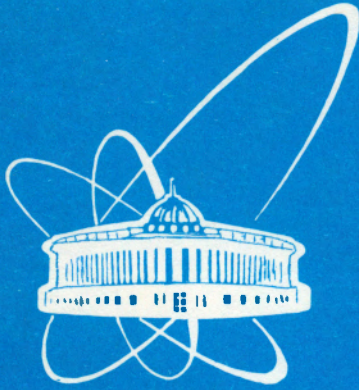98-290

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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EXISTENCE THRESHOLD FOR THE AC-DRIVEN DAMPED NONLINEAR SCHRÖDINGER SOLITONS

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1 Introduction

The externally driven, damped nonlinear Schrödinger (NLS) equation,

$$i\Psi_t + \Psi_{xx} + 2|\Psi|^2\Psi = -i\gamma\Psi - he^{i\Omega t}, \qquad (1)$$

arises in a variety of fields including plasma and condensed matter physics, nonlinear optics and superconducting electronics. In some of these applications (e.g. the theory of rf-driven waves in plasma [1]; the description of the optical soliton propagation in a diffractive or dispersive ring cavity in the presence of an input forcing beam [2]) Eq.(1) has a direct interpretation. In others — like for instance in charge-density-wave conductors with external electric field [3]; shear flows in nematic liquid crystals [4]; easy-axis ferromagnets in an external magnetic field parallel to the easy axis [5]; ac-driven long Josephson junctions [6], and periodically forced Frenkel-Kontorova chains [7] — it occurs as an amplitude equation for small and slowly changing solutions of the externally driven, damped sine-Gordon equation:

$$q_{tt} + \lambda q_t - q_{xx} + \sin q = \Gamma\cos(\omega t).$$

Without loss of generality Ω in Eq.(1) can be normalized to unity; hence, the driver's strength h and dissipation coefficient γ are the only two essential control parameters. Given some h and γ, a fundamental question is what nonlinear attractors will arise at this point of the (γ, h)-plane. In their pioneering paper [8] Kaup and Newell considered Eq.(1) on the infinite line under the vanishing boundary conditions at infinity. By means of the Inverse Scattering-based perturbation theory, these authors have demonstrated that for small h and γ Eq.(1) exhibits two soliton solutions phase-locked to the frequency of the driver. As h is decreased for the fixed γ, the two solitons approach each other and eventually merge in a turning point for $h = (2/\pi)\gamma$ [8]. Consequently, this value plays the role of a threshold; no solitons exist below $h = (2/\pi)\gamma$. Later the same existence threshold was reobtained by Terrones, McLaughlin, Overman and Pearlstein [9] in a regular perturbative construction of solutions to (1) in powers of h and γ (see also [10]).
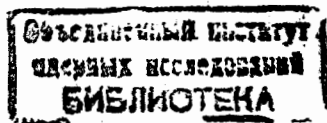
In ref.[11] equation (1) was studied, numerically, in the full range of h and γ. It was found that the two soliton solutions persist for γ up to approximately 0.7. For each $\gamma \leq 0.7$ there is a turning point at some $h = h_{\text{thr}}$ at which one branch of solitons turns into another, and which plays the role of the lower boundary of the existence region [12]. Amazingly, Kaup and Newell's approximate relation $h_{\text{thr}} = (2/\pi)\gamma$ was found to remain valid even for not very small γ. For example, for $\gamma = 0.48$ the ratio h_{thr}/γ was different from $2/\pi$ by only one part in a thousand [11].

A completely different approach was put forward by Kollmann, Capel and Bountis [13] who regard Eq.(1) as the continuous limit of a discrete NLS equation which they study by means of fixed point analysis and Melnikov-function method. In particular, the lower boundary was obtained from the tangential intersection of the invariant manifolds of a hyperbolic fixed point. A remarkable accuracy of Kaup and Newell's linear law detected in [11] as well as conclusions of their own Melnikov-function analysis prompted the authors of [13] to suggest that the relation $h_{\text{thr}} = (2/\pi)\gamma$ is *exact*, at least for sufficiently small γ.

The aim of the present note is to demonstrate that this relation is, in fact, *not* exact, and the actual reason why it appears to be so accurate for small γ is simply because the coefficient of the next term in the expansion of $h_{\text{thr}}(\gamma)$ in powers of γ is *anomalously small*. We do this by reconstructing the two solitons in the vicinity of the lower boundary of their existence domain by means of a singular (rather than regular) perturbation expansion.

