ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E17-98-251

G.Jackeli, N.M.Plakida

CHARGE DYNAMICS AND OPTICAL CONDUCTIVITY OF THE $t-J$ MODEL

Submitted to «Physical Review B»

1998

I. INTRODUCTION

Among the other unconventional normal state properties of high-T_c superconductors an anomalous charge dynamics has also been detected in the optical measurements of the underdoped samples [1]. Namely, a non-Drude fall off of the low-frequency absorption indicating a linear ω–dependence of the relaxation rate and an anomalous mid-infrared (MIR) band with a typical energy ~ 0.1 eV have been observed [2,3].

It is widely believed that unusual properties of the superconducting cuprates are due to the strong electron correlations [1]. The minimal model to describe correlation effects in the cuprates is the $t - J$ model. While a number of analytical works have been done to investigate spin dynamics within the $t - J$ model, only few of authors have studied charge dynamics [4–6]. In Refs. [4,5] charge fluctuations have been studied by slave boson and Hubbard operator (HO) formalism within the leading order of $1/N$ expansion, respectively. It was found that the density fluctuations at large momenta shows a sharp high energy peak corresponding to the collective mode which reduces to the sound mode in the long-wavelength limit [4,5]. Later, the authors of Ref. [6] showed that next order corrections in $1/N$ expansion leads to the broadening of the high energy peak due to incoherent motion of bear holes. Similar features of the density response have been previously observed in exact diagonalization studies of small clusters [7].

In the present paper we investigate charge fluctuation spectrum of the $t-J$ model in the paramagnetic state with short-range antiferromagnetic (AFM) correlations. We develop a self-consistent theory for the dynamical charge susceptibility (DCS) by applying the memory function method in terms of HO's. The employment of HO technique has a twofold advantage; By using the equations of motion for the HO's we automatically take into account scattering of electrons on spin and charge fluctuations originated from the strong correlations, as it has first been pointed out by Hubbard [8]. Moreover HO formalism allows us to preserve rigorously the local constraint of no doubly occupancy.

We calculate the memory function within the mode coupling approximation (MCA) in terms of the dressed particle-hole and spin fluctuations. Similarly to the antiferromagnetic Fermi liquid approach [9], we treat fermionic and localized spin excitations as independent degrees of freedom. We show that the memory function involves two contributions. The first one stems from the hopping term and describes a particle-hole contribution from the itinerant hole subsystem to the DCS. The second one involves scattering processes of electrons on charge and spin fluctuations and comes from both kinematik and exchange interactions.

Further, we perform an analytical analysis of different limiting behavior of DCS to show that the essential features observed in the exact diagonalization studies [7] can be reproduced within the present formalism. We find out that for small q the DCS is mainly governed by the sound mode. Although unrenormalized sound velocity is larger than Fermi velocity, unlike to the Fermi liquid theory, the "self-energy" corrections, lead to softening of the sound. The renormalized sound falls down into particle-hole continuum getting a finite damping due to the decay into pair excitations. In the short-wavelength limit momenta density fluctuation spectrum mainly consists of a broad high-energy peak. At large enough wave vectors the peak is dispersed out from coherent particle-hole continuum and broadens due to high energy $\sim t$ transitions involving the incoherent band of the one-particle excitations. The contribution from the particle-hole excitations from the coherent bandwidth leads to a some low-energy structure in the charge fluctuation spectrum.

We also discuss the optical conductivity $\sigma(\omega)$. For low frequencies we analyze $\sigma(\omega)$ in terms of the generalized Drude law. We show that there is a mass enhancement of order $m^*/m \simeq 6$, due to the electron scattering on spin fluctuations. These scattering processes also leads to a frequency dependent relaxation rate which exhibits a crossover from $\omega^{3/2}$ behavior at low frequencies, $\omega < 2|\mu|$, to a linear ω-dependence for $\omega > 2|\mu|$. We also discuss a possible origin of MIR band.

The paper is structured as follows. In the next section we give the basic definitions and sketch the memory function formalism. In Sec. III we employ MCA to calculate a memory function. The dynamical charge susceptibility and optical conductivity are discussed in Secs. IV and V, respectively. The last section summarizes our main results.

II. MODEL AND MEMORY FUNCTION FORMALISM

The $t-J$ model expressed in terms of HO's, $X_i^{\alpha\beta} = |i,\alpha\rangle\langle i,\beta|$, reads as

$$H = H_t + H_J = -\sum_{i,j} t_{ij} X_i^{\sigma 0} X_j^{0\sigma} + \frac{1}{4}\sum_{i,j,\sigma} J_{ij}\{X_i^{\sigma\bar\sigma} X_j^{\bar\sigma\sigma} - X_i^{\sigma\sigma} X_j^{\bar\sigma\bar\sigma}\}, \tag{1}$$

