of (36) from ij to mm', with $m, m' = 0, 1, ..., M-1$. The arising nonlocal Gaussian fermionic action is of the form:

$$S_0(L) = \sum_{m=0}^{M-2} L_m\,(L_{m+1} + \dots + L_{M-1}) = \sum_{m=0}^{M-2}\sum_{m'=m+1}^{M-1} L_mL_{m'}\,. \tag{40}$$

It can be expanded over either periodic or aperiodic Fourier exponentials. Assuming the aperiodic Fourier substitution:

$$L_m = \frac{1}{\sqrt{M}}\sum_{p=0}^{M-1} L_p\,e^{i\frac{2\pi}{M}m(p+1/2)} = \frac{1}{\sqrt{M}}\sum_{p=0}^{M-1} L_{M-1-p}\,e^{-i\frac{2\pi}{M}m(p+1/2)}, \tag{41}$$

we find a particularly simple expression:

$$S_0(L) = \sum_{p=0}^{M-1} \frac{L_pL_{M-1-p}}{e^{i\frac{2\pi}{M}m(p+1/2)} - 1}\,. \tag{42}$$

Assuming the periodic Fourier decomposition:

$$L_m = \frac{1}{\sqrt{M}}\sum_{p=0}^{M-1} L_p\,e^{i\frac{2\pi}{M}mp} = \frac{1}{\sqrt{M}}\sum_{p=0}^{M-1} L_{M-p}\,e^{-i\frac{2\pi}{M}mp}, \tag{43}$$

we obtain a similar though somewhat more sophisticated representation with a special role of the $p = 0$ mode:

$$S_0(L) = \sum_{p=1}^{M-1}\left[-L_0L_p + \frac{(L_p - L_0)(L_{M-p} - L_0)}{e^{i\frac{2\pi}{M}p} - 1}\right], \tag{44}$$

where $L_0 = L_{p=0}$. The sums in (42) and (44) can be symmetrized by means of the identity:

$$\frac{1}{e^{ip}-1} = \frac{1}{2}\left[\frac{1}{i\tan(p/2)} - 1\right]. \tag{45}$$

In the above identities it is essential that L_m are the purely anticommuting fermionic forms in Grassmann variables. There are two remarkable features that can be readily observed in the Fourier sums like (42) and (44). First, we may note that (i) though the action (40) is highly nonlocal in the real space, it becomes diagonal in the momentum space. The second interesting feature (ii) is the $1/ip$ singularity in the p-mode of the action near $p = 0$, as $p \to 0$. This $1/ip$ singularity is the essentially fermionic effect, related to the fact that fermions anticommute. The reason for (ii) is that under $p \leftrightarrow -p$ symmetrization fermions just select the skew-symmetric part of the kernel (45), that is, $1/2i\tan(p/2)$, while the contribution of the symmetric part of that kernel vanishes. The situation will be the opposite for bosons. For the application of the above considerations to the 2DIM in a nonzero magnetic field see also the discussion in Sect. 9.

7 Two variables per site

In this section we consider some further modifications for the lattice fermionic interpretation of the 2D Ising model. Eliminating part of the fermionic variables from the basic Gaussian integral (19) for Q, we obtain a reduced Gaussian fermionic integral for the same partition function, Q, but now with only two fermionic variables per site, see (47). In order to eliminate extra fermionic variables in (19), we intend to apply the identity given below in (46). Let a, b be independent Grassmann variables, then:

$$\int db\, da\, e^{ab+aL_1+bL_2} = \int db\, da\, e^{ab}(1+aL_1)(1+bL_2) = \exp\, L_2L_1\,, \tag{46}$$

where L_1, L_2 are arbitrary linear forms in some other Grassmann variables, not involved in the integration, but anticommuting with a, b. Integrating out from (19) the a_{mn}, b_{mn} fields by means of identity (46), we obtain a reduced Gaussian integral for (19) expressed in terms of the remaining variables a^*_{mn}, b^*_{mn}. Let us change the notation for the fields: $a^*_{mn}, b^*_{mn} \to c_{mn}, -c^*_{mn}$, respectively, $da^*_{mn}db^*_{mn} \to -dc_{mn}dc^*_{mn} \to dc^*_{mn}dc_{mn}$, the reduced integral for Q then appears in the form:

$$Q = \int \prod_{m=1}^{L}\prod_{n=1}^{L} dc^*_{mn}dc_{mn}\, \exp \sum_{m=1}^{L}\sum_{n=1}^{L} \Big[c_{mn}c^*_{mn} + (c_{mn}+c^*_{mn})\,(t^{(1)}_{mn}c_{m-1n} - t^{(2)}_{mn}c^*_{mn-1})$$
$$- t^{(1)}_{mn}t^{(2)}_{mn}c_{m-1n}c^*_{mn-1} \Big]\,, \tag{47}$$

where c_{mn}, c^*_{mn} are Grassmann variables, $c_{0n} = 0$, $c^*_{m0} = 0$. The integral (47) is equivalent to (9) and (19), assuming the free boundary conditions in all the cases. Since the inhomogeneous distribution of the bond coupling parameters is still preserved, all the information on the thermodynamic functions as well the correlation functions of the 2D Ising model on a rectangular lattice net is still contained in (47). The integral (47) (as well as (19)) may be of interest with respect to the problem of quenched disorder in the 2D Ising model [32, 33, 26].