The outline of this note is as follows. We start by discussing the regular asymptotic expansion as h and $\gamma \to 0$ (section 2). The procedure is similar to the one in [9]; the only difference is that since we now deal with solutions decaying at infinities ($\Psi_x \to 0$) rather than periodic as in [9], we will be able to find perturbative corrections in closed form. In section 3 we explain why the perturbation series of Ψ breaks down as h approaches the turning point, and replace it by a singular expansion. This will allow us to find the next terms in the expansion of $h_{\text{thr}}(\gamma)$. Some concluding remarks are made in section 4 followed by a brief summary of our results.

2 Regular perturbation expansion

By making a substitution $\Psi(x,t) = \psi(x,t)e^{it}$ Eq.(1) can be reduced to an autonomous equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = -i\gamma\psi - h. \tag{2}$$

We will be interested in time-independent solutions of Eq.(2); these satisfy the stationary equation

$$\psi_{xx} + 2|\psi|^2\psi - \psi = -i\gamma\psi - h \tag{3}$$

with the boundary conditions

$$\psi(x) \to \psi_0 \quad \text{as } |x| \to \infty.$$

We start with developing a regular perturbation expansion away from the turning point. As the authors of [9], we assume that we are approaching the origin on the (γ, h)-plane along a straight line $h = \hbar\gamma$ (where $\hbar$ is a proportionality coefficient, not to be confused with Planck's constant.) Letting

$$\psi = (u + iv)e^{-i\alpha}, \tag{4}$$

where α is some constant phase that can be conveniently chosen at a later stage, we expand

$$u = u_0 + \gamma u_1 + ..., \quad v = v_0 + \gamma v_1 + ... \tag{5}$$

and substitute into Eq.(3). The coefficient of γ^0 gives the unperturbed stationary NLS equation with a well-known soliton solution

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{sech} x.$$

Here θ is a free parameter. Next, at the order $O(\gamma^1)$ one gets

$$\hat{H}_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \hbar\cos\alpha - v_0 \\ \hbar\sin\alpha + u_0 \end{pmatrix}, \tag{6}$$

where the Hermitean operator

$$\hat{H}_0 \equiv (-\partial^2 + 1)\hat{I} - 2 \begin{pmatrix} v_0^2 + 3u_0^2 & 2u_0 v_0 \\ 2u_0 v_0 & u_0^2 + 3v_0^2 \end{pmatrix}; \qquad (7)$$

$\partial = \partial/\partial x$ and $\hat{I}$ is the 2×2 identity matrix. In order for the equation (6) to be solvable, its right-hand side needs to be orthogonal to the vector $(v_0, -u_0)^T$, the eigenfunction of the operator $\hat{H}_0$ associated with the zero eigenvalue. (This zero eigenvalue results from the U(1) phase-invariance of the unperturbed NLS equation.) The orthogonality gives a relation between α and θ,

$$\pi \hbar \sin(\theta - \alpha) = 2, \qquad (8)$$

implying that only one of the two parameters (say, θ) can be chosen freely. It does not matter what exactly we choose for θ; the net phase of the leading-order approximation depends only on $(\theta - \alpha)$ and this is fixed by Eq.(8). The meaning of this relation is straightforward. For $h = \gamma = 0$, the NLS equation has a family of soliton solutions, $\psi = e^{i(\theta - \alpha)}\mathrm{sech}x$, with $(\theta - \alpha)$ arbitrary. However, if we want to continue the solution along the line $h = \hbar\gamma$, the unperturbed solution that we need to start with has the phase given by Eq.(8).

It is convenient to take $\theta = \pi/2$; this makes the linear operator (7) diagonal. The constant phase α is then determined by

$$\cos \alpha = \frac{2}{\pi}\frac{1}{\hbar}. \qquad (9)$$

In fact, there are two values of α defined by this equation, one positive and one negative. The positive $\alpha = \alpha_+$ corresponds to the soliton $\psi^{(+)}$ and the negative $\alpha = \alpha_-$ defines the soliton $\psi^{(-)}$. Since the left-hand side cannot exceed 1, the right-hand side gives the well-known formula for the lower boundary of the domain of existence of the two solitons: $\hbar \geq \hbar_{\mathrm{thr}} = 2/\pi$ [8, 9, 10]. (In the next section we will obtain a more precise formula for this threshold.)