where the indices 0 and $\sigma = \pm 1$ correspond to a hole and an electron with spin $\sigma/2$, respectively, $t_{ij} = t$ and $J_{ij} = J$ for the nearest-neighbor (n.n.) sites on a planar lattice. The HO's can be either Bose-like or Fermi-like and obey the following on-site multiplication rules $X_i^{\alpha\beta} X_i^{\gamma\delta} = \delta_{\beta\gamma} X_i^{\alpha\delta}$ and the commutation relations

$$\left[X_i^{\alpha\beta} X_j^{\gamma\delta}\right]_{\pm} = \delta_{ij}\left(\delta_{\beta\gamma} X_i^{\alpha\delta} \pm \delta_{\delta\alpha} X_i^{\gamma\beta}\right), \tag{2}$$

where the upper sign stands for the case when both HO's are Fermi-like otherwise the lower sign should be adopted. In the $t-J$ model only singly occupied sites are retained and the completeness relation for the HO's reads as

$$X_i^{00} + \sum_\sigma X_i^{\sigma\sigma} = 1. \tag{3}$$

The spin and density operators are expressed by HO's as

$$S_i^\sigma = X_i^{\sigma\bar\sigma}, \quad S_i^z = \frac{1}{2}\sum_\sigma \sigma X_i^{\sigma\sigma}, \quad n_i = \sum_\sigma X_i^{\sigma\sigma}. \tag{4}$$

The dynamical charge susceptibility $N_{\mathbf{q}}(\omega)$ is given by a Fourier transformed two-time retarded Green function (GF) [10]

$$N_{\mathbf{q}}(\omega) = -\langle\langle n_{\mathbf{q}} | n_{-\mathbf{q}}\rangle\rangle_\omega = i\int_0^\infty dt e^{i\omega t}\langle [n_{\mathbf{q}}(t), n_{-\mathbf{q}}]\rangle. \tag{5}$$

To calculate $N_{\mathbf{q}}(\omega)$ we employ the memory formalism as discussed in [11,12]. First we introduce density-density relaxation function

$$\Phi_q(\omega) = ((n_{\mathbf{q}}|n_{-\mathbf{q}}))_\omega = -i\int_0^\infty dt e^{i\omega t}(n_{\mathbf{q}}(t)|n_{-\mathbf{q}}), \tag{6}$$

where the Kubo-Mori scalar product is defined as

$$(A(t), B) = \int_0^\beta d\lambda \langle A(t - i\lambda) B \rangle, \tag{7}$$

with $\beta = 1/T$. The DCS $N_{\mathbf{q}}(\omega)$ is coupled to the relaxation function $\Phi_q(\omega)$ by the equation

$$N_{\mathbf{q}}(\omega) = N_{\mathbf{q}} - \omega \Phi_q(\omega), \tag{8}$$

where $N_{\mathbf{q}} = N_{\mathbf{q}}(0)$ is the static susceptibility.

We introduce the memory function $M^0_{\mathbf{q}}(\omega)$ for the relaxation function $\Phi_{\mathbf{q}}(\omega)$ as

$$\Phi_{\mathbf{q}}(\omega) = \frac{N_{\mathbf{q}}}{\omega - M^0_{\mathbf{q}}(\omega)/N_{\mathbf{q}}}. \tag{9}$$

By adopting the equation of motion method for the relaxation function $\Phi_{\mathbf{q}}(\omega)$ one finds, that the memory function $M^0_{\mathbf{q}}(\omega)$ is given by the irreducible part of the relaxation function for "currents" [13,14]

$$M^0_{\mathbf{q}}(\omega) = ((j_{\mathbf{q}}|j_{-\mathbf{q}}))^{irr}_{\omega}. \tag{10}$$

The "current" operator $j_{\mathbf{q}}$ in the site representation reads as

$$j_i = \dot{n}_i = -i[n_i, H] = -i\sum_{j,\sigma} t_{ij}(X^{\sigma 0}_j X^{0\sigma}_i - \text{H.c.}). \tag{11}$$

The Heisenberg part of the Hamiltonian (1) conserves the local particle number and thus gives no contribution to the "current" operator (11). To treat properly a contribution from H_J term we go one step further and similarly to Eq.(9) we introduce the memory function $M_{\mathbf{q}}(\omega)$ for the relaxation function for "currents" $M^0_{\mathbf{q}}(\omega)$ (10), by the following equation

$$M^0_{\mathbf{q}}(\omega) = \frac{m_{\mathbf{q}}}{\omega - M_{\mathbf{q}}(\omega)/m_{\mathbf{q}}}, \tag{12}$$

where

$$m_{\mathbf{q}} = -\langle\langle j_{\mathbf{q}}|j_{-\mathbf{q}}\rangle\rangle_{\omega=0} = i\langle [j_{\mathbf{q}}, n_{-\mathbf{q}}]\rangle \tag{13}$$

is the first momentum of DCS and the memory function $M_{\mathbf{q}}(\omega)$ is given by the irreducible part of relaxation function for "forces ":

$$M_{\mathbf{q}}(\omega) = ((F_{\mathbf{q}}|F_{-\mathbf{q}}))^{irr}_{\omega} \tag{14}$$

with

$$F_{\mathbf{q}} = \dot{j}_{\mathbf{q}} = -i[j_{\mathbf{q}}, H]. \tag{15}$$

Further, to close the system of equations [see Eqs.(8),(9), and (12)] we employ a mode coupling approximation for the memory function $M_{\mathbf{q}}(\omega)$.

