ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E17-98-7

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VIBRATIONS AND OSCILLATORY INSTABILITIES OF GAP SOLITONS

Submitted to «Physics Review Letters»

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1998

In the late seventies and early eighties, the theory of elementary particles [1] and condensed matter physics (in particular, the Su-Schrieffer-Heeger polyacetylene model [2]) stimulated a wide interest in particle-like solutions of classical spinor field equations. Recently there has been a remarkable upsurge of the interest; the localized solutions of spinor and spinor-like systems have made a come-back under the new name of *gap solitons*.

The term "gap solitons" usually refers to spatially localized nonlinear waves occurring in systems with a gap in the linear spectrum. Due to the presence of the gap, the solitons can propagate without losing their energy to resonantly excited radiation waves [3, 4]. An example of the gap-soliton bearing system is given by an optical fiber with periodically varying refractive index [3]; here the gap is produced by the Bragg reflection and resonance of the waves along the grating. Another class of gap solitons arises in two-wave resonant optical materials with a $\chi^{(2)}$ susceptibility and diatomic crystal lattices (see Ref.[4] and references therein). Finally, in the already mentioned polyacetylene model [2], the gap in the electron spectrum is due to the electron-phonon interaction and effective period-doubling of the lattice.

The aim of this Letter is to analyze the *stability* of gap solitons. Previous analytical studies of the spinor soliton stability faced serious obstacles (cf. [5]), while results of computer simulations were contradictory (cf. [6, 7]). As a result, no stability or instability criterion is available to date. The main difficulty of the previous analyses was that they were all based on a postulate that stable solutions must render the energy minimum. In the actual fact, however, the minimality of energy is not necessary for stability in systems with indefinite metrics (e.g. in spinor systems) [8].

As far as *optical* gap solitons are concerned, they have been commonly deemed stable following recent computer simulations carried out for certain particular parameter values [9]. In this Letter we demonstrate that the gap solitons *can* be unstable, elucidate the mechanism of instability and demarcate the stability/instability regions on the plane of their parameters.

In nonlinear optics the gap solitons are usually analyzed within the coupled-mode theory [3] which reduces to a system of coupled equations for the amplitudes of the forward- and backward-propagating waves:

$$\begin{aligned} i\,(u_t + u_x) + v + \left(|v|^2 + \rho|u|^2\right) u &= 0, \\ i\,(v_t - v_x) + u + \left(|u|^2 + \rho|v|^2\right) v &= 0. \end{aligned} \tag{1}$$

Here x is the coordinate along the grating, t time and ρ a parameter. In the periodic Kerr medium one typically has $\rho = 1/2$ [3]; in other problems of the fiber optics ρ may range up to infinity [10]. In the case $\rho = 0$ Eqs.(1) yield the massive Thirring model of the field theory [11]. In this case Eqs.(1) are invariant with respect to the Lorentz transformations $(x,t) \to (X,T)$, where

$$X = \frac{x - Vt}{\sqrt{1-V^2}}, \quad T = \frac{t - Vx}{\sqrt{1-V^2}},$$

with u and v transforming as components of the Lorentz spinor [see Eq.(2) below]. Although in the general case ($\rho \neq 0$) the Lorentz symmetry is broken, its artefact is that the simplest form of the soliton solution is still in terms of the boosted variables X and T [9]:

$$\begin{aligned} u &= \alpha W(X) e^{y/2 + i\varphi(X) - i\cos\theta\, T}, \\ v &= -\alpha W^*(X) e^{-y/2 + i\varphi(X) - i\cos\theta\, T}, \end{aligned} \tag{2}$$

where

$$\varphi(X) = 2\alpha^2 \rho \sinh(2y) \arctan\left\{\tan\left(\frac{\theta}{2}\right) \tanh[(\sin\theta) X]\right\},$$

$$W(X) = \frac{\sin\theta}{\cosh\left[(\sin\theta)X - i\frac{\theta}{2}\right]},$$

