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I.V.Barashenkov, E.V.Zemlyanaya

STABLE COMPLEXES OF PARAMETRICALLY DRIVEN, DAMPED NONLINEAR SCHRÖDINGER SOLITONS

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Motivation. Bound states of solitons and solitary pulses are attracting increasing attention in nonlinear optics [1, 2, 3, 4, 5], dynamics of fluids [6, 7, 8, 9] and excitable media [10]. Stable bound states can compete with free solitons as alternative attractors. This is detrimental in nonlinear optics, for example, where the interaction between adjacent pulses poses limitations to the stable operation of transmission lines and information storage elements. Unstable solitonic complexes are not meaningless either; they serve as intermediate states which the system visits when in the spatio-temporal chaotic regime.

In this Letter we consider solitonic complexes in the parametrically driven damped nonlinear Schrödinger equation:

$$i\Psi_t + \Psi_{xx} + 2|\Psi|^2\Psi - \Psi = -i\gamma\Psi + h\overline{\Psi}. \tag{1}$$

This equation describes the effect of phase-sensitive amplifiers in optical systems [11]; the nonlinear Faraday resonance in water [12, 13, 8, 9]; convection in binary mixtures [14] and nematic liquid crystals [15]; magnetization waves in easy-plane ferromagnets with a rf magnetic field in the easy plane [16] and synchronized oscillations in parametrically driven Frenkel-Kontorova chains [17].

Malomed noted that since solitons of Eq.(1) decay monotonically as $|x| \to \infty$, they cannot form bound states via the tail-overlap mechanism [3]. A variational analysis demonstrated that in the undamped case ($\gamma = 0$), a strong overlapping of solitons cannot lead to their binding either [7]. However, oscillatory and stationary soliton associations were observed in experiments with solitons in an oscillating water trough [18, 8, 9] and subsequently reproduced in numerical simulations of the NLS equation (1) [9].

These physical and computer experiments have raised several challenging questions. Firstly and most importantly, an open problem is the very mechanism of the complex formation. Next, it was observed that stationary ("standing") complexes exist only at large separations [8]; it has therefore remained unclear whether these complexes are genuinely stationary or do diverge slowly due to an exponentially small repulsion. A related question is whether soliton associations can arise only on finite intervals under periodic boundary conditions and how essential the interval finiteness and periodicity are for their stability. (Note that the experiments of [8, 9] were carried out on relatively short intervals [19]. On the other hand, periodic chains of solitons can form stable stationary states even in situations where a finite number of solitons do not bind [5].) Lastly, numerical simulations can detect only stable complexes; however, the description of the phase space is incomplete without knowledge of all unstable complexes and their bifurcations.

The purpose of this Letter is to answer some of these questions and gain insight into the other ones. We focus on stationary complexes here. Oscillating complexes arise as Hopf bifurcations of the latter; from this point of view stationary complexes are more fundamental objects.

Variational approximation. Two coexisting stationary solitons of Eq.(1) are given by

$$\psi_\pm(x) = A_\pm e^{-i\theta_\pm}\operatorname{sech}(A_\pm x) \tag{2}$$

where

$$A_\pm = \sqrt{1 + h\cos 2\theta_\pm}; \quad \cos 2\theta_\pm = \pm\sqrt{1 - \frac{\gamma^2}{h^2}}.$$

The soliton ψ_- is always unstable [16] and hence is usually disregarded. We will attempt to approximate a complex of two solitons ψ_+ by a trial function of the form

$$\Psi(x,t) = \psi(x - x_0)e^{ik(x-x_0)} + \psi(x + x_0)e^{-ik(x+x_0)}, \tag{3}$$

where $\psi(x) = Ae^{-i\theta}\operatorname{sech}(Ax)$ and parameters x_0, k, θ and A are allowed to depend on time. The evolution of the parameters can be found if one notices [20] that Eq.(1) follows from the stationary action principle $\delta S = 0$, where $S = \int \mathcal{L} e^{2\gamma t} dt$ and the Lagrangian

$$\mathcal{L} = \operatorname{Re}\int \left(i\Psi_t\overline{\Psi} - |\Psi_x|^2 + |\Psi|^4 - |\Psi|^2 - h\Psi^2 \right) dx. \tag{4}$$