III. MODE COUPLING APPROXIMATION

First we express the memory function in terms of the irreducible part of time-dependent corelation function for "forces" by means of the fluctuation-dissipation theorem [11]

$$M_{\mathbf{q}}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{e^{\beta\omega'} - 1}{\omega'(\omega - \omega' + i\eta)} \times \int_{-\infty}^{\infty} dt e^{-i\omega' t} \langle F_{-\mathbf{q}}(t)|F_{\mathbf{q}}\rangle^{irr}. \tag{16}$$

The "force" operator given by

$$F_i = \sum_{m,j,\sigma\sigma'} \left[\Pi^{\sigma\sigma'}_{i,m,j} - \Pi^{\sigma\sigma'}_{m,i,j} + \text{H.c.}\right], \tag{17}$$

$$\Pi^{\sigma\sigma'}_{m,i,j} = t_{im} \; [\; t_{mj}(X^{\sigma 0}_i X^{0\sigma}_j \delta_{\sigma\sigma'} - X^{\sigma 0}_i X^{0\sigma'}_j B^{\sigma'\sigma}_m) + J_{mj} X^{\sigma 0}_i X^{0\sigma'}_m B^{\sigma'\sigma}_j], \tag{18}$$

where the Bose-like operator

$$B^{\sigma'\sigma}_i = X^{\bar\sigma\bar\sigma}_i \delta_{\sigma\sigma'} - X^{\bar\sigma\sigma}_i \delta_{\bar\sigma\sigma'} = \left[\frac{1}{2}n_i - \sigma S^z_i\right]\delta_{\sigma\sigma'} - S^{\bar\sigma}_i \delta_{\bar\sigma\sigma'} \tag{19}$$

describes electron scattering on spin and charge fluctuations.

The sum in Eq.(17) contains the products of HO's from the same site. As follows, such products give no contribution to the memory function, while being decoupled they do contribute. That is result of the complexity of HO's algebra (2). To show this let us consider the term given by Eq.(18); since (i,m) and (m,j) are n.n. pairs $i \neq m$ and $m \neq j$, however i can be equal to j. For the latter case, $i = j$, the first term in Eq.(18) is linear in density operator ($X_i^{\sigma 0}X_i^{0\sigma} = X_i^{\sigma\sigma}$) and thus gives no contribution to the irreducible part of corelation function for "forces" [14]. As for the second term (18), one can easily verify that in the case $i = j$ it is canceled out by its counter part from the sum (17). Finally, the last term in Eq.(18) vanishes for $i = j$ since $X_i^{\sigma 0}B_i^{\sigma'\sigma} = 0$ due to the constraint. Therefore we have to substruct these terms from the "force" operator. As a result we come to the following expression in the momentum space

$$F_{\mathbf{q}} = -\frac{1}{\sqrt{N}}\sum_{\mathbf{k},\sigma} G_{\mathbf{k},\mathbf{q}} X_{\mathbf{k}+\mathbf{q}}^{\sigma 0} X_{\mathbf{k}}^{0\sigma} - \frac{1}{N}\sum_{\mathbf{k},\mathbf{p},\sigma\sigma'} M_{\mathbf{k},\mathbf{q},\mathbf{p}} X_{\mathbf{k}+\mathbf{q}-\mathbf{p}}^{\sigma 0} X_{\mathbf{k}}^{0\sigma'} B_{\mathbf{p}}^{\sigma'\sigma}, \tag{20}$$

the vertexes $G_{\mathbf{k},\mathbf{q}}$ and $M_{\mathbf{k},\mathbf{q},\mathbf{p}}$ are given by

$$G_{\mathbf{k},\mathbf{q}} = g_{\mathbf{k},\mathbf{q}} - \overline{g}_{\mathbf{k},\mathbf{q}} \tag{21}$$

$$M_{\mathbf{k},\mathbf{q},\mathbf{p}} = \sum_i [m_{\mathbf{k},\mathbf{q},\mathbf{p}}^{(i)} - \overline{m}_{\mathbf{k},\mathbf{q},\mathbf{p}}^{(i)}] \tag{22}$$

where

$$g_{\mathbf{k},\mathbf{q}} = (zt)^2\gamma_{\mathbf{k},\mathbf{q}}^2, \quad m_{\mathbf{k},\mathbf{q},\mathbf{p}}^{(1)} = \frac{z^2tJ}{2}\gamma_{\mathbf{p}}[\gamma_{\mathbf{k}-\mathbf{p},\mathbf{q}} - \gamma_{\mathbf{k},\mathbf{q}}]$$
$$m_{\mathbf{k},\mathbf{q},\mathbf{p}}^{(2)} = (zt)^2[\gamma_{\mathbf{k},\mathbf{q}}\gamma_{\mathbf{k}+\mathbf{q}-\mathbf{p}} - \gamma_{\mathbf{k}-\mathbf{p},\mathbf{q}}\gamma_{\mathbf{k}}]. \tag{23}$$