and $\alpha^{-2} = 1 + \rho\cosh(2y)$. Here the angle θ ($0 < \theta < \pi$) parametrizes the detuning frequency of the soliton within the spectrum gap: $\Omega = \cos\theta$. The parameter y determines the soliton's velocity: $V = \tanh y$. (In relativity y is known as rapidity.) At the upper edge of the spectrum gap, i.e. in the limit $\Omega \to 1$ ($\theta \to 0$) and assuming $|V| \ll 1$, Eq.(2) approaches the small-amplitude NLS soliton [3]: $W(X) \to \theta\,\mathrm{sech}[\theta(X - i/2)]$. At the

lower edge, i.e. in the limit $\Omega \to -1$ ($\theta \to \pi$), the gap soliton has a finite amplitude and decays as a power law: $W(X) = i/(X + i/2)$. These two limits are referred to as the "low intensity" and "high intensity" limits [3].

Linearizing Eq.(1) about the stationary soliton (2) and choosing the perturbation in the form

$$\begin{aligned} u &= \left[\alpha W(X) + z_1(X) e^{i\lambda T}\right] e^{y/2 + i\varphi(X) - i\cos\theta\, T}, \\ v &= \left[-\alpha W^*(X) + z_2(X) e^{i\lambda T}\right] e^{-y/2 + i\varphi(X) - i\cos\theta\, T}, \\ u^* &= \left[\alpha W^*(X) + z_3(X) e^{i\lambda T}\right] e^{y/2 - i\varphi(X) + i\cos\theta\, T}, \\ v^* &= \left[-\alpha W(X) + z_4(X) e^{i\lambda T}\right] e^{-y/2 - i\varphi(X) + i\cos\theta\, T}, \end{aligned}$$

gives an eigenvalue problem

$$\hat{\mathcal{H}} \mathbf{z} = \lambda J \mathbf{z}, \quad J = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \tag{3}$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$ and the hermitean operator $\hat{\mathcal{H}}$ is defined by

$$\begin{aligned} \hat{\mathcal{H}} &= i\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \frac{d}{dX} + \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} (|W|^2 + \cos\theta) + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \\ &+ \alpha^2 \begin{pmatrix} \rho e^{2y}|W|^2 & -W^2 & \rho e^{2y} W^2 & -|W|^2 \\ -W^{*2} & \rho e^{-2y}|W|^2 & -|W|^2 & \rho e^{-2y} W^{*2} \\ \rho e^{2y} W^{*2} & -|W|^2 & \rho e^{2y}|W|^2 & -W^{*2} \\ -|W|^2 & \rho e^{-2y} W^2 & -W^2 & \rho e^{-2y}|W|^2 \end{pmatrix}. \end{aligned} \tag{4}$$

(Here σ_0, σ_1 and σ_3 are the Pauli matrices.) Eq.(3) has four zero eigenvalues arising from symmetries of the nonlinear system (1) and four branches of the continuous spectrum pertaining to real λ. We will only need two branches with $\lambda > 0$; the associated eigenfunctions can be specified by their asymptotic behaviour as $X \to -\infty$:

$$\mathbf{Z}^{(1)}(X,k) \to (0,0,1,-r)^T e^{ikX} \quad \text{for} \quad \lambda = \lambda_1(k) = \sqrt{1+k^2} - \cos\theta, \tag{5}$$

$$\mathbf{Z}^{(2)}(X,k) \to (1,r,0,0)^T e^{ikX} \quad \text{for} \quad \lambda = \lambda_2(k) = \sqrt{1+k^2} + \cos\theta, \tag{6}$$

normalsize where $r = r(k) = \sqrt{1+k^2} + k$. The continuous spectrum solutions describe radiations propagating on the solitonic background.