Substituting the ansatz (3) into Eq.(4) and integrating x out yields a finite-dimensional Lagrangian

$$\mathcal{L} = 4A(1 - \sigma_s)\dot{\theta} + 2\eta\dot{A} + Ap\dot{z} - Az\sigma_s\dot{p} - H, \tag{5}$$

where $z = 2Ax_0$ and $p = 2k/A$. The Hamiltonian $H = H_0 + H_{\text{int}}$ comprises the free solitons contribution,

$$H_0(\theta, A, p) = 4A - \frac{4}{3}A^3 + A^3p^2 + \frac{2\pi Ah\cos(2\theta)p}{\sinh(\pi p/2)}, \tag{6}$$

and the Hamiltonian of the soliton-soliton interaction:

$$\begin{aligned} H_{\text{int}}(\theta, A, p, z) = \frac{4\pi A^3}{\sinh(\pi p/2)} \\ \cdot\left\{ \left[\frac{2p\cos(pz/2)}{\sinh z}\right]_z - \left[\frac{\sin(pz/2)}{\sinh z}\right]_{zz} + \frac{4\sin(pz/2)}{\sinh^3 z} + \frac{\sin(pz/2)}{A^2\sinh z} \right\} \\ + \frac{16A^3}{\sinh z}\left[\frac{z}{\sinh z}\right]_z + \frac{8\pi A^3}{\sinh(\pi p)\sinh z}\left[\frac{\sin(pz)}{\sinh z}\right]_z \\ + \frac{4hAz}{\sinh z}\cos\left(\frac{pz}{2} + 2\theta\right). \end{aligned} \tag{7}$$

The quantities σ and η are given by

$$\sigma_c(p,z) + i\sigma_s(p,z) = -\pi\frac{e^{ipz/2}}{\sinh(\pi p/2)\sinh z};$$

$$\eta(p,z) = \left[1 - \frac{\pi p/2}{\tanh(\pi p/2)} - \frac{z}{\tanh z}\right]\sigma_c + \frac{\pi\sigma_s\coth z}{\tanh(\pi p/2)}.$$

The variations in z and θ yield, respectively:

$$2\gamma Ap + \partial_z H = 0; \tag{8}$$

$$8\gamma A(1 - \sigma_s) + \partial_\theta H = 0. \tag{9}$$

(Here and below we restrict ourselves to stationary solutions.) Since the second term in (8) decays rapidly as $z \to \infty$, p has to be small. We expand all other variables in powers of p: $\theta = \sum\theta^{(n)}p^n$, $A = \sum A^{(n)}p^n$, $z = \sum z^{(n)}p^n$; $n = 0, 1, \ldots$. Then Eq.(9) gives, at the order p^0: $\theta^{(0)} = \theta_+$. The next order produces

$$h\sin(2\theta) = \gamma - p\frac{h\cos 2\theta_+}{2}\,\frac{z^2}{z + \sinh z} + O(p^2). \tag{10}$$

Varying with respect to A we get $4\gamma\eta = -\partial_A H$. Noting that $\eta = O(p)$ and writing $H = \sum H^{(n)}(\theta, A, z)p^n$, the leading order is given by

$\partial_A H^{(0)} = 0$, which amounts to

$$A^2 = \frac{(1 + h\cos 2\theta_+)(z + \sinh z)}{\sinh z + 3z + 6[z(\cosh z - 3)/\sinh z]_z}. \qquad (11)$$