For the homogeneous lattice, $t^{(1)}_{mn}, t^{(2)}_{mn} \to t_1, t_2$, the integral (47) becomes:

$$Q = \int \prod_{m=1}^{L}\prod_{n=1}^{L} dc^*_{mn}dc_{mn}\, \exp \sum_{m=1}^{L}\sum_{n=1}^{L} \Big[c_{mn}c^*_{mn} - t_1\, c_{m-1n}c^*_{mn} - t_2\, c_{mn}c^*_{mn-1}$$
$$- t_1t_2\, c_{m-1n}c^*_{mn-1} + t_1\, c_{mn}c_{m-1n} + t_2\, c^*_{mn-1}c^*_{mn} \Big]\,. \tag{48}$$

This integral can be calculated by analogy with (21). We assume again the periodic closing conditions for fermions and pass to the momentum space by Fourier substitution:

$$c_{mn} = \frac{1}{\sqrt{L^2}} \sum_{p=0}^{L-1}\sum_{q=0}^{L-1} c_{pq}\, e^{-i\frac{2\pi p}{L}m + i\frac{2\pi q}{L}n}\,, \quad c^*_{mn} = \frac{1}{\sqrt{L^2}} \sum_{p=0}^{L-1}\sum_{q=0}^{L-1} c^*_{pq}\, e^{+i\frac{2\pi p}{L}m - i\frac{2\pi q}{L}n}\,. \tag{49}$$

The choice of the signs of pq is here adopted for future convenience in (50). The orthogonality relations (24) are to be taken into account. In the momentum space, the integral (48) becomes:

$$Q = \int \prod_{p=0}^{L-1}\prod_{q=0}^{L-1} dc^*_{pq}dc_{pq}\, \exp \Big\{ \sum_{p=0}^{L-1}\sum_{q=0}^{L-1} \Big[c_{pq}c^*_{pq}\,(1 - t_1\, e^{i\frac{2\pi p}{L}} - t_2\, e^{i\frac{2\pi q}{L}} - t_1t_2\, e^{i\frac{2\pi p}{L} + i\frac{2\pi q}{L}})$$
$$+ t_1\, e^{i\frac{2\pi p}{L}}\, c_{L-pL-q}c_{pq} + t_2\, e^{i\frac{2\pi q}{L}}\, c^*_{pq}c^*_{L-pL-q} \Big] \Big\}\,. \tag{50}$$

Then we have to make the $p, q \leftrightarrow L-p, L-q$ symmetrization of the action in order to single out explicitly the independent subsets of the variables. The integral then decouples into a product of simplest Gaussian fermionic integral factors:

$$Q^2_{pq} = \int dc^*_{pq}dc_{pq}dc^*_{L-pL-q}dc_{L-pL-q}\, \exp \Big[c_{pq}c^*_{pq}\,(1 - t_1\, e^{i\frac{2\pi p}{L}} - t_2\, e^{i\frac{2\pi q}{L}} - t_1t_2\, e^{i\frac{2\pi p}{L} + i\frac{2\pi q}{L}}) +$$

$$+ c_{L-pL-q}c^*_{L-pL-q}\,(1 - t_1\, e^{-i\frac{2\pi p}{L}} - t_2\, e^{-i\frac{2\pi q}{L}} - t_1t_2\, e^{-i\frac{2\pi p}{L} - i\frac{2\pi q}{L}}) +$$
$$+ 2\,i\,t_1\, \sin\frac{2\pi p}{L}\, c_{L-pL-q}c_{pq} + 2\,i\,t_2\, \sin\frac{2\pi q}{L}\, c^*_{pq}c^*_{L-pL-q} \Big]\,. \tag{51}$$

This integral factor can be evaluated making use of the elementary rules like (1) and/or (46). Alternatively, if we decide to interpret this integral as the determinant, then we have to present the action in the form: $S = a\,A\,a^*$, where A is a two by two matrix. This is possible, for instance, assuming the correspondence: $a_1, a_2, a^*_1, a^*_2 \leftrightarrow c_{pq}, c^*_{L-pL-q}, c^*_{pq}, -c_{L-pL-q}$. The calculation is very simple in any case, and we find:

$$Q^2_{pq} = \Big|1 - t_1\, e^{i\frac{2\pi p}{L}} - t_2\, e^{i\frac{2\pi q}{L}} - t_1t_2\, e^{i\frac{2\pi p}{L} + i\frac{2\pi q}{L}}\Big|^2 - 4t_1t_2\, \sin\frac{2\pi p}{L}\, \sin\frac{2\pi q}{L} =$$
$$= (1+t_1^2)(1+t_2^2) - 2t_1(1-t_2^2)\, \cos\frac{2\pi p}{L} - 2t_2(1-t_1^2)\, \cos\frac{2\pi q}{L}\,. \tag{52}$$

The squared partition function follows as the product of factors (52) over the whole momentum-space lattice. The factor in the final line of (52) is the same as in (29). So, we come again to the same results for the partition function and the free energy of the standard 2D Ising model on a rectangular lattice that have been commented already in Sect. 5. An interesting new feature in (52) is the trigonometric expression in the first line, wherefrom it is easy to recognize the all possible critical modes (zeroes of Q^2_{pq}) in the ferromagnetic as well as antiferromagnetic regimes. Assuming p, q to be normalized to the 2π interval, there are four such possible critical modes: $(p,q) = (0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π). In the ferromagnetic case, the only possible critical mode is that with $p = q = 0$, and the criticality condition is given by (33) (notice that $0 \equiv 2\pi$). The other three modes, being always positive in the ferromagnetic case ($t_{1,2} > 0$), define the possible critical points in the antiferromagnetic cases. For a related discussion also see [20, 25].