In the Thirring case ($\rho = 0$) the set of the neutral and continuum eigenfunctions is complete [12] so that any additional eigenvalues are absent. However, as ρ deviates from zero, new eigenvalues can detach from the edges of the continuous spectrum. (In fact, this is the only way bifurcations may occur in nearly integrable systems provided the perturbation preserves symmetries of the integrable limit [13].) To see whether this is indeed the case, we expand solutions to (3) over the complete set of the Thirring eigenfunctions:

$$z(X) = \sum_{i=1}^{2} \int_{-\infty}^{\infty} \frac{a_i(k) \mathbf{Z}^{(i)}(X,k)}{\lambda - \lambda_i(k)} dk + (\ldots), \tag{7}$$

where (...) stands for terms which remain bounded as λ approaches $\lambda_1(0) = 1 - \cos\theta$ and $\lambda_2(0) = 1 + \cos\theta$. Using the orthogonality relations between the Thirring eigenfunctions [12], Eq.(3) can be reduced to a system of two integral equations:

$$a_i(k) = \frac{(-1)^i \rho}{4\pi\sqrt{1+k^2} r(k)} \sum_{j=1}^{2} \int_{-\infty}^{\infty} \frac{K_{ij}(k,k') a_j(k')}{\lambda - \lambda_j(k')} dk' \tag{8}$$

($i = 1, 2$), where

$$\begin{aligned} K_{ij} &= C_1 \int_{-\infty}^{+\infty} \Big[W^2 (Z_1^{(i)*} + Z_4^{(i)*})(Z_2^{(j)} + Z_3^{(j)}) + W^{*2} (Z_2^{(i)*} + Z_3^{(i)*})(Z_1^{(j)} + Z_4^{(j)}) \\ &+ |W|^2 (Z_1^{(i)*} + Z_4^{(i)*})(Z_1^{(j)} + Z_4^{(j)}) + |W|^2 (Z_2^{(i)*} + Z_3^{(i)*})(Z_2^{(j)} + Z_3^{(j)}) \Big] dX \\ &+ \sinh(2y) \int_{-\infty}^{+\infty} \Big[W^2 (Z_1^{(i)*} Z_3^{(j)} - Z_4^{(i)*} Z_2^{(j)}) + W^{*2} (Z_3^{(i)*} Z_1^{(j)} - Z_2^{(i)*} Z_4^{(j)}) \\ &+ |W|^2 (Z_1^{(i)*} Z_1^{(j)} - Z_2^{(i)*} Z_2^{(j)} + Z_3^{(i)*} Z_3^{(j)} - Z_4^{(i)*} Z_4^{(j)}) \Big] dX, \quad C_1 = \cosh(2y), \end{aligned}$$

and we have denoted $Z_n^{(i)*} = Z_n^{(i)*}(X,k)$ and $Z_n^{(j)} = Z_n^{(j)}(X,k')$. The edges of the continuum branches, $\lambda_1(0)$ and $\lambda_2(0)$, are well separated unless $\theta \approx \pi/2$. Consequently, if θ is not very close to $\pi/2$ we can get away with a single-mode approximation and disregard the nonresonant branch. Let, first, $0 < \theta \ll \pi/2$ and assume that a new eigenvalue detaches from the edge of the (inner) branch λ_1: $\lambda = \lambda_1(0) - \frac{1}{2}\kappa^2$. Sending $\rho \to 0$ and disregarding the branch λ_2, we obtain $|\kappa| = \frac{1}{2}\rho K_{11}(0,0)$. The Thirring eigenfunctions pertaining to $k = 0$ satisfy $Z_1^{(1)}(X,0) = -Z_2^{(1)*}(X,0) \equiv z_1^{(1)}(X)$ and $Z_3^{(1)}(X,0) = -Z_4^{(1)*}(X,0) \equiv z_2^{(1)}(X)$; this follows from (3) and (5). Using these symmetry properties we finally arrive at

$$|\kappa| = -\frac{\rho}{2} \cosh(2y) \int_{-\infty}^{+\infty} \left[W(z_1^{(1)*} - z_2^{(1)}) - \text{c.c.} \right]^2 dX. \tag{9}$$

Since the right-hand side in Eq.(9) is positive, we conclude that a small deviation from the integrable case $\rho = 0$ does indeed bring about a new real eigenvalue $\lambda < \lambda_1(0)$. This additional eigenvalue represents a vibration mode of the gap soliton with $0 < \theta \ll \pi/2$.