Now for $\theta = \pi/2$ the equations (6) become

$$L_0 u_1(x) = \hbar \cos \alpha - v_0(x); \qquad (10)$$
$$L_1 v_1(x) = \hbar \sin \alpha, \qquad (11)$$

where $v_0(x) = \mathrm{sech}x$ and L_0 and L_1 are the well-known Schrödinger operators with familiar spectral properties:

$$L_0 = -\partial^2 + 1 - 2\mathrm{sech}^2 x; \qquad (12)$$
$$L_1 = -\partial^2 + 1 - 6\mathrm{sech}^2 x. \qquad (13)$$

The operator L_1 is invertible on even functions; in particular,

$$L_1^{-1}\mathrm{sech}\, x = \frac{1}{2}(x \tanh x - 1)\mathrm{sech}x, \qquad (14)$$

$$L_1^{-1}\mathrm{sech}^3 x = -\frac{1}{4}\mathrm{sech}x \qquad (15)$$

and

$$L_1^{-1}1 = 1 - 2\mathrm{sech}^2 x. \qquad (16)$$

Hence Eq.(11) is readily solved:

$$v_1 = \hbar \sin \alpha(1 - 2\mathrm{sech}^2 x). \qquad (17)$$

The condition (9) being in place, Eq.(10) is solved as well:

$$u_1 = \mathcal{U}_1(x) + A\mathrm{sech}x, \qquad (18)$$

where

$$\mathcal{U}_1(x) = \frac{2}{\pi} + \frac{1}{2}\tanh x \sinh x + \frac{1}{\pi} \times \{j(x)\,\mathrm{sech}x -$$
$$-(x\,\mathrm{sech}x + \sinh x)\arcsin(\tanh x) - 1\}, \qquad (19)$$
$$j(x) \equiv \int_0^x \xi\,\mathrm{sech}\xi\,d\xi,$$

and A is an arbitrary constant which is to be fixed at higher orders of the expansion. Hence we proceed to $O(\gamma^2)$ to find

$$L_0 u_2 = (2v_0 u_1 - 1)v_1, \qquad (20)$$
$$L_1 v_2 = 2v_0(u_1^2 + 3v_1^2) + u_1. \qquad (21)$$

Equation (20) is solvable if its right-hand side is orthogonal to $\mathrm{sech}x$. Substituting from (17)-(18), this condition fixes the constant A:

$$A = A^{(0)} \equiv \frac{4}{\pi}\int \mathcal{U}_1(x)\,\mathrm{sech}^2 x(1 - 2\mathrm{sech}^2 x)dx, \qquad (22)$$

where we have used Eqs.(15-16). (Here we have written $A^{(0)}$ for A so as to emphasize that this is now a fixed number; this number will reappear in the singular expansion below.) Eq.(20) is now solved in the form

$$u_2 = \mathcal{U}_2(x) + B\operatorname{sech} x. \qquad (23)$$

The constant B is to be fixed at the γ^3-level, where we obtain the equation

$$L_0 u_3 = 2\{u_1(v_1^2 + u_1^2) + 2v_0(u_2 v_1 + u_1 v_2)\} - v_2. \qquad (24)$$

The solvability condition for eq.(24) gives us B:

$$B = (\pi\hbar\sin\alpha)^{-1} \times$$
$$\times \int v_0\{2u_1(u_1^2 + v_1^2) + 4v_0(u_1 v_2 + \mathcal{U}_2 v_1) - v_2\}\, dx. \qquad (25)$$

So far our treatment followed the lines of Terrones *et al* [9]; the only difference is that our $v_0, u_1, v_1, ...$ are given by explicit formulas. Using (19) in (22) and integrating numerically, we identify the constant $A^{(0)}$ which completes the determination of the first-order corrections: $A^{(0)} = -2.4378 \times 10^{-1}$.