8 Continuum Limit

In this section we consider the continuum-space limit (low momenta sector) of the exact lattice theory near T_c. A suitable starting point is the integral (48) for Q. In what follows, we assume the homogenous case and ferromagnetic interactions. Let $x_{mn} = c_{mn}, c^*_{mn}$, we define lattice derivatives in a natural way: $\partial_m x_{mn} = x_{mn} - x_{m-1n}$, $\partial_n x_{mn} = x_{mn} - x_{mn-1}$. Substituting $c_{m-1n} = c_{mn} - \partial_m c_{mn}$, $c^*_{mn-1} = c^*_{mn} - \partial_n c^*_{mn}$ into (48), we find the action in the form:

$$S = \sum_{mn} \Big[\underline{m}\, c_{mn}c^*_{mn} - \lambda_1\, c^*_{mn}\partial_m c_{mn} + \lambda_2\, c_{mn}\partial_n c^*_{mn} - t_1\, c_{mn}\partial_m c_{mn}$$
$$+ t_2\, c^*_{mn}\, \partial_n c^*_{mn} - t_1\, t_2\, (\partial_m c_{mn})(\partial_n c^*_{mn}) \Big]\,, \tag{53}$$

with the following set of parameters:

$$\underline{m} = (1 - t_1 - t_2 - t_1t_2)\,, \quad \lambda_1 = t_1(1+t_2)\,, \quad \lambda_2 = t_2(1+t_1)\,. \tag{54}$$

The lattice action (53) is still the exact expression. In this action one can already distinguish the typical field-theoretical like structures, with the mass term and kinetic part. Evidently, the parameter $\underline{m}$ plays the role of mass, while λ_1, λ_2 and t_1, t_2 are the kinetic coefficients. The critical point can be readily guessed to be $\underline{m} = 0$, in agreement with (33). Let us take the formal limit to the continuum space:

$$mn \to x = (x_1, x_2)\,, \qquad \sum_{mn} \to \int d^2x = \int dx_1 dx_2\,,$$

$$\partial_m \to \partial_1 = \partial / \partial x_1 , \quad \partial_n \to \partial_2 = \partial / \partial x_2 ,$$

$$c_{mn} , c^*_{mn} \to \psi(x), \bar{\psi}(x) \to \psi , \bar{\psi} . \tag{55}$$

The continuum-limit counterpart for the lattice action (53) then appears in the form:

$$S = \int d^2x \left[\underline{m} \psi\bar{\psi} - \lambda_1 \psi \partial_1 \bar{\psi} + \lambda_2 \psi \partial_2 \bar{\psi} - t_1 \psi \partial_1 \psi + t_2 \bar{\psi} \partial_2 \bar{\psi} \right] . \tag{56}$$

This is the Majorana-like continuum action for two-component massive fermions. In the above continuum action we have dropped an interesting second-order momentum term with $\partial_1 \partial_2$. The mass and other parameters are the same as in (54). In presenting the action in the final form, we have as well applied the rule $\int d^2x (a\partial b) = \int d^2x (b\partial a)$, where $\partial = \partial_1, \partial_2$ and a, b are any fermionic fields. This simple rule can be checked by integration by parts, taking into account that fermions anticommute and neglecting the boundary effects. Alternatively, one can check the above rule in lattice interpretation. In (56) the momenta operators ∂_1, ∂_2 in all cases act to the right. The continuum-space action (56) captures the basic features of the exact lattice theory with action (48) in the low-momentum sector near the critical point, which is responsible for the critical-point singularities in the thermodynamic functions and the large-distance behaviour of correlations. In the momentum space, this corresponds to approximation like $e^{ip} - 1 \simeq ip$, $e^{iq} - 1 \simeq iq$, assuming also the ultraviolet cut-off in the momentum integrals, $|p| \leq k_0$, with k_0 of order 1 (or say $\pi/4$) or less.

The Majorana like action (56), however, is not in the canonical form. It can be brought into a canonical form by a suitable linear transformation of the fields, eliminating the undesirable kinetic terms like $\psi_1 \partial_1 \bar{\psi}$, $\psi \partial_2 \bar{\psi}$. In the canonical form, the 2D Majorana action (56) is given as follows:

$$S = \int d^2x \left[\bar{m} \psi_1 \psi_2 + \psi_1 \frac{1}{2} (\partial_1 + i \partial_2) \psi_1 + \psi_2 \frac{1}{2} (-\partial_1 + i \partial_2) \psi_2 \right] , \tag{57}$$

with the new Majorana components, ψ_1, ψ_2, and the rescaled mass:

$$\bar{m} = \frac{1 - t_1 - t_2 - t_1 t_2}{\sqrt{2 (t_1 t_2)_c}} . \tag{58}$$

In order to pass from (56) to (57), we have to transform the fermionic fields and the momenta operators ∂_1, ∂_2 in a suitable way. Here we comment shortly on this transformation. For the first step, making use of the rescaling of the fermionic fields like $\psi \to e^{\frac{\mu}{2}} \psi$, $\bar{\psi} \to e^{-\frac{\mu}{2}} \bar{\psi}$, with properly chosen μ, we write the action (56) in a more symmetric form:

$$S = \int d^2x \left[\underline{m} \psi_1 \bar{\psi}_2 + \psi (-\lambda_1 \partial_1 + \lambda_2 \partial_2) \bar{\psi} + \omega_0 (\psi (-\lambda_1 \partial_1) \psi + \bar{\psi} (\lambda_2 \partial_2) \bar{\psi}) \right] , \tag{59}$$

with new kinetic parameter:

$$\omega_0 = \sqrt{\frac{t_1 t_2}{\lambda_1 \lambda_2}} = \frac{1}{\sqrt{(1 + t_1)(1 + t_2)}} \quad \to \quad (\omega_0)_c = \frac{1}{\sqrt{2}} . \tag{60}$$