Next, let $\pi/2 \ll \theta < \pi$ and assume that an eigenvalue $\lambda = \lambda_2(0) - \frac{1}{2}\kappa^2$ detaches from the branch λ_2 (which is now the inner branch.) The same asymptotic approach as above produces $|\kappa| = -\frac{1}{2}\rho K_{22}(0,0)$. Making use of the symmetry relations $Z_1^{(2)}(X,0) = Z_2^{(2)*}(X,0) \equiv z_1^{(2)}(X)$ and $Z_3^{(2)}(X,0) = Z_4^{(2)*}(X,0) \equiv z_2^{(2)}(X)$, this becomes

$$|\kappa| = -\frac{\rho}{2} \cosh(2y) \int_{-\infty}^{+\infty} \left[W(z_1^{(2)*} + z_2^{(2)}) + \text{c.c.} \right]^2 dX. \tag{10}$$

Since the right-hand side is negative, we have arrived at a contradiction. Thus the birth of a vibration mode can not occur for $\pi/2 \ll \theta < \pi$.

Finally, the case $\theta \approx \pi/2$ has to be analyzed within the full two-mode system (8); in this case *both* continuous branches are resonant. We let $\theta = \pi/2 + \epsilon$ and look for a new eigenvalue as $\lambda = \min\{\lambda_1(0), \lambda_2(0)\} - \frac{1}{2}\kappa^2$. Assuming $\mathrm{Re}\,\kappa > 0$ and $\rho \to 0$, the system (8) can be reduced to an algebraic equation for κ:

$$4\kappa\sqrt{\kappa^2 + 4\epsilon} - 2\rho \left[K_{11}(0,0)\kappa - K_{22}(0,0)\sqrt{\kappa^2 + 4\epsilon} \right] - \rho^2 D(0,0) = 0, \tag{11}$$

where $D(0,0) = K_{11}(0,0)K_{22}(0,0) - K_{12}(0,0)K_{21}(0,0)$. [Here we have assumed $\epsilon > 0$; for $\epsilon < 0$ one should simply transpose κ and $\sqrt{\kappa^2 + 4|\epsilon|}$ in Eq.(11).] To find the coefficients in (11), we first derive the edge eigenfunctions $z_n^{(i)}(X)$ for $\theta = \pi/2$:

$$z_1^{(1)}(X) = -\mathrm{sech}(2X) \frac{\cosh X + i \sinh X}{\cosh X - i \sinh X};$$
$$z_2^{(1)}(X) = i \tanh(2X) \frac{\cosh X - i \sinh X}{\cosh X + i \sinh X};$$

$z_1^{(2)}(X) = (z_2^{(1)})^*$; $z_2^{(2)}(X) = -(z_1^{(1)})^*$. Using these formulas, we obtain

$$\begin{aligned} K_{nn}(0,0) &= 2[\pi + (-1)^n 2] \cosh(2y), \quad n = 1, 2; \\ K_{12}(0,0) &= K_{21}(0,0) = -4 \sinh(2y), \end{aligned} \tag{12}$$

and the analysis of the roots of Eq.(11) becomes straightforward. When $\epsilon < \epsilon_{cr}$, where $\epsilon_{cr}(V) = [\rho D(0,0)/4K_{22}(0,0)]^2$, there are 4 real roots,

$\kappa_1 < \kappa_4 < \kappa_2 < 0 < \kappa_3$ (Fig.1). The positive root κ_3 corresponds to the above-mentioned vibration mode that continues from $\theta = 0$ (see Fig.2). The negative root κ_2 becomes positive for ϵ between ϵ_{cr} and some ϵ_{osc} where κ_2 merges with κ_3 (Fig.1). That is, in this narrow region the gap soliton has two vibration modes. At $\epsilon = \epsilon_{osc}$ the two modes resonate, κ and λ become complex and the oscillatory instability sets in (Fig.2). In the vicinity of the bifurcation the instability growth rate is given by $\mathrm{Im}\lambda = -\mathrm{Re}\kappa\,\mathrm{Im}\kappa \sim \sqrt{\epsilon - \epsilon_{osc}}$.