Let us now send $\hbar \to (2/\pi)\gamma$. The formula (18) for $u_1(x)$ is not affected and the expression (22) for A remains valid as well. Therefore, the solvability of Eq.(20) is ensured and u_2 can be written in the form (23). The constant B is expected to be identifiable from Eq.(25). However, for $\hbar \to 2/\pi$ we have $\sin\alpha \to 0$ and so this formula gives $B = \infty$ unless

$$\int v_0(4v_0 u_1 v_2 + 2u_1^3 - v_2)dx = 0. \qquad (26)$$

(Here we have used that $v_1 \to 0$ as $\sin\alpha \to 0$.) In general the condition (26) is *not* in place, and therefore the regular expansion blows up.

3 Singular perturbation expansion at the turning point

The reason for the breaking down of the expansion is that it was implicitly assumed in Eq.(5) that $v_1 = O(1)$ whereas in the actual fact, in the limit $\hbar \to 2/\pi$ we have $v_1 \to 0$. Thus, let us now explicitly take this fact into account by writing

$$u = \gamma u_1 + \gamma^2 u_2..., \quad v = v_0 + \gamma^2 v_2 + ... \qquad (27)$$

where $v_0 = \operatorname{sech} x$ and

$$\hbar = \hbar_0 + \hbar_1\gamma + \hbar_2\gamma^2 + ..., \quad \hbar_0 = \frac{2}{\pi}. \qquad (28)$$

Here we have fixed $\theta = \pi/2$ and $\alpha = 0$ straight away. Substituting into (3), the first order in γ yields eq.(10) where we should only replace $\hbar \to \hbar_0$. Its solution is given by the same eq.(18-19) as before, with A an undetermined constant. At the order γ^2 we obtain

$$L_0 u_2 = \hbar_1,$$

and hence $\hbar_1 = 0$ and $u_2 = B\operatorname{sech} x$. We also obtain the equation for v_2 which is always solvable:

$$v_2 = L_1^{-1}(u_1 + 2v_0 u_1^2). \qquad (29)$$

Finally, the γ^3-level yields

$$L_0 u_3 = 2(2v_0 u_1 v_2 + u_1^3) - v_2 + \hbar_2,$$

whose solvability condition is given by

$$\int v_0 \left(4v_0 u_1 v_2 + 2u_1^3 - v_2 + \hbar_2\right) dx = 0. \qquad (30)$$

We will show now that it is only this equation (30) that fixes the constant A in Eq.(18).

Substituting u_1 from (18) and v_2 from (29), Eq.(30) reduces to a quadratic equation for the unknown A:

$$A^2 - 2PA + Q - \pi\hbar_2 = 0, \qquad (31)$$

where after some algebra the coefficients are found to be

$$P = -2\hbar_0^2 + \frac{\hbar_0}{2}\int\{\hbar_0 - \mathcal{U}_1(x)\}dx \qquad (32)$$

and

$$Q = \int \{\mathcal{U}_1^2 - \mathcal{U}_1(1 + 2\mathcal{U}_1 \text{sech} x) \times$$
$$\times [h_0(1 - 2\text{sech}^2 x) + \text{sech} x\,(1 - x\tanh x)]\} dx. \tag{33}$$

In the derivation of (32-33) we used Eq.(14) and the identity

$$4L_1^{-1}(v_0^2 \mathcal{U}_1) = L_1^{-1}(h_0 - v_0) - \mathcal{U}_1; \tag{34}$$

this is a straightforward consequence of Eqs.(9),(10) and the fact that the Schrödinger operators (12)-(13) differ by $4\,\text{sech}^2 x$:

$$L_0 = L_1 + 4v_0^2(x). $$

Since there is a cubic term in u_1 in Eq.(30), one could expect the resulting equation for A to be cubic; however the coefficient in front of A^3 is easily shown to vanish. Another observation is that the coefficient P coincides with the constant $A^{(0)}$ [Eq.(22)] obtained in the regular expansion. To see that, one only needs to use the identity (34) once again.