Exactly at T_c, independently of the rate of the lattice anisotropy, we have $(\omega_0)_c = \frac{1}{\sqrt{2}}$, since $(1 - t_1 - t - t_1 t_2)_c = 0$. Taking into account that the continuum-limit formulation by itself is reliable only near T_c, in what follows we put: $\omega_0 = 1/\sqrt{2} = (\omega_0)_c$.

The action (59) is already in a suitable form to be transformed into the canonical Majorana action (57). The general idea is to introduce the new fields by a linear substitution like

$$\psi = u (\gamma a + \bar{\gamma} \bar{a}) , \quad \bar{\psi} = u (\eta a + \bar{\eta} \bar{a}) , \tag{61}$$

where γ, $\bar{\gamma}$, η, $\bar{\eta}$ are free parameters (four complex numbers) and $a, \bar{a}$ are the new anticommuting components (we shall pass $a, \bar{a} \to \psi_1, \psi_2$ at next stages). Substituting (61) into (59), we then look for the uniformization condition that the undesirable terms like $a \partial_1 \bar{a}$, $a \partial_2 \bar{a}$ do not appear in transformed action. In essence, the idea is similar to that of the Bogoliubov transformation in the theories of superfluidity [34] and superconductivity [35]. It appears that the uniformization requirement, in any case, implies $\gamma\bar{\gamma} = \eta\bar{\eta}$ (the rule $a\partial b = b\partial a$ is not to be forgotten at this stage). We then put $\gamma\bar{\gamma} = \eta\bar{\eta} = 1$, assuming the remaining normalization parameter, u, to be fixed by the condition $\psi\bar{\psi} = a\bar{a}$, whence $d\bar{\psi}\, d\psi = d\bar{a}\, da$, which is the condition that transformation (61) is canonical, that is, the fermionic measure is unchanged. In this way we come to the uniformization condition in the form:

$$(\gamma\bar{\eta} + \bar{\gamma}\eta) + 2\omega_0 = 0 , \quad \omega_0 = \sqrt{\frac{t_1 t_2}{\lambda_1 \lambda_2}} , \tag{62}$$

while u is then fixed by the condition:

$$u^2 (\gamma\bar{\eta} - \bar{\gamma}\eta) = 1 . \tag{63}$$

The momenta ∂_1, ∂_2 also will be transformed, in general, under the transformation of the fields like (61). The correspondent relations are not shown, however, since in our particular case, with $\omega_0 = 1/\sqrt{2}$, this momenta transformation appears to be identical. Assuming $\omega_0 = 1/\sqrt{2}$, we find that a possible realization for (61)-(63) is the following substitution:

$$\psi = \frac{a + \bar{a}}{\sqrt{2i \sin(\pi/4)}} , \quad \bar{\psi} = - \frac{a\, e^{i(\pi/4)} + e^{-i(\pi/4)} \bar{a}}{\sqrt{2i \sin(\pi/4)}} , \tag{64}$$

which corresponds to $\gamma = 1, \eta = -\frac{1}{\sqrt{2}}(1 + i)$ in (62) and (63). Substituting (64) into (59), the action appears in the form:

$$S = \int d^2x \left[\underline{m}\, a\bar{a} + \tfrac{1}{2} a (\lambda_1 \partial_1 + i \lambda_2 \partial_2) a + \tfrac{1}{2} \bar{a} (-\lambda_1 \partial_1 + i \lambda_2 \partial_2) \bar{a} \right] , \tag{65}$$

where $a, \bar{a}$ are the new Majorana fields, and $\psi\bar{\psi} \to a\bar{a}$, $d\bar{\psi} d\psi \to d\bar{a}\, da$. A remarkable feature is that the momenta components ∂_1, ∂_2 in (65) are not effected, they are the same as in the original action (56) and in (59), which is provided by a special value of the kinetic parameter, $\omega_0 \to (\omega)_c = 1/\sqrt{2}$. The axis in the $d^2x \leftrightarrow d^2p$ space will be rescaled and rotated, in general, by the uniformization transformation under substitution like (61) with $\omega_0 \neq 1/\sqrt{2}$. The uniformization condition (62) by itself still provide some freedom corresponding to the gauge rotation of the fields: $a \to a\, e^{i\alpha}$, $\bar{a} \to \bar{a}\, e^{-i\alpha}$ in (65), or $\psi_1 \to \psi_1 e^{i\alpha}$, $\psi_2 \to \psi_2 e^{-i\alpha}$ in (57), accompanied by the covariant orthogonal rotation of the reference frame of momenta $\lambda_1 \partial_1, \lambda_2 \partial_2$. In (64) and (65) this freedom is fixed in such a way that the axis are not rotated.