The numerical analysis of the eigenvalue problem (3) shows that the above bifurcation pattern persists for finite ρ. In Fig.3a we have demarcated the boundary of the stability domain in the (Ω, V)-plane for $\rho = 1/2$. The asymptotic approximation for the oscillatory bifurcation curve, $\Omega = \cos(\epsilon_{osc}(V) + \pi/2)$, is also shown for comparison (dashed curve). Fig.3b is a similar bifurcation and stability chart for $\rho = \infty$.

As we increase θ further on, another pair of complex eigenvalues detaches from the edge of the continuous spectrum. In contrast to the first bifurcation, this happens near the edge of the "outer" branch of the continuous spectrum (Fig.2). There is no intermediate region with two real discrete eigenvalues – the detaching eigenvalues are complex at once. The growth rate of this secondary instability is smaller than that of the primary one (see Fig.2b).

Here it is important to emphasize that these oscillatory instabilities cannot be detected within the variational approach. One notices that (2) is a stationary point of the functional $\mathcal{L} = H - VP - \omega N$, where the conserved Hamiltonian, momentum and energy are given by

$$H = \frac{1}{2}\int_{-\infty}^{\infty}\left[i(uu_x^* + v^*v_x) - 2vu^* - |uv|^2 - \frac{\rho}{2}\left(|u|^4 + |v|^4\right) + c.c.\right]dx \quad (13)$$

$$P = \frac{i}{2}\int_{-\infty}^{\infty}(uu_x^* + vv_x^* - c.c.)\,dx, \quad (14)$$

$$N = \int_{-\infty}^{\infty}\left(|u|^2 + |v|^2\right)dx, \quad (15)$$

respectively, and $\omega = \Omega\sqrt{1-V^2}$. The idea of the variational (or energetic) approach to stability would be to prove that the soliton minimizes Hamiltonian for the fixed P and N, or equivalently, that $\delta^2\mathcal{L}$ is positive definite. However, writing $\delta^2 L = (\mathbf{z}, \hat{\mathcal{H}}\mathbf{z})$ with $\hat{\mathcal{H}}$ as in (4), and noting that the continuous spectrum of the eigenvalue problem $\hat{\mathcal{H}}\mathbf{z} = E\mathbf{z}$ extends from minus to plus infinity, one immediately concludes that the form $\delta^2 L$ is bounded neither from above nor from below.