Solutions of (31) are given by

$$A^{(\pm)} = A^{(0)} \pm \sqrt{\pi\left(h_2 - h_2^{(0)}\right)}, \tag{35}$$

where

$$h_2^{(0)} \equiv \frac{1}{\pi}(Q - P^2). \tag{36}$$

If $h_2 > h_2^{(0)}$, we have two solutions $\psi^{(\pm)}$ which are only different in the coefficients $A^{(\pm)}$. If $h_2 < h_2^{(0)}$, we have no solutions at all. The value $h_2 = h_2^{(0)}$ is therefore the turning point. Doing numerically the integral in (33) we find $Q = 6.4665 \times 10^{-2}$. Recalling that P coincides with Eq.(22), $P = A^{(0)} = -2.4378 \times 10^{-1}$, Eq.(36) gives $h_2^{(0)} = 1.667 \times 10^{-3}$. Finally, the coefficient A corresponding to the turning point coincides with the off-turning point value, Eq.(22): $A = A^{(0)}$.

It is worth noting here that if $h_2 = 0$, Eq.(30) is formally coincident with Eq.(26). This does not mean, however, that soliton solutions exist for $h_2 = 0$, and that these solutions can be found by regular expansions (4-5). The point is that in Eq.(26) the function $u_1(x)$ has the coefficient A which has already been fixed by Eq.(22), whereas Eq.(30) is an equation for *unknown* A.

Next, how close to h_{thr} does the regular expansion stop working and has to be replaced by the singular one? For $h = 2/\pi + h_2\gamma^2$ Eq.(9) produces

$$\alpha_{\pm} = \pm\sqrt{\pi h_2}\,\gamma + O(\gamma^3) \tag{37}$$

and so the regular expansions for $\psi^{(\pm)}$ read

$$\psi^{(\pm)} = i\,\text{sech} x + \gamma\mathcal{U}_1(x) +$$
$$+ \gamma\left(A^{(0)} \pm \sqrt{\pi h_2}\right)\text{sech} x + O(\gamma^2) \tag{38}$$

with $\mathcal{U}_1$ as in (19), whereas the correct, singular expansion is

$$\psi^{(\pm)} = i\,\text{sech} x + \gamma\mathcal{U}_1(x) + \gamma A^{(\pm)}\text{sech} x + O(\gamma^2), \tag{39}$$

where $A^{(\pm)}$ are given by Eq.(35). Comparing (38) to (39), one concludes that the difference between the regular and singular expansions is negligible provided $h_2 \gg h_2^{(0)} \sim 2 \times 10^{-3}$. Otherwise the difference cannot be ignored.

4 Concluding remarks and conclusions

1. In the undamped case ($\gamma = 0$) for any $h \in (0, \sqrt{2/27})$ Eq.(3) has two explicit solutions [14]:

$$\psi^{(\pm)}(x) = \psi_0\left\{1 + \frac{2\sinh^2\alpha}{1 \pm \cosh(\mathcal{A}x)\cosh\alpha}\right\}, \tag{40}$$

where

$$\psi_0 = \left\{2(1 + 2\cosh^2\alpha)\right\}^{-1/2}$$

is the asymptotic value of $\psi^{(\pm)}(x)$ as $|x| \to \infty$; the parameter α is defined by inverting the relation