In Eqs.(21) and (22) $\overline{g}(\overline{m})$ denote $g(m)$ averaged over the Brillouin zone and are given by

$$\overline{g}_{\mathbf{k},\mathbf{q}} = 2zt^2(1-\gamma_{\mathbf{q}}), \; \overline{m}_{\mathbf{k},\mathbf{q},\mathbf{p}}^{(1)} = \frac{ztJ}{2}[\gamma_{\mathbf{k},\mathbf{q}} - \gamma_{\mathbf{k}-\mathbf{p},\mathbf{q}}]$$
$$\overline{m}_{\mathbf{k},\mathbf{q},\mathbf{p}}^{(2)} = 2zt^2\gamma_{\mathbf{p}-\mathbf{q},\mathbf{q}}, \tag{24}$$

where $\gamma_{\mathbf{k},\mathbf{q}} = \gamma_{\mathbf{k}+\mathbf{q}} - \gamma_{\mathbf{k}}$, $\gamma_{\mathbf{q}} = 1/2[\cos(q_x) + \cos(q_y)]$ and $z = 4$ for 2-dimensional square lattice. This form of the renormalized vertexes (21)-(22) insures that all the operators in the products of Eq.(20) are from different sites. Therefore they can be simply permuted within the decoupling procedure.

To calculate the irreducible part of the time-dependent corelation function in the right-hand side of Eq.(16) we apply the mode-coupling approximation [12] in terms of an independent propagation of dressed particle-hole pairs and charge-spin fluctuations. The proposed approximation is defined by the following decoupling of the time-dependent correlation functions

$$\langle X_{\mathbf{k}-\mathbf{q}}^{\sigma 0}(t)X_{\mathbf{k}}^{0\sigma}(t)|X_{\mathbf{k}'+\mathbf{q}}^{\sigma' 0}X_{\mathbf{k}'}^{0\sigma'}\rangle \simeq$$
$$\delta_{\sigma,\sigma'}\delta_{\mathbf{k}-\mathbf{q},\mathbf{k}'}\langle X_{\mathbf{k}-\mathbf{q}}^{\sigma 0}(t)X_{\mathbf{k}'}^{0\sigma'}\rangle\langle X_{\mathbf{k}}^{0\sigma}(t)X_{\mathbf{k}'+\mathbf{q}}^{\sigma' 0}\rangle, \tag{25}$$

$$\langle X_{\mathbf{k}-\mathbf{q}-\mathbf{p}}^{\sigma 0}(t)X_{\mathbf{k}}^{0\sigma'}(t)B_{\mathbf{p}}^{\sigma'\sigma}(t)|X_{\mathbf{k}'+\mathbf{q}-\mathbf{p}'}^{s 0}X_{\mathbf{k}'}^{0s'}B_{\mathbf{p}'}^{s's}\rangle$$
$$\simeq \delta_{\sigma,s'}\delta_{\sigma',s}\delta_{\mathbf{k}-\mathbf{q}-\mathbf{p},\mathbf{k}'}\delta_{\mathbf{p},-\mathbf{p}'}$$
$$\langle X_{\mathbf{k}-\mathbf{q}-\mathbf{p}}^{\sigma 0}(t)X_{\mathbf{k}'}^{0s'}\rangle\langle X_{\mathbf{k}}^{0\sigma'}(t)X_{\mathbf{k}'+\mathbf{q}-\mathbf{p}'}^{s 0}\rangle\langle B_{\mathbf{p}}^{\sigma'\sigma}(t)B_{\mathbf{p}'}^{s's}\rangle. \tag{26}$$

By using the decoupling scheme and the spectral representation for the two-time retarded GF's [10] we obtain for the memory function

$$M_{\mathbf{q}}(\omega) = \frac{1}{\omega}[\Pi(\mathbf{q},\omega) - \Pi(\mathbf{q},0)],$$
$$\Pi(\mathbf{q},\omega) = \Pi_1(\mathbf{q},\omega) + \Pi_2(\mathbf{q},\omega), \tag{27}$$

where $\Pi_1(\mathbf{q},\omega)$ and $\Pi_2(\mathbf{q},\omega)$ stems from the first and the second term of Eq.(20), respectively. Their imaginary parts are given by

$$\Pi_1''(\mathbf{q},\omega) = \frac{-2\pi}{N}\sum_{\mathbf{k}} G_{\mathbf{k},\mathbf{q}}^2 \int_{-\infty}^{\infty} d\omega_1 n_{\omega,\omega_1} \times A_{\mathbf{k}}(\omega_1)A_{\mathbf{k}+\mathbf{q}}(\omega_1+\omega), \tag{28}$$