This relation defines A as a monotonically growing function of z. Next, the variation with respect to p produces equation $2\gamma A z \sigma_s = \partial_p H$ whose leading order is

$$p\left\{A\frac{h\cos(2\theta_+)z^4}{\sinh z(z+\sinh z)} + 2H^{(2)}\right\} = 0, \qquad (12)$$

where we have used Eq.(10). One readily checks that the expression in the curly brackets is linear in A^2 and hence Eq.(12) defines another function $A(z)$:

$$A^2[1 + S_1 + S_2 + S_3] = h\cos 2\theta_+\left[\frac{\pi^2}{6} + \frac{z^3}{2(z+\sinh z)}\right] + \frac{z(\pi^2+z^2)}{6\sinh z} \qquad (13)$$

where

$$S_1 = \frac{(\pi^2 - 18)z + z^3}{6\sinh z},$$

$$S_2 = \frac{(\pi^2 + 3z^2)\cosh z - 4(\pi^2 + 3z^2)}{3\sinh^2 z},$$

$$S_3 = \frac{z(\pi^2 + z^2)(4\cosh z - 1)}{3\sinh^3 z}.$$

In Eqs.(11),(13) A and z stand for $A^{(0)}$ and $z^{(0)}$. The curves (11) and (13) intersect at the point $(\tilde{z}, \tilde{A})$. For example, for $\gamma = 0.565, h = 0.9$ we have $\tilde{z} = 4.60, \tilde{A} = 1.14$.

Finally, returning to the equation (8), its p^0-order is given by $\partial_z H^{(0)}_{\text{int}} = 0$, where $H^{(0)}_{\text{int}}$ is the leading order of the potential of the soliton-soliton interaction. This function of z is known not to have nontrivial extrema [7]; however this does not mean that our variational equations do not have stationary solutions. We only need to retain one more power of p in Eq.(8):

$$p\left(2\gamma A + \partial_z H^{(1)}_{\text{int}}\right) + \partial_z H^{(0)}_{\text{int}} = 0. \qquad (14)$$

(Here we are regarding $\partial_z H^{(0)}_{\text{int}}$ as a function of $z^{(0)}$, $A^{(0)}$ and $\theta^{(0)}$ and discarding p^1-corrections to this function which are negligible comparing to the first term in (14).) The second term cannot vanish for finite z but as z grows, it rapidly becomes smaller than the first term. Consequently, the leading order is given not by $\partial_z H^{(0)}_{\text{int}} = 0$ but by Eq.(14) which determines the stationary value of p:

$$\tilde{p} = \frac{1}{2\gamma A}\frac{\partial_z H^{(0)}_{\text{int}}}{1 - (z^2/\sinh z)_z}, \qquad (15)$$

where $z = \tilde{z}$ and $A = \tilde{A}$. For example, for $\gamma = 0.565$ and $h = 0.9$ Eq.(15) yields $\tilde{p} = -0.12$.

Thus, we have demonstrated the existence of a stationary two-soliton bound state on the infinite line. Below the complex $\Psi_{(++)}$ is reobtained numerically and Fig.1 compares it to the variational approximation (3). It is seen that $\tilde{z}$ gives a reasonable approximation for the actual intersoliton separation but as we proceed to the comparison of the *shapes* of the numerical and variational solution, the agreement deteriorates. The approximation could be improved by decoupling the solitons' amplitudes from their widths and adding the chirp variable; however, even taken in its present form the variational description allows to draw several principal conclusions. First, Eq.(15) shows that strong dissipation is imperative for the existence of complexes; no bound states can arise for γ zero or small. Second, the phase variation is an essential ingredient of the complex formation mechanism. Had we not included a nonzero p, we would have arrived at the equation $\partial_z H^{(0)}_{\text{int}} = 0$ which has no nontrivial roots. A related observation concerns the recipe suggested in the undamped case [7] where the time-dependent version of Eq.(12), $\dot{z} = p\{...\}$, was used to express p through $\dot{z}$. By eliminating p the authors of [7] reduced the finite-dimensional dynamics to a single equation for a particle in a potential field $H^{(0)}_{\text{int}}(z)$. Technically, this recipe cannot be utilized in the case at hand because the contents of the curly brackets in Eq.(12) vanishes exactly at the stationary point. An implication of this fact is that the formation of complexes in our case *cannot* be explained by Malomed's two-particle mechanism [2, 3, 4, 21] where one soliton is captured in a potential well formed by its mate. Lastly, the variable