$$h = \frac{\sqrt{2}\cosh^2\alpha}{(1 + 2\cosh^2\alpha)^{3/2}},$$

Fig.1
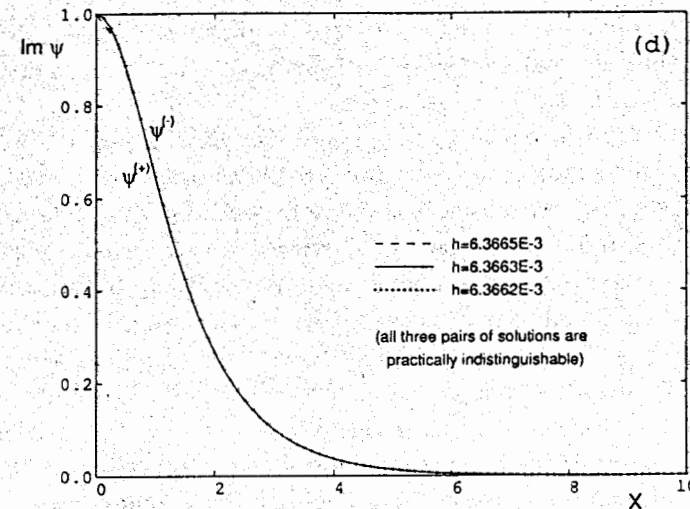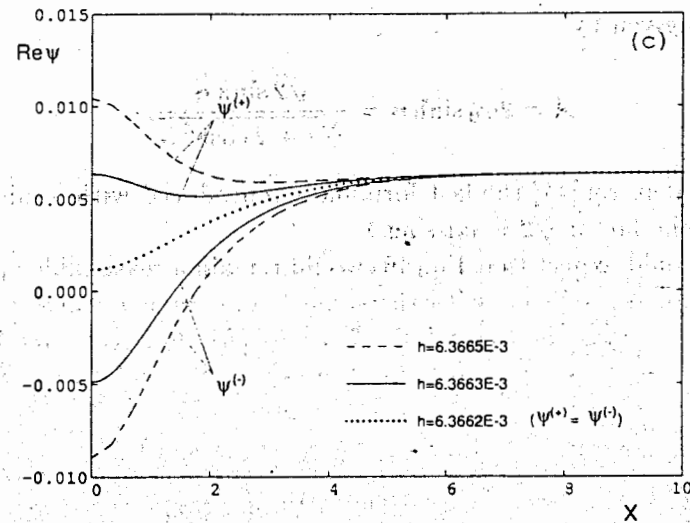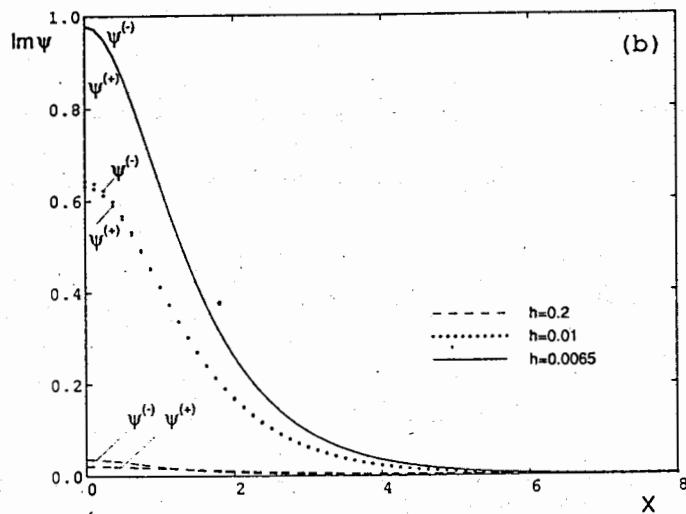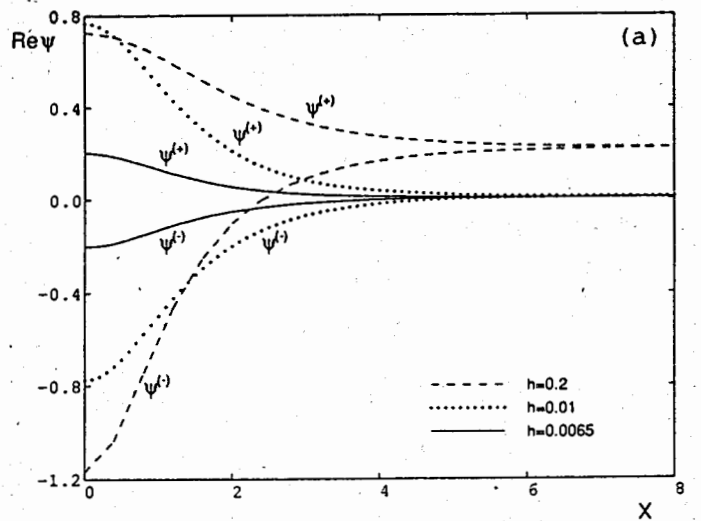
Fig.1. Soliton transformation for small γ ($\gamma = 0.01$). (a,b): For h far above the turning point $h_{\text{thr}} = 6.3662 \times 10^{-3}$ the imaginary parts are close to zero and the two solitons are well approximated by the two undamped solitons (40) (dashed lines). As h approaches the turning point, the real parts decrease and converge while imaginary parts grow (dotted then solid lines.) (c,d): In the immediate vicinity of h_{thr} the hump in the $\psi^{(+)}$ profile rapidly transforms into a dip and the two solitons collapse into one (dashed then solid then dotted line.)

and $\mathcal{A}$ is given by

$$\mathcal{A} = 2\psi_0 \sinh \alpha = \frac{\sqrt{2}\sinh \alpha}{\sqrt{1 + 2\cosh^2 \alpha}}.$$

(Note that in ref.[11] the last formula is reproduced with a misprint; namely, the factor $\sqrt{2}$ is missing.)