$$\Pi_2''(\mathbf{q},\omega) = \frac{-2\pi}{N^2}\sum_{\mathbf{k},\mathbf{p}} M_{\mathbf{k},\mathbf{q},\mathbf{p}}^2 \iint_{-\infty}^{\infty} d\omega_1 d\omega_2 N_{\omega,\omega_1,\omega_2} \times A_{\mathbf{k}+\mathbf{q}-\mathbf{p}}(\omega-\omega_1+\omega_2)A_{\mathbf{k}}(\omega_2)\chi_{cs}''(\mathbf{p},\omega_1), \tag{29}$$

where $n_{\omega,\omega_1} = n(\omega_1) - n(\omega_1+\omega)$, $N_{\omega,\omega_1,\omega_2} = [1 + N(\omega_1) + N(\omega-\omega_1)]n_{\omega_2,\omega-\omega_1}$ with $n(\omega)$ and $N(\omega)$ being Fermi and Bose distribution functions respectively, and

$$A_{\mathbf{k}}(\omega) = -\frac{1}{\pi}\mathrm{Im}\langle\langle X_q^{0\sigma} \mid X_q^{\sigma 0}\rangle\rangle_\omega, \tag{30}$$

is a single-particle spectral function which does not depend on spin σ in the paramagnetic state. We have also introduced the unified spin-charge fluctuation spectrum

$$\chi''_{cs}(\mathbf{q},\omega) = \frac{1}{4\pi}N''_{\mathbf{q}}(\omega) + \chi''_{\mathbf{q}}(\omega), \tag{31}$$

with

$$\chi_{\mathbf{q}}(\omega) = -\frac{1}{\pi}\langle\langle \mathbf{S_q}|\mathbf{S_{-q}}\rangle\rangle_\omega \tag{32}$$

being the dynamical spin susceptibility. In obtaining Eq.(29) we have also used the identity $\langle\langle S^{\sigma}_{\mathbf{q}}|S^{\bar\sigma}_{-\mathbf{q}}\rangle\rangle_\omega = 2\langle\langle S^z_{\mathbf{q}}|S^z_{-\mathbf{q}}\rangle\rangle_\omega$ which holds in the paramagnetic state.

As it is clear from Eq.(27) the memory function involves two contributions; the first one (28) stems from the H_t term (1) and describes the particle-hole contribution, while the second one comes from both H_t and H_J parts and involves electron scattering on spin and charge fluctuations.

Since the charge fluctuations are suppressed for $\delta \to 0$ (δ being the hole concentration) $N''_{\mathbf{q}}(\omega)$ is at least of order of δ (or even smaller). Therefore, considering the leading in δ contributions we retain only the scattering from spin fluctuations.

For further discussion it is more convenient to integrate out the fermionic degrees of freedom $(\mathbf{k},\omega_2)$ in $\Pi''_2(\mathbf{q},\omega)$ (29). That results

$$\Pi''_2(\mathbf{q},\omega) = \frac{1}{N}\sum_{\mathbf{p}}\int_{-\infty}^{\infty} d\omega_1\,[1 + N_{\omega_1} + N_{\omega-\omega_1}] \times \tilde{\Pi}''_{\mathbf{q-p}}(\omega-\omega_1)\chi''_{\mathbf{p}}(\omega_1), \tag{33}$$

where we have introduced an effective spectral function

$$\tilde{\Pi}''_{\mathbf{q-p}}(\omega) = \frac{-2\pi}{N}\sum_{\mathbf{k}} M^2_{\mathbf{k,q,p}}\int_{-\infty}^{\infty} d\omega_1 n_{\omega,\omega_1} \times A_{\mathbf{k+q-p}}(\omega+\omega_1)A_{\mathbf{k}}(\omega_1), \tag{34}$$

for particle-hole excitations coupled to a particular $(\mathbf{p},\omega)$ state of spin fluctuations.

To conclude the section we calculate the first momentum of DCS (13). By performing the commutation between the density and "current" (11) operators we readily get

$$m_{\mathbf{q}} = 4ztN_1(1-\gamma_{\mathbf{q}}), \tag{35}$$

where

$$N_m = \frac{1}{N}\sum_{\mathbf{q}}\gamma^m_{\mathbf{q}}\langle X^{\sigma 0}_{\mathbf{q}}X^{0\sigma}_{\mathbf{q}}\rangle \tag{36}$$

is the particle-hole corelation function.