Now, let us rescale the momenta in (65) as follows: $\partial_1 \to (\lambda_2/\lambda_1)^{1/2} \partial_1$, $\partial_2 \to (\lambda_1/\lambda_2)^{1/2} \partial_2$, which is canonical transformation ($d^2x \to d^2x$, $d^2p \to d^2p$). The rescaled momenta are those that finally appear in the canonical action (57). By this rescaling, we gain a new overall kinetic factor, let us call it ω_1, given by:

$$\omega_1 = \sqrt{\lambda_1 \lambda_2} = \sqrt{t_1 t_2 (1 + t_1)(1 + t_2)} \quad \to \quad (\omega_1)_c = \sqrt{2 (t_1 t_2)_c} , \tag{66}$$

it is reasonable to fix ω_1 at $T = T_c$ as is indicated above. For the last step, we remove ω_1 from the kinetic part by the rescaling of the fields (first changing notation for the components): $a, \bar{a} \to \psi_1, \psi_2 \to \psi_1/\sqrt{\omega_1}$, $\psi_2/\sqrt{\omega_1}$, and obtain the Majorana action in the canonical form (57). Respectively, the mass $\underline{m}$ from (56) will get renormalized to give the rescaled mass $\bar{m}$ (58).

The canonical two-component Majorana action (57) can be written as well in matrix notation. Noting that $\bar{m}\,\psi_1\psi_2 = \frac{1}{2}\bar{m}(\psi_1\psi_2-\psi_2\psi_1)$ and introducing the matrix structure at each space point in the d^2x integral, we write:

$$S_{\rm major} = \frac{1}{2}\int d^2x \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}^T \left[\begin{pmatrix}\partial_1 + i\partial_2 & \bar{m}\\ -\bar{m} & -\partial_1 + i\partial_2\end{pmatrix}\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{67}$$

where $(\,)^T$ stands for transposition of spinor. In terms of the standard Pauli matrices, $\sigma_1, \sigma_2, \sigma_3$, the matrix kernel of this action (the 'inverse propagator', or 'equation of motion') can be written in the form: $[\,\bar{m}\,(i\,\sigma_2) + \partial_1\,(\sigma_3) + i\,\partial_2\,(1)\,]$, or in the form: $(i\,\sigma_2)\,[\bar{m} + \partial_1\,(\sigma_1) + \partial_2\,(\sigma_2)]$. Thus, we find:

$$S_{\rm major} = \tfrac{1}{2}\int d^2x\,\bar{\Psi}\,[\,\bar{m} + \hat{\partial}\,]\,\Psi\,, \quad \bar{\Psi} = \Psi^T(i\,\sigma_2), \quad \hat{\partial} = \gamma_1\,\partial_1 + \gamma_2\,\partial_2\,, \tag{68}$$

with the 2D γ-matrices $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$. This is the 2D Majorana action in the relativistic field-theoretical form. The conjugated Majorana spinors $\bar{\Psi}$ and Ψ in (68) are built in fact from the same component fields, ψ_1, ψ_2, so they are not the truly independent fields in the path integral. By doubling the number of fermions in the Majorana representation we can pass to the Dirac action with four independent components:

$$S_{\rm dirac} = \tfrac{1}{2}\int d^2x\,\bar{\Psi}\,(x)\,[\,\bar{m} + \hat{\partial}\,]\,\Psi\,(x)\,, \tag{69}$$

where $\Psi = (\psi_1, \psi_2)$ and $\bar{\Psi} = (\psi_1^*, \psi_2^*)^T$ are now charged Dirac spinors with four independent anticommuting components: $\psi_1, \psi_2, \psi_1^*, \psi_2^*$. By convention, one can assume ψ_1^*, ψ_2^* to be complex conjugates of ψ_1, ψ_2. The propagator $\bar{m} + \hat{\partial}$ in (69) is the same as in (68). To obtain the Dirac action (69), we take two identical copies S' and S'' of the Majorana action (68) and write: $S_{\rm dirac} = (S' + S'')_{\rm majorana}$. Introducing the new Dirac fields by means of substitution:

$$\Psi = \frac{1}{\sqrt{2}}(\Psi' + i\,\Psi''), \qquad \bar{\Psi} = \frac{1}{\sqrt{2}}(\bar{\Psi}' - i\,\bar{\Psi}''), \tag{70}$$

we obtain the action (69). Mathematically, the transformation from (68) to (69) is in essence the same that we have considered in relation to identity (4) in Sect. 2, which establishes the connection between the fermionic Gaussian integrals of the first and second kind. The transformations from lattice to continuum in 2DIM are also discussed in [21, 25].

9 Critical-point singularities

The field-theoretical formulation for the 2D Ising model near T_c is a suitable representation to discuss the thermodynamic singularities near the transition point. Assuming that we start with the Dirac interpretation (squared partition function) and noting that in the momentum space $\det(\bar{m} + \hat{\partial}) = \bar{m}^2 + p^2$, the singular part of the free energy readily follows in the form:

$$-\beta\, f_{\rm sing} = \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\,\ln(\bar{m}^2 + p^2) = \frac{1}{8\pi}\,\bar{m}^2 \ln\frac{\rm const}{\bar{m}^2} + (...)\,, \tag{71}$$

where the mass $\bar{m}$ is that given in (58). This is the exact expression for the most singular part of the free energy of the 2DIM in a zero magnetic field ($h = 0$). The same asymptotics follows from the exact solution. Noting that near the critical point $\bar{m} \sim \tau = |T - T_c|/T_c$, let us assume $\bar{m}$ for conventional temperature. The internal (average) energy is then given by:

$$(E)_{\bar{m}\ \rm sing} = \frac{\partial}{\partial \bar{m}}(-\beta f_{\rm sing}) = \int \frac{d^2p}{4\pi^2}\,\frac{\bar{m}}{\bar{m}^2 + p^2} = \frac{\bar{m}}{4\pi}\ln\frac{\rm const}{\bar{m}^2} + (...)\,, \tag{72}$$

and the dimensionless specific heat is:

$$(C/k)_{\bar{m}\ \rm sing} = \frac{1}{4\pi}\,\ln\frac{\rm const}{\bar{m}^2} + (...) = \frac{1}{2\pi}\,\ln\frac{\rm const}{|\bar{m}|} + (...)\,, \tag{73}$$

where we put the subscript $\bar{m}$ to remember that the derivatives are taken with respect to the conventional temperature $\bar{m}$. For the isotropic lattice, $\bar{m} \simeq 4\,(b - b_c) \simeq 4b_c\tau$, with $b_c = \frac{1}{2}\ln(1 + \sqrt{2})$, wherefrom one can recover, for instance, the specific-heat asymptotics (34) with the correct value of the amplitude: $(C/k)_{\rm sing} = A_c\,|\ln|\tau||$, $A_c = (8/\pi)\,b_c^2$.

The asymptotics (71)-(73) are to be compared with the hypothetical form of the critical-point singularities in the same functions in a nonzero magnetic field near T_c, which subject we intend to discuss, in short, in the remaining part of this section. We are interesting merely in what may be the singular behaviour of the specific heat in a nonzero magnetic field along the critical isotherm, that is, when the temperature is fixed exactly at T_c and the deviation from the critical point is realized by a small nonzero magnetic field, $h \neq 0$. To start with, let us write the expected form for the singular part of the free energy near the critical isotherm, in the regime of the "strong" magnetic field, $\tau^{15/8} \ll h \ll 1$:

$$-\beta\, f_{\rm sing} = \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\,\ln\left(\bar{m}^2 + p^2 + \frac{\lambda^2}{p^2}\right) + (...)\,, \tag{74}$$

with $\lambda \propto h\,M(\tau, h)$, where $M(\tau, h)$ is magnetization, $\tau \to 0$. More precisely, both τ and h are assumed to be small, but we are interesting in the situation near the critical isotherm, $\tau = 0, h \neq 0$, and introduce infinitesimal deviation from T_c with respect to the temperature, $\tau \ll h^{8/15}$, merely to perform the differentiation, then we put $\tau \to 0$. The same form of the free energy can be considered for "weak" field, $h^{8/15} \ll \tau$. In this case, however, the choice of λ as function of τ, h may be more sophisticated. In particular, this choice may be nontrivial in the ordered phase, where the effects of the external field are superimposed on the effects of the inherent molecular field [36].

A somewhat unusual perturbation term λ^2/p^2 which appears in the propagator in (74) is the result of an approximation in the mixed spin-fermion representation for $Q_{h\neq 0}$. The insertion of the $h > 0$ weights like $1 + h\,\sigma_{mn}$ into the factorized density matrix in (16) prevents the exact solution, as it was already commented in Sect. 6. We then have elaborated the ordered products of factors from (16) into an exponential form, cf. the discussion in Sect. 6, and then applied the simplest approximation of the Hartree-Fock type for the spin subsystem. In particular, this kind of approximation implies $\lambda \simeq hM(\tau, h)$. The nonlocal Gaussian exponentials, like those considered in Sect. 6, then appear in the action. This, roughly, corresponds to the modification of the Majorana action of the following kind:

$$S = \tfrac{1}{2}\int d^2x\,\bar{\Psi}\,[\bar{m} + \hat{\partial} + \lambda/\hat{\partial}\,]\,\Psi\,. \tag{75}$$

This form of the action is not be understood too literally, the less singular λ-corrections are ignored (or incorporated in λ). The main statement is that the free energy appears with the perturbed propagator as is given in (74). It may be noted that the parameter λ in (74) and (75) is rather charge then mass. The free energy in the form (74) might be of interest also at D$\neq$2. Now, let us assume (74) to be true and consider what follows.

In the strong-coupling regime ($\lambda \gg \frac{1}{2}\bar{m}^2$) the internal energy per site is given by:

$$(\,E\,)_{\bar{m}\ \rm sing} = \frac{\bar{m}}{4\pi}\int \frac{p^2\,dp^2}{\lambda^2 + \bar{m}^2p^2 + p^4} + \frac{1}{8}\,\frac{\partial\lambda}{\partial\bar{m}} + (...)\,. \tag{76}$$

Respectively, the specific heat at the critical isotherm ($\bar{m} \to 0$) appears in the form:

$$(C_{\bar{m}})_{\rm sing} = \frac{1}{4\pi}\int \frac{p^2\,dp^2}{\lambda^2+(p^2)^2} + (\ldots) = \frac{1}{8\pi}\ln\frac{\rm const}{\lambda^2} + (\ldots)$$
$$= \frac{1}{2\pi}\,|\ln\sqrt{\lambda}| + (\ldots)\,, \quad \sqrt{\lambda} \propto h^{8/15} \to 0\,, \quad \bar{m} = 0\,, \tag{77}$$