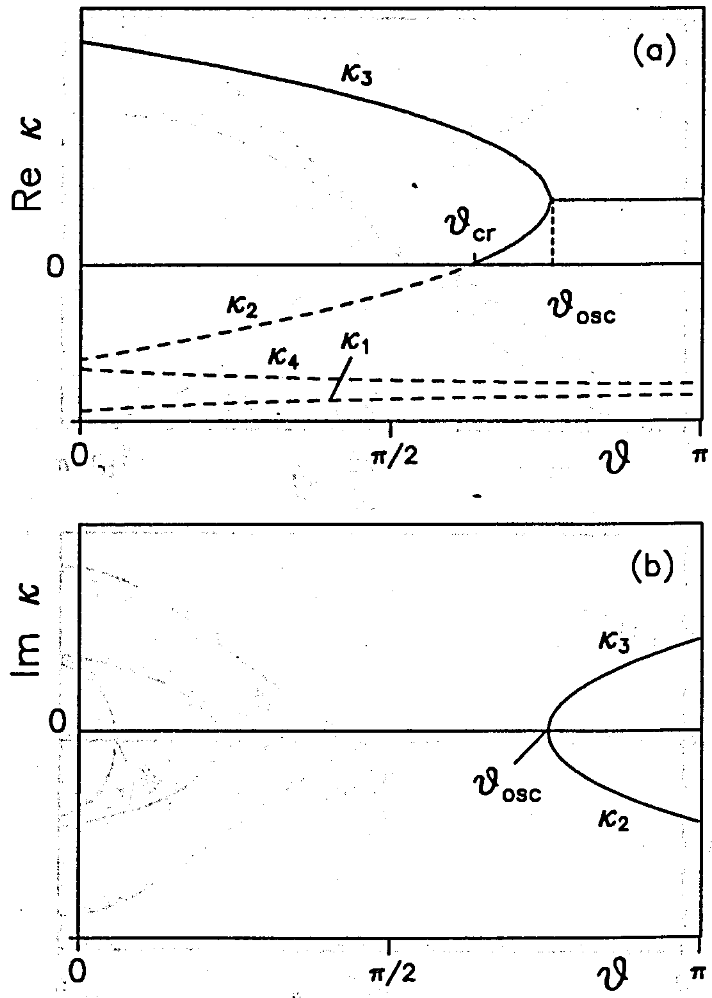

Fig. 1 Roots of the algebraic equation (11) for $\rho = 1/2$. Note that only roots with $\mathrm{Re}\kappa > 0$ (solid lines) correspond to eigenvalues of Eq.(3); roots with $\mathrm{Re}\kappa < 0$ (dashed lines) do not represent any λ. A large value of $\theta_{osc} = 2.38$ is explained by a large velocity: $V = 0.9$. The horizontal zero line has been moved down for visual clarity; the actual $\theta_{cr} = 2.36$.