One could expect that Eq.(40) would remain a reasonable approximation for the solitons with the same h and small *nonzero* γ — in other words, that the two solutions can be smoothly continued from the h-axis to the (γ, h)-plane. On the other hand, we know that for the given value of h the soliton $\psi^{(+)}$ merges with $\psi^{(-)}$ at some γ_{thr} [defined, approximately, by $h = (2/\pi)\gamma_{\mathrm{thr}} + h_2^{(0)}\gamma_{\mathrm{thr}}^3$]. If h is small then this γ_{thr} is also small, so that the point of the merger is very close to the h-axis and hence intuitively one could expect solutions at this point to be close to the undamped solitons (40). However, it is not quite obvious how this proximity can be reconciled with the fact that the two real solutions (40) are rather far from each other; $\psi^{(+)}$ has a hump and $\psi^{(-)}$ has a dip.

The situation near the turning point can be clarified by invoking the asymptotic expansions (4-5) and (27-28). For the sake of illustration, we have also computed the two solitons numerically for a fixed small γ ($\gamma = 0.01$) and varying h; results are shown in Fig.1. In agreement with Eq.(4-5), for not very small h ($h \geq 0.1$) the imaginary parts of the solitons are seen to be almost zero [dashed lines in Fig.1(b)] while the real parts change slowly with the variation of h. For these h the pair of real solutions (40) does indeed provide a good approximation for the corresponding $\psi^{(\pm)}$ with $\gamma = 0.01$. However, as h goes down, the real parts start changing (decreasing) more vigorously while the imaginary parts begin to grow; consequently, the approximation deteriorates. Near the turning point the real parts of both solitons become much smaller than their imaginary parts [see Fig. 1(c,d)]. Nevertheless, for h not very close to the turning point (more specifically, for $h \geq 6.3665 \times 10^{-3}$), the real part of $\psi^{(+)}$ still has a hump and real part of $\psi^{(-)}$ still has a dip [Fig. 1(a)]. This justifies our usage of the notations $\psi^{(+)}$ and $\psi^{(-)}$ for $\gamma \neq 0$. Finally, in a very near vicinity of the turning point $h_{\mathrm{thr}} = 6.3662 \times 10^{-3}$, the

hump of the real part quickly transforms into the dip [Fig.1(c)].

This evolution of the two solitons (in particular, the rapid change near the turning point) can be easily understood in terms of the singular expansion (27-28). Near the turning point, where h_2 is close to $h_2^{(0)}$, we can define a small ϵ by writing

$$\pi h_2 = \pi h_2^{(0)} + \epsilon. \tag{41}$$

The corresponding A's in Eq.(18) are then given by

$$A^{(\pm)} = A^{(0)} \pm \sqrt{\epsilon}. \tag{42}$$

Recalling that to the leading order in γ it is only this coefficient A that determines the dependence of solutions on h, Eq.(42) gives the rate of change of their real parts:

$$\frac{\partial u}{\partial h_2} = \pm \frac{\pi}{2}\frac{\gamma}{\sqrt{\epsilon}}\operatorname{sech} x + O(\gamma^2). \tag{43}$$

As it should be the case near the turning point, the derivative (43) becomes very large as $\epsilon \to 0$ implying a very rapid transformation of the soliton. Away from the neighbourhood of the turning point, in the region of the applicability of the regular expansion (4-5), the rate of the transformation of the solitons $\psi_\pm$ is given by

$$\frac{d\alpha}{dh} = \frac{2}{\pi h}\frac{1}{\sqrt{h^2 - (2/\pi)^2}}.$$

Similarly to Eq.(43), this shows that as $h \to 2/\pi$, the two solitons transform increasingly fast.

Thus if we want to use the two real solitons (40) as approximations for their respective ($\gamma \neq 0$)-counterparts, we should keep in mind that this approximation is valid only far away from the turning point γ_{thr}. Since for small h the turning point is close to the h-axis (i.e. $\gamma_{\mathrm{thr}} \sim (\pi/2)h$ is also small), the validity of the approximation will be restricted to *very* small γ, $\gamma/h \ll \pi/2$.