where $\lambda(0,h) \propto h\,M(0,h)$, or $\lambda(0,h) \propto h^{16/15}$, and we have passed in the final line to $\sqrt{\lambda} \propto h^{8/15} \to 0$ in order to make the amplitude to be equal to that in (73). It is known from scaling and other considerations that $M(0,h) \propto h^{1/15}$, wherefrom $\lambda(0,h) \propto h^{16/15}$. Comparing (77) with (73), we see that under given approximation the specific heat along the critical isotherm is logarithmic and can be formally recovered from (73) by replacing the thermal mass $\bar{m} = \bar{m}(\tau,0) \sim \tau$ by the "magnetic mass" $\sqrt{\lambda}(0,h) \sim h^{8/15}$. The amplitude in (77) remains the same (with respect to the mass parameters) as in (73). The specific heat (77) is obtained by $\partial^2/\partial\bar{m}^2$, where $\bar{m}$ is given in (58). Formally, the asymptotics (77) includes the case of the anisotropic lattice as well. For the isotropic lattice, $\bar{m} \simeq 4\,(b-b_c)$, $b = J/kT$, and we have to multiply the amplitude from (77) by factor $16b_c^2$ to obtain the true specific heat along the critical isotherm:

$$(C/k)_{\rm sing} = \frac{8}{\pi}\,b_c^2\,|\ln h^{8/15}| = E_c\,|\ln h|\,, \quad E_c = \frac{8}{15}\,A_c = \frac{64}{15\pi}\,b_c^2\,, \tag{78}$$

where $A_c = (8/\pi)b_c^2$ is the thermal critical amplitude along the critical isobar, $\tau \neq 0, h = 0$, $A_c = 0.494538589$, while $E_c = (8/15)A_c = 0.263753914$ is the amplitude along the critical isotherm, $\tau = 0, h \neq 0$, as it appears within given approximation. Here $b_c = \frac{1}{2}\ln(1+\sqrt{2}) = 0.440686793$ is the inverse critical temperature, $b_c = J/kT_c$. It may be interesting to check (78) by the Monte-Carlo experiments and other numerical methods. The specific heat (77) is obtained, formally, by differentiating with respect to $\bar{m}$ placed in front of the integral in (76) and then taking the limit $\bar{m} \to 0$. The other less singular corrections to the specific heat are ignored in (77) and (78).

Curiously, we could guess (77) from the most crude phenomenological considerations, simply replacing the "thermal" mass $m_\tau \sim \tau$ from (73) by the "magnetic" mass $\bar{m}_h \sim h^{8/15}$. This replacement can not be done, however, at least in a simple form, in the free energy like (71), since this will yield the expressions with the logarithmic corrections in the functions related to the magnetization at the critical isotherm, which hardly is the case. The unusual form of the magnetic-field correction λ^2/p^2 in the propagator in (74), versus a naive modification of mass term in $\bar{m}^2 + p^2$, is in fact favorable with respect to the known data about the Ising model. Merely, this concerns the absence of the logarithms, observed or expected, in the field derivatives of the free energy.

The 2D Ising model at T_c can also be considered in terms of the conformal field theory (CFT) axiomatics [37, 38, 39, 40]. Zamolodchikov [40] has conjectured the existence of the eight masses $m_j \sim h^{8/15}$ ($j = 1,2,\ldots 8$) in the perturbed CFT assumed to be in the same universality class as the 2DIM at the critical isotherm, $\tau = 0, h \neq 0$. A remarkable feature is that the ratios of these masses are predicted from the symmetries as the exact numbers up to the overall normalization constant: $m_2/m_1 = 2\cos\frac{1}{5}\pi$, $m_3/m_1 = 2\cos\frac{1}{30}\pi$, etc [40]. The nature of these masses from the point of view of the original lattice formulation of 2DIM is yet not well understood. If these masses are thought out as the result of some kind of fine splitting of the λ term in the propagator in (74), their effect on the behaviour of the correlations might be different, as compared with the thermal mass effect, since λ^2 is not the same that $\bar{m}^2$ in (74). If so, the naive expectation that the asymptotics of the two-point correlation functions will be given, by analogy with thermal decay of correlations, by the sum of the terms like $K_0(m_i\,R)$,

where K_0 is modified Bessel function, may be not the case. It is difficult to make definite predictions, however, at present stage, what may be the modifications. The approximations like (74) seem to be two crude in this respect. It might be conjectured, for instance, that some of the masses (probably all except the lightest or the heaviest one) might have imaginary parts and will then contribute only either more rapidly decaying additive corrections to the leading term (with extra factors $1/R$) or the corrections with the oscillating formfactors (with the same periods $R_i \sim m_i^{-1}$ as the decay rates in the accompanying exponentials) to the term with 'normal' decay, like $K_0(m_1R)$, which is what can also be expected from common scaling.

In principle, taking the free energy in the form (74) as it is, one can try to analyze other thermodynamic functions. However, this will claim for further fine detailing of the meaning of λ as a function of both τ and h. In particular, the effects related to the possible spontaneous ordering are to be taken into account properly below T_c. An interesting feature is that at a special line $\lambda = \frac{1}{2}\bar{m}^2$ the free energy (74) reproduces, in essence, the same results (71)-(73) as at $\lambda = 0$, that is, at $h = 0$. This might be an evidence for the possibility to incorporate the effects of the spontaneous ordering in this scheme. We are going to discuss these subjects in a more detail elsewhere. In fact, the line $\lambda = \frac{1}{2}\bar{m}^2$ distinguishes between the weak-field and strong-field regimes, with respect to τ, in the integral (74). At this boundary, $\tau \sim h^{8/15}$, this is just what one can expect for this boundary from scaling and other considerations [36].