IV. DYNAMICAL CHARGE SUSCEPTIBILITY

The Eqs.(8),(9), and (12) result in the following form of DCS

$$N_{\mathbf{q}}(\omega) = -\frac{m_{\mathbf{q}}}{\omega^2 - [\Pi(\mathbf{q},\omega) - \Pi(\mathbf{q},0)]/m_{\mathbf{q}} - \Omega^2_{\mathbf{q}}}, \tag{37}$$

where $m_{\mathbf{q}}$ is the first momentum of DCS given by Eq.(35), and $\Omega^2_{\mathbf{q}} = m_{\mathbf{q}}/N_{\mathbf{q}}$ is a mean field (MF) spectrum for the density fluctuations. The memory function formalism does not provide itself the static susceptibility $N_{\mathbf{q}}$. The latter one is calculated within the same approximation scheme as for the static spin susceptibility [15]. That results to the following form of MF spectrum

$$\Omega^2_{\mathbf{q}} = 2z^2t^2C_{\mathbf{q}}(1-\gamma_{\mathbf{q}}) \tag{38}$$

where

$$C_{\mathbf{q}} = \frac{1}{z}(1-\frac{n}{2}) + N_2 - \frac{J}{2zt}N_1(1+z\gamma_{\mathbf{q}}) \tag{39}$$

and the parameters N_m are defined by Eq.(36). Here we note that the MF spectrum $\Omega_{\mathbf{q}}$ resembles the dispersion of undamped collective mode in the charge channel found in the leading order of $1/N$ expansion [4].

In a proper analysis, Eqs. (27)-(29) and (37) should be treated self-consistently with the equations for the single-particle spectral function $A_{\mathbf{k}}(\omega)$ [16] and the spin susceptibility

$\chi_{\mathbf{q}}(\omega)$ [15], the problem, to our knowledge, can be solved only numerically. Here we show, that the main features of charge fluctuation spectrum observed in the exact diagonalization studies [7] can be, at least qualitatively, reproduced in an analytical way based on the physically justified anzats for the single-particle spectral function and the dynamical spin susceptibility. First we discuss one-particle spectral function $A_{\mathbf{k}}(\omega)$.

Actually, the spectral characteristics of the $t-J$ model have been investigated by various analytical and numerical approaches [17]. Those results led to the consensus that the single-particle spectrum involves a narrow quasiparticle (QP) band of coherent states and a broad continuum of the incoherent states. The corresponding spectral function can be represented as

$$A_{\mathbf{k}}(\omega) = A_{\mathbf{k}}^{\rm coh}(\omega) + A_{\mathbf{k}}^{\rm inc}(\omega), \tag{40}$$

where the QP part is given by

$$A_{\mathbf{k}}^{\rm coh}(\omega) = Z_{\mathbf{k}}\delta(\omega - \epsilon_{\mathbf{k}}), \tag{41}$$

with $Z_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ being the QP weight and dispersion referred to the chemical potential μ, respectively. While the incoherent part $A_{\mathbf{k}}^{\rm inc}(\omega)$ is little effected by the doping, the coherent band structure strongly depends on the magnetic background. Namely, in the low doping regime (ordered phase) the QP dispersion is determined by the hopping within a given AFM sublattice [17] and Fermi surface (FS) consists of small hole-pockets centered around $(\pm\pi/2, \pm\pi/2)$. While for a moderate doping (paramagnetic state) the dispersion reflects the dominance of n.n. hopping and there exists large, electronic FS which enclose a fraction of the Brillouin zone equal to the electron concentration n [18] . For the latter doping regime, the exact diagonalization results are well fitted by the simple tight-binding dispersion with some effective hopping amplitude $t_{\rm eff}$ which scales with J (for $\delta = 0.1$ and $J = 0.4t$ $t_{\rm eff} = 0.24t$ [18]). Hence, in the paramagnetic phase we can put $\varepsilon_{\mathbf{k}} = -zt_{\rm eff}\gamma_{\mathbf{k}} - \mu$.

Nearly structureless incoherent part is predominantly distributed bellow the QP band (in the electronic picture) and we approximate $A_{\mathbf{k}}^{\rm inc}(\omega)$ as follows

$$A_{\mathbf{k}}^{\rm inc}(\omega') = \frac{1}{\Gamma}\theta(-\omega')\theta(W_{\rm inc} + \omega'), \tag{42}$$

where $W_{\rm inc} \simeq 5t$ is the incoherent bandwidth and ω' is measured from the bottom of the QP band. The spectral weights of the incoherent band Γ and QP $Z_{\mathbf{k}}$ are provided by the sum rules

$$\frac{1}{N}\sum_{\mathbf{k}}\int_{-\infty}^{\infty} d\omega A_{\mathbf{k}}(\omega) = 1 - \frac{n}{2},$$
$$\frac{1}{N}\sum_{\mathbf{k},\sigma}\int_{-\infty}^{0} d\omega A_{\mathbf{k}}(\omega) = n, \tag{43}$$

and are given by

$$Z = \frac{2\delta}{1+\delta}, \quad \Gamma = \frac{2(1+\delta)}{(1-\delta)^2}W_{\rm inc}, \tag{44}$$

where $Z = \langle Z_{\mathbf{k}}\rangle$ is the averaged QP weight and it coincides with one obtained within the Gutzwiller approximation [19].