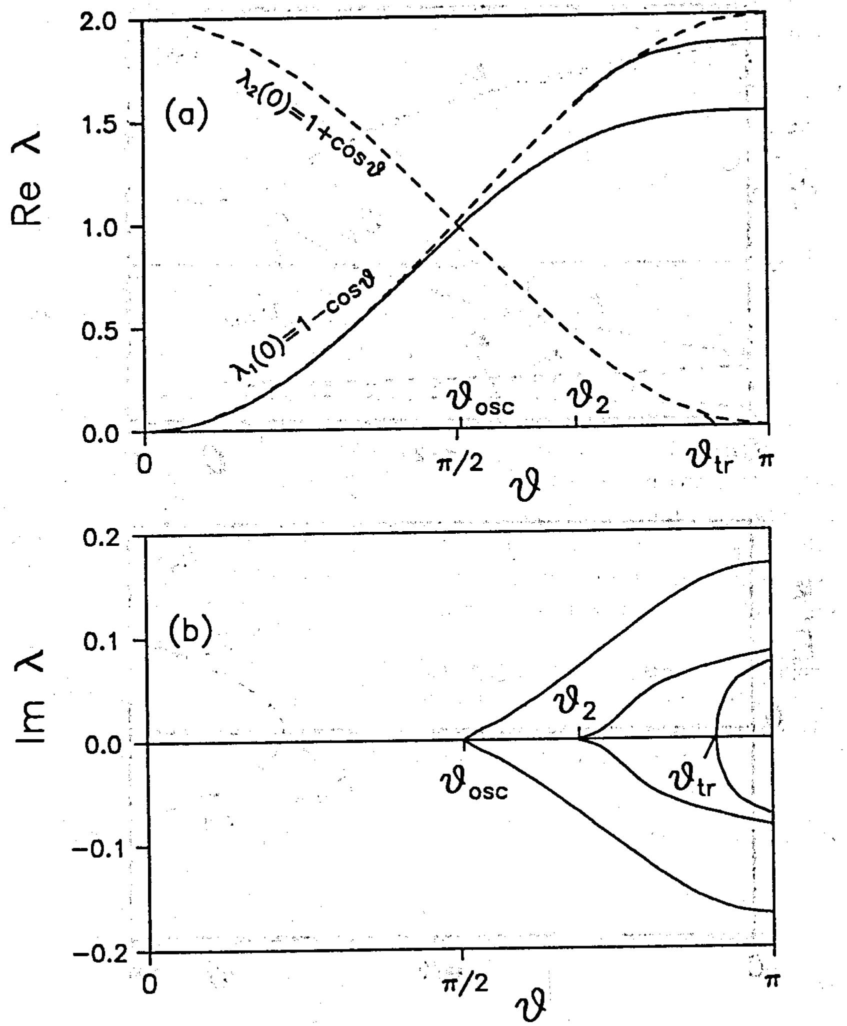

Fig. 2 Numerically found eigenvalues. Dashed lines indicate the edges of the continuous spectrum, $\lambda_{1,2}(0) = 1 \mp \cos\theta$. A real eigenvalue detaches from λ_1 at $\theta = 0$ and another real eigenvalue detaches from λ_2 at $\theta = \theta_{cr} > \pi/2$ (not clearly visible). At $\theta = \theta_{osc}$ the two collide and the oscillatory instability sets in. Another complex doublet emerges from λ_1 at $\theta = \theta_2$. Finally, one more real eigenvalue detaches from λ_2 and moves on the imaginary axis at θ_{tr}.