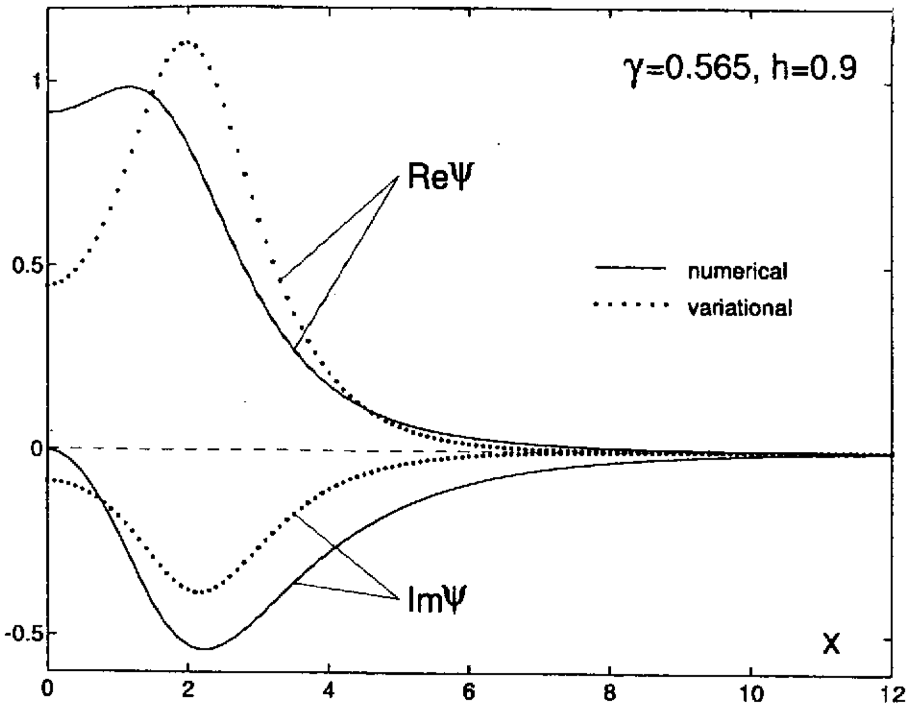

Fig.1 The variational ansatz (3) with stationary values $\theta = \theta_+, z = \tilde{z}, A = \tilde{A}$ and $p = \tilde{p}$ (dotted) and the numerically obtained complex $\Psi_{(++)}$ (solid curves).

amplitude A is another essential ingredient. If we had not made provision for the variation of the "number of particles" $N = \int |\Psi|^2 dx$, the resulting equations for z and p would have had stationary points only for large $h > h_c = \sqrt{1+\gamma^2}$. (We have checked this.) In this region all localized solutions are unstable against radiation waves [16].

Numerical solutions. We used a predictor-corrector continuation algorithm with a fourth-order Newtonian solver to obtain stationary solutions of Eq.(1). As a bifurcation measure we adopted the energy functional

$$E = \mathrm{Re} \int \left\{ |\Psi_x|^2 + |\Psi|^2 - |\Psi|^4 + h\Psi^2 \right\} dx, \qquad (16)$$

which is conserved when $\gamma = 0$. Our findings are summarized in Fig.2 where we have also included information on the stability of solutions. This was studied by examining eigenvalues of the linearized problem

$$\mathcal{H}y = \mu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y,$$

$$\mathcal{H} = -\partial_x^2 + \begin{pmatrix} A_+^2 - 6u^2 - 2v^2 & \gamma - 4uv \\ \gamma - 4uv & A_-^2 - 6v^2 - 2u^2 \end{pmatrix}, \qquad (17)$$

where $u + iv \equiv \Psi(x)e^{i\theta_+}$; $(\delta u, \delta v)^T = e^{\lambda t} y(x)$, $\lambda = \mu - \gamma$.