10 Conclusions

In the above discussion, the two-dimensional Ising model (2DIM) has been treated as a theory of free fermions on a lattice. The use was made of the anticommuting (Grassmann) variables and integrals. The fermionization procedure is based on the mirror-ordered fermionic factorization of the density matrix. Following this method, the original spin-variable partition function Q with arbitrary inhomogeneous set of bond coupling parameters was transformed into a Gaussian fermionic integral. The subsequent discussion includes the momentum-space analysis and the exact solution for the standard (translationally invariant) rectangular 2D Ising lattice, the free fermion representation for Q with two variables per site, the Majorana-Dirac field theory interpretation of the 2DIM near T_c (continuum limit). The effects of the long-range fermionic correlations in a nonzero magnetic field and the behaviour of the specific heat along the critical isotherm also have been discussed. Grassmann variables provide a powerful tool to analyze the 2DIM. In physical aspect, it seems to be important to understand better the mechanism of the spontaneous ordering in 2DIM in terms of fermions. The fermionic interpretation of the 2D Ising model provides grounds for this model to be treated in a common range with some other typical models in condensed matter physics and quantum field theory.

References

[1] L. Onsager, Phys. Rev. **65**, 117 (1944).
[2] B. Kaufman, Phys. Rev. **76**, 1232 (1949).
[3] C. N. Yang, Phys. Rev. **85**, 808 (1952).
[4] M. Kac, J. C. Ward, Phys. Rev. **88**, 1332 (1952).
[5] E. W. Montroll, R. B. Potts, J.C. Ward, J. Math. Phys. **4**, 308 (1963).
[6] C. A. Hurst, H. S. Green, J. Chem. Phys. **33**, 1059 (1960).
[7] T. D. Schultz, D. C. Mattis, E. H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
[8] T. T. Wu, B. M. McCoy, C. A. Tracy, E. Barouch, Phys. Rev. B **13**, 316 (1976).
[9] K. Huang, Statistical Mechanics (Wiley, New York, 1963).

[10] H. S. Green, C. A. Hurst, Order-Disorder Phenomena (Interscience, New York, 1964).
[11] B. M. McCoy, T. T. Wu, The Two-Dimensional Ising Model (Harward U. Press, Cambridge, Mass., 1973).
[12] D. C. Mattis, The Theory of Magnetism II (Springer, Berlin, 1985).
[13] F. A. Berezin, Russ. Math. Surveys, **24**, No. 3, 1 (1969).
[14] J. B. Zuber, C. Itzykson, Phys. Rev. D **15**, 2875 (1977).
[15] E. Fradkin, M. Srednicki, L. Susskind, Phys. Rev. D **21**, 2885 (1980).
[16] S. Samuel, J. Math. Phys. **21**, 2806 (1980).
[17] C. Itzykson, Nucl. Phys. B **210** [FS6], 448 (1982).
[18] V. N. Plechko, Sov. Phys. Doklady, **30**, 271 (1985).
[19] V. N. Plechko, Theor. Math. Phys. **64**, 748 (1985).
[20] V. N. Plechko, Physica A, **152**, 51 (1988).
[21] C. Itzykson, J.-M. Drouffe, Statistical Field Theory (Cambridge U. Press, Cambridge, 1989).
[22] A. M. Tsvelik, Quantum Field Theory in Condensed Matter Physics (Cambridge U. Press, Cambridge, 1995).
[23] A. I. Bugrij, V. N. Shadura, Phys. Let. B, **150**, 171 (1990).
[24] V. N. Plechko, Fermions on Irregular 2D Ising Lattices. In: Selected Topics in Statistical Mechanics, Ed. by A. A. Logunov et al (World Scientific, Singapore, 1990) p. 489.
[25] V. N. Plechko, J. Phys. Studies, **1**, 554 (1997).
[26] V. N. Plechko, Phys. Lett. A, **239**, 289 (1998); (E) **245**, 563 (1998).
[27] F. A. Berezin, The Method of Second Quantization (Academic, New York, 1966).
[28] E. Caianiello, Nuovo Cim. Suppl., **14**, 177 (1959).
[29] E. R. Caianiello, Combinatorics and Renormalization in Quantum Field Theory (Benjamin, London, 1973).
[30] L. Onsager, In: Critical Phenomena in Alloys, Magnets and Superconductors, Ed. by R. E. Mills, E. Ascher, R. I. Jaffe (McGraw-Hill, New York, 1971), p. XIX-XXIV; p. 3–12.
[31] L. P. Kadanoff, H. Ceva, Phys. Rev. B **3**, 3918 (1971).
[32] B. N. Shalaev, Phys. Rep. **237**, 129 (1994).
[33] Vik. S. Dotsenko, Sov. Phys. Usp. **38**, 310 (1995).
[34] N. N. Bogolubov, J. Phys. USSR **11**, 23 (1947).
[35] N. N. Bogolubov, JETP **34**, 58 (1958).
[36] V. N. Plechko, Phys. Lett. A **89**, 373 (1982).
[37] C. Itzykson, H. Saleur, J.-B. Zuber, Eds., Conformal Invariance and Applications to Statistical Mechanics (World Scientific, Singapore, 1988).
[38] J. L. Cardy, In: Fields, Strings and Critical Phenomena, Les Houches, Session XLIX, Eds. E. Brézin, J. Zinn-Justin (Elsevier, Amsterdam, 1989).
[39] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
[40] A. B. Zamolodchikov, Adv. Stud. Pure Math. **19**, 641 (1989).

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