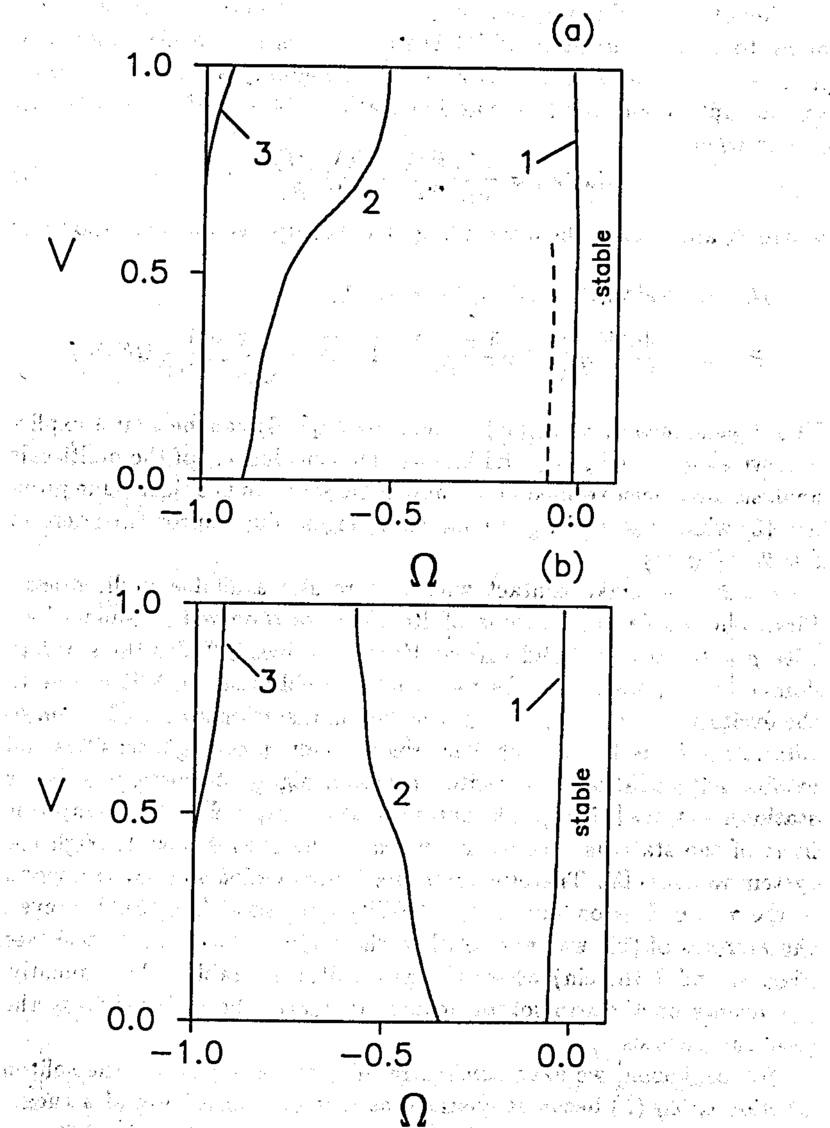

Fig. 3 Stability charts for $\rho = 1/2$ (a) and $\rho = \infty$ (b). Curves 1 and 2 are the lines of the primary and secondary oscillatory bifurcations, and the line 3 demarcates the onset of the translational instability. The dashed line is the asymptotic approximation for the primary oscillatory instability.

Nevertheless, the integrals of motion are not entirely useless. They allow to detect *translational* bifurcations where a real eigenvalue approaches zero and then passes on to the imaginary axis. According to the asymptotic multiscale expansion method [14, 15], the zero crossing occurs when

$$D(\omega, V) = \frac{\partial N_s}{\partial \omega}\frac{\partial P_s}{\partial V} - \frac{\partial N_s}{\partial V}\frac{\partial P_s}{\partial \omega} = 0, \qquad (16)$$

where P_s and N_s are the invariants (14-15) computed on the soliton (2):

$$N_s = 4\alpha^2 N_0(\Omega) \equiv 4\alpha^2 \left(\frac{\pi}{2} - \arcsin\Omega\right),$$

$$P_s = \frac{4\alpha^4 V}{\sqrt{1-V^2}}\left(1+\rho\frac{5+V^2}{1-V^2}\right)\sqrt{1-\Omega^2} - \frac{16\rho\alpha^4 V}{\sqrt{(1-V^2)^3}}\Omega N_0(\Omega).$$

The dependence $\Omega = \Omega_{tr}(V)$ defined by Eq.(16) can be found explicitly; we show it in Fig.3 (solid line 3). The conclusions of the multiscale analysis have been verified in the numerical study of the eigenvalue problem (3) which revealed a pure imaginary eigenvalue approaching zero at $\theta \approx \theta_{tr}$ (Fig.2b).

Finally we make contact with two results available in literature. First, the stable gap solitons of Ref.[9] were observed in simulations with $\rho = 1/2, \theta = \pi/2$ and various V; these values fall into the stability domain of Fig.3(a). The observed soliton oscillations [9, 16] are due to the excitation of the vibration mode. Second, stationary solutions *on finite intervals* are known to be unstable for high incoming intensities and exhibit self-pulsations and switchings from highly- to low-transmissive stationary states [17, 18]. The gap solitons correspond to the asymptotic limit of the stationary solutions in which the energy flow through the system vanishes [3]. Therefore, our present discussion should correspond to the $\gamma \to 0, L \to \infty$ limit of the stability analysis of Ref.[18]. However, the analysis of [18] was restricted to the region $V = 0$ and $\Omega > 0$ (see Figs. 6 and 7 therein) where the gap soliton is stable. Consequently, our results on the gap soliton instability cannot be deduced from the previous analysis.

In conclusion, we have demonstrated that for any $\rho > 0$ the soliton solution of Eq.(1) becomes unstable as Ω is decreased beyond a (negative) critical value. The instability is caused by the resonance between the soliton's vibration mode and two branches of the long-wavelength radiation. This "triple-resonance" mechanism is different from previously encountered mechanisms of oscillatory instability (cf.[19]).

Useful discussions with I.V. Amirkhanov, Yu.S. Kivshar, T.I. Lakoba, V.A. Osipov, and C.M. de Sterke are gratefully appreciated. We thank I.V. Puzynin for his support of the investigation and T.A. Strizh for her assistance with the administration of the project. This research was supported by the FRD of South Africa and the URC of UCT. E.Z. was supported by a RFFR grant (#RFFR 97-01-01040).

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Received by Publishing Department
on January 26, 1998.