Based on the above form of the spectral function (40)-(42), the equal time correlation functions $N_{1,2}$ (39) is estimated to be

$$N_1 \simeq \frac{2}{\pi^2}nZ, \quad N_2 \simeq \frac{n}{2z} \cdot \tag{45}$$

Finally we assume that the spin susceptibility $\chi_{\mathbf{q}}(\omega)$ (32) is peaked at the AFM wave vector $\mathbf{Q} = (\pi, \pi)$ [20]. Then, at $T = 0$, we approximate $\Pi_2''(\mathbf{q}, \omega)$ (33) as follows

$$\Pi_2''(\mathbf{q}, \omega) \simeq \int_0^{\omega} d\omega_1 \tilde{\Pi}_{\mathbf{q}-\mathbf{Q}}''(\omega - \omega_1)\chi''(\omega_1), \tag{46}$$

where

$$\chi(\omega) = \frac{1}{N}\sum_{\mathbf{p}}\chi_{\mathbf{p}}''(\omega) \tag{47}$$

is the local spin susceptibility. Since the detailed form $\chi(\omega)$ is not essential for our study we use the following relaxation type form

$$\chi(\omega) = \frac{\chi}{1 - i\omega/\omega_0} \tag{48}$$

which seems to be in accordance with the exact diagonalization data [7,21]. In Eq.(48) $\omega_0 \propto J$ is the energy scale of spin fluctuations and χ is the static susceptibility provided by the sum rule

$$\int_0^{\omega_c} d\omega \chi(\omega) = \frac{3}{4}n, \tag{49}$$

with high-energy cutoff $\omega_c = 2J$. Eqs. (48) and (49) results

$$\chi = \frac{3\pi n}{2\omega_0 \ln[1 + (\omega_c/\omega_0)^2]}. \tag{50}$$

A. Long–wavelength limit

First we discuss small $\mathbf{q}, \omega$ limit of DCS. In the long wave-length limit, $q \to 0$, MF spectrum (38) reduces to the sound mode $\Omega_{\mathbf{q}} = v_s q$ with the sound velocity $v_s = zt\sqrt{C_0/2}$ larger than the Fermi velocity $v_F(\mu) = zt_{\rm eff}\sqrt{(1-\tilde{\mu}^2)/2}$ ($\tilde{\mu} = |\mu|/zt_{\rm eff} \simeq \pi\delta/4$ for small δ).

At small q the vertex functions (21)-(24) are given by

$$G_{\mathbf{k},\mathbf{q}} = 2t^2[2z(\hat{\mathbf{q}}\nabla_{\mathbf{k}}\gamma_{\mathbf{k}})^2 - 1]q^2, \tag{51}$$

$$M_{\mathbf{k},\mathbf{Q},\mathbf{p}} = z(z-1)tJ(\hat{\mathbf{q}}\nabla_{\mathbf{k}}\gamma_{\mathbf{k}})q \tag{52}$$

with $\hat{\mathbf{q}} = \mathbf{q}/q$.

First we consider the real part of "self-energy" $\Pi(\mathbf{q},\omega)$ (27)-(29). Since for small q the vertex functions $G_{\mathbf{k},\mathbf{q}} \sim q^2$ and $M_{\mathbf{k},\mathbf{Q},\mathbf{p}} \sim q$ we proximate $\Pi'(\mathbf{q},\omega) \simeq \Pi_2'(\mathbf{q},\omega)$ to keep the leading in small q contributions. Moreover for small ω we expand $\Pi_2'(\mathbf{q},\omega)$ as

$$\Pi_2'(\mathbf{q},\omega) \simeq \Pi_2'(\mathbf{q},0) - \alpha_{\mathbf{q}}\omega^2, \tag{53}$$

where $\alpha_{\mathbf{q}} > 0$ and is given by

$$\alpha_{\mathbf{q}} = -\frac{1}{2}\frac{d^2\Pi_2'(\mathbf{q},\omega)}{d\omega^2}\bigg|_{\omega=0} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Pi_2''(\mathbf{q},\omega)}{\omega^3}d\omega. \tag{54}$$

Since $\Pi_2''(\mathbf{q},\omega)$ (46) is an odd function of ω there is no linear in ω term in the expansion (53). The Eqs.(37) and (53) results in the following form of DCS for small q, ω

$$N_{\mathbf{q}}(\omega) \simeq \frac{-m_{\mathbf{q}}/(1+\lambda)}{\omega^2 - \tilde{v}_s^2 q^2 + 2i\omega\Gamma_{\mathbf{q}}}, \tag{55}$$

where

$$\lambda = \lim_{\mathbf{q}\to 0}\frac{\alpha_{\mathbf{q}}}{m_{\mathbf{q}}},\ \tilde{v}_s = \frac{v_s}{\sqrt{1+\lambda}},\ \Gamma_{\mathbf{q}} = \frac{-\Pi''(\mathbf{q},\omega)}{2m_{\mathbf{q}}(1+\lambda)\omega}, \tag{56}$$

$\tilde{v}_s$ and $\Gamma_{\mathbf{q}}$ are the renormalized sound velocity and sound damping, respectively.