The phase variable $\chi(x) = -\mathrm{Arg}\Psi$ turns out to be useful in the identification of different complexes. For example, the phase of the solution identified as $\Psi_{(+-+)}$ is close to θ_- around the central soliton (Ψ_-) and θ_+ around two side solitons (Ψ_+) (Fig.3). The separation of the constituents in this complex is large even for small h ($2x_0 \sim 60$ for $h \sim \gamma$). (All numbers are for $\gamma = 0.565$). As $h \to h_c = \sqrt{1+\gamma^2}$ and the width of the central soliton increases ($1/A_- \to \infty$), the inter-soliton separation grows to infinity. If we continue along the branch $\Psi_{(-+-)}$ towards larger h, the separation decreases from $2x_0 \sim 60$ at $h \sim \gamma$ to $2x_0 \sim 20$ near the turning point $h = 0.86742$. Turning left and upwards, the separation keeps on decreasing, the central soliton gradually disappears and the complex is made into $\Psi_{(--)}$. After one more turning point at $h = 0.83504$, as we continue to the right, the amplitudes of the constituent solitons start to grow and the complex gradually transforms into $\Psi_{(++)}$. It is interesting to note here that

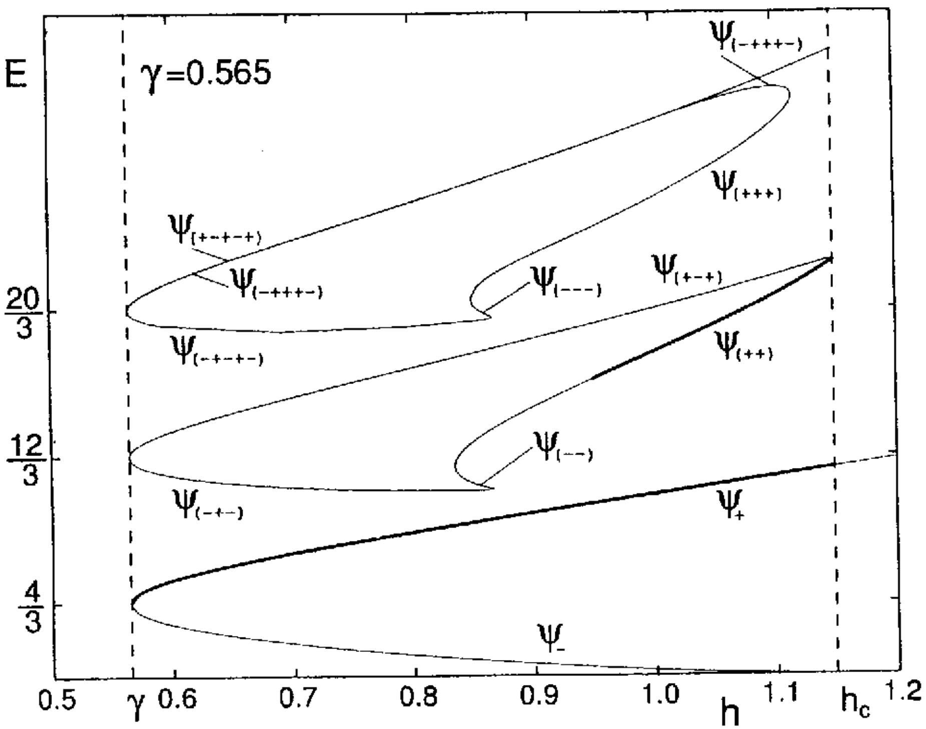

Fig.2 Existence and stability diagram of one, two and three-soliton stationary solutions. Thick respectively thin lines depict stable respectively unstable branches.