2. It is important to emphasize that the perturbative expansion constructed in this note is asymptotic, i.e. it only tends to $h_{\mathrm{thr}}(\gamma)$ as

$\gamma \to 0$. To see that the series does not necessarily have to converge for finite γ, take $\gamma = 0.48$ in which case the numerical analysis [11] shows that the difference $h_{\text{thr}}/\gamma - (2/\pi)$ is approximately 10^{-3}. The discrepancy between this numerical value and the perturbative result $(1.667 \times 10^{-3})\gamma^2$ is of the order 6×10^{-4}, which is a good agreement. However if we wanted to improve the accuracy even further, by adding the next term $\hbar_3 \gamma^3$ to the expansion $h_{\text{thr}} = \sum \hbar_n \gamma^{n+1}$, then in order to account for the above discrepancy we would have to take $\hbar_3 \sim 6 \times 10^{-3}$. The coefficient $\hbar_3$ being several times larger than the previous coefficient $\hbar_2^{(0)}$ is an indication of the divergence of the series.

3. Finally, we briefly summarize the main points of this work.

(**a.**) The lower boundary of the existence domain of the two solitons is given by the following asymptotic expression (as $\gamma \to 0$):

$$h_{\text{thr}} = \frac{2}{\pi}\gamma + (1.667 \times 10^{-3})\gamma^3 + O(\gamma^4). \tag{44}$$

(**b.**) For h away from the above threshold, more precisely for $h - (2/\pi)\gamma \gg 0.002\gamma^3$, the solitons are given by the asymptotic expansion Eq.(4-5) where v_1 and u_1 are given by explicit expressions (17)-(19) with $A^{(0)} = -2.4378 \times 10^{-1}$.

(**c.**) For h close to the turning point, $h = (2/\pi)\gamma + \hbar_2 \gamma^3$ with $\hbar_2 \sim 0.002$ the second, γ^2-order of the regular expansion (4-5) becomes greater than the first order, and the expansion breaks down. In this case the two solitons are given by the singular expansion (39) with $A^{(\pm)}$ as in Eq.(35) and $\hbar_2^{(0)} = 1.667 \times 10^{-3}$.

5 Acknowledgments

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Барашенков И.В., Земляная Е.В.   E17-98-290
Порог существования солитонов в нелинейном уравнении
Шредингера с диссипацией и накачкой

Известно, что солитоны нелинейного уравнения Шредингера с диссипацией и периодическим возбуждением существуют лишь в случае, если амплитуда накачки превышает приблизительно $(2/\pi)\,\gamma$, где γ — коэффициент диссипации. Недавние исследования показали, что формула $h_{\text{thr}} - (2/\pi)\,\gamma$ описывает порог возникновения солитонов гораздо точнее, чем можно было бы ожидать, исходя из того обстоятельства, что она получена лишь в главном порядке теории возмущений. На этом основании было высказано предположение, что указанная формула является точной, т.е. справедлива во всех порядках. В настоящей работе вычислен следующий порядок в разложении $h_{\text{thr}}\,(\gamma)$ и показано, что действительной причиной этого явления является аномальная малость коэффициента следующего члена разложения: $h_{\text{thr}} = (2/\pi)\,\gamma + 0{,}002\,\gamma^3$.

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Barashenkov I.V., Zemlyanaya E.V.   E17-98-290
Existence Threshold for the AC-Driven Damped Nonlinear
Schrödinger Solitons

It has been known for some time that solitons of the externally driven, damped nonlinear Schrödinger equation can only exist if the driver's strenght, h, exceeds approximately $(2/\pi)\,\gamma$, where γ is the dissipation coefficient. Although this perturbative result was expected to be correct only to the leading order in γ, recent studies have demonstrated that the formula $h_{\text{thr}} = (2/\pi)\,\gamma$ gives a remarkably accurate description of the soliton's existence threshold prompting suggestions that it is, in fact, *exact*. In this note we evaluate the next order in the expansion of $h_{\text{thr}}\,(\gamma)$ showing that the actual reason for this phenomenon is simply that the next-order coefficient is anomalously small: $h_{\text{thr}} = (2/\pi)\,\gamma + 0.002\,\gamma^3$.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.