First we consider renormalization factor λ. The estimation of λ is interesting by itself, since it represents the interaction induced electron mass enhancement factor [see Sec. V] and can be related to experiment [22]. The approximation (40) for the one-particle spectral function $A_{\mathbf{k}}(\omega)$ leads to the two different contributions to the effective spectral function for particle-hole excitations $\tilde{\Pi}''_{\mathbf{q}-\mathbf{Q}}(\omega)$ (34). The first one $\tilde{\Pi}''_{\mathbf{q}-\mathbf{Q}}(\omega)^{\text{c-c}}$ is due to the transitions within the QP band and the remaining part $\tilde{\Pi}''_{\mathbf{q}-\mathbf{Q}}(\omega)^{\text{i-c}}$ are provided by the incoherent-coherent transitions.

First, considering $\tilde{\Pi}''_{\mathbf{q}-\mathbf{p}}(\omega)^{\text{c-c}}$ (34), (51) we come to the following expression for small q

$$\tilde{\Pi}''_{\mathbf{q}-\mathbf{Q}}(\omega)^{\text{c-c}} = \frac{-2\pi\Lambda Z^2}{N}q^2\sum_{\mathbf{k}}(\hat{\mathbf{q}}\nabla_{\mathbf{k}}\gamma_{\mathbf{k}})^2 \times[n(\epsilon_{\mathbf{k}}) - n(\epsilon_{\mathbf{k}+\omega})]\delta(\omega - \epsilon_{\mathbf{k}+\mathbf{q}-\mathbf{Q}} + \epsilon_{\mathbf{k}}), \tag{57}$$

with $\Lambda = [z(z-1)tJ]^2$. As it has been previously discussed by several authors [23], the particle-hole spectral function for tight-binding electrons exhibits a crossover at frequency $\omega = 2|\mu|$. Namely, for $\omega < 2|\mu|$ it is peaked at the incommensurate wave vectors $\mathbf{Q}_* = (\pi\pm\delta^*,\pi), (\pi,\pi\pm\delta^*)$ where the displacement δ^* for small μ is given by $\delta^* \simeq \mu/t_{\rm eff}$. While for $\omega > 2|\mu|$ particle-hole spectral function gets its maximum value at AFM wave vector $\mathbf{Q}$ and follows the nested Fermi liquid scaling [24]. Hence we consider the cases $\omega < 2|\mu|$ and $\omega > 2|\mu|$ separately.

In the case $\omega < 2|\mu|$ we put $\epsilon_{\mathbf{k}+\mathbf{q}-\mathbf{Q}} = \epsilon_{\mathbf{k}-\mathbf{Q}_*}$ in Eq.(57) and for $\delta^* \ll 1$ expand $\epsilon_{\mathbf{k}+\mathbf{Q}_*} \simeq -\epsilon_{\mathbf{k}} - \delta^*\partial\epsilon_{\mathbf{k}}/\partial k_x$. Moreover, since the dominant contribution in the integration in Eq.(54) comes from $\omega \sim 0$, at $T = 0$ we approximate $n(\epsilon_{\mathbf{k}}) - n(\epsilon_{\mathbf{k}+\omega}) \simeq \omega\delta(\epsilon_{\mathbf{k}})$ and as a result we get

$$\tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{c-c}} = \frac{-2\pi\Lambda t Z^2}{W_{\text{coh}}^2|\mu|} q^2 \omega I_1(\omega)\theta(2|\mu|-\omega), \tag{58}$$

where $W_{\text{coh}} = 2zt_{\text{eff}}$ is the coherent bandwidth, and

$$I_1(\omega) = \frac{2\sin^2 k_x^*}{\pi^2|\cos k_x^* \sin k_y^*|}$$

with $\sin k_x^* = 1 - \omega/(2|\mu|)$ and $\cos k_y^* = 2\tilde{\mu} - \cos k_x^*$. For $\omega \ll 2|\mu|$ we have $I_1(\omega) \simeq 2\sqrt{\omega/|\mu|}/\pi^2$ and thus

$$\tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{c-c}} \simeq \frac{-4\Lambda t Z^2}{\pi W_{\text{coh}}^2}\sqrt{\frac{\omega}{|\mu|}} q^2. \tag{59}$$

Then from Eqs. (46) and (59) for the velocity renormalization factor λ (56) we get

$$\lambda_1^{c-c} \simeq \frac{\sqrt{2}\Lambda Z^2 \chi}{15\pi^2 z t_{\text{eff}}^2 N_1 \omega_0} \tag{60}$$

where the QP weight Z and n.n particle-hole correlator N_1 are given by Eqs.(44) and (45). For the actual values of the parameters $J = 0.4t$, $t_{\text{eff}} = 0.24t$, and $\delta = 0.1$ we obtain $\lambda_1^{c-c} \simeq 4$.

Next we consider the case $\omega > 2|\