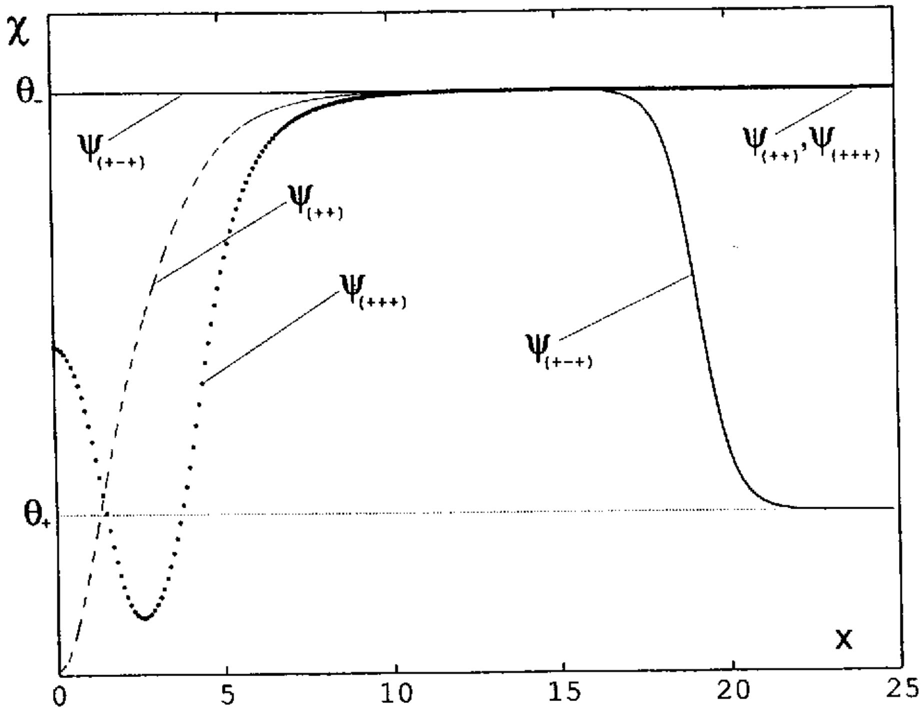

Fig.3 The phase $\chi(x)$ of the complex $\Psi_{(+-+)}$ (solid line). The Ψ_+ solitons in this complex are centered at $x_0 \approx \pm 37$ while the variation of χ is confined mainly to $17 < |x| < 21$. Also shown is the phase of $\Psi_{(++)}$ (dashed) and $\Psi_{(+++)}$ (dotted).

the asymptotic phase of the solution remains equal to θ_- and not θ_+ as could have been expected of the complex of two Ψ_+ solitons (Fig.3). At $h = 0.9435$ the complex undergoes an "inverse" Hopf bifurcation where a pair of unstable complex-conjugate eigenvalues crosses from $\mathrm{Re}\lambda > 0$ to $\mathrm{Re}\lambda < 0$ half-plane. The remaining portion of the $\Psi_{(++)}$ branch (thick line in Fig.2) represents the only stable bound state in the system; all other complexes were found to be unstable. As h increases, the intersoliton separation grows but remains finite all way up to $h = h_c$.

Some insight into the structure of stationary complexes can be gained by noting the law of the number of particles variation:

$$\dot{N} = 2h \int \rho\{\sin(2\chi) - \sin(2\theta_\pm)\} dx. \tag{18}$$

Here $\Psi = \sqrt{\rho} e^{-i\chi}$. For stationary complexes the integral in the right-hand side has to vanish. This can be easily achieved when solitons bind at a very large separation, like in $\Psi_{(+-+)}$. In this case the variation of χ should mainly be confined to regions where ρ is almost zero (Fig.3). The resulting contribution to the integral (18) can be offset by small deviations of χ from $\theta_\pm$ around the centers of the solitons, and indeed, a closer inspection reveals that $\sin(2\chi) - \sin(2\theta_\pm)$ assumes small negative values around the core of each soliton bound in $\Psi_{(+-+)}$.

Formation mechanism. As we have mentioned, the binding mechanism is more involved here than just a balance of repulsion and attraction between the two solitons. Details are yet to be elucidated in numerical simulations of the time-dependent equation (1) while here we shall only emphasize its main ingredients. First of all, noting that the amplitude of each soliton is given by Eq.(11), one can check that the total number of particles in the configuration (3) is a monotonically growing function of z. Consequently oscillations of the separation between the two solitons are completely characterized by oscillations of N. The dynamics of the latter is described by Eq.(18) where the right-hand side is very sensitive to variations of the phase (in particular, of our p-variable.) If the complex is in its stage of expansion, at a certain moment of time the phase $\chi(x, t)$ will pass

through a configuration rendering the integral in (18) zero. The expansion will then switch to contraction – until the phase is again such that $\dot{N} = 0$. In the stable region of h and γ the oscillations will settle to the stationary complex $\Psi_{(++)}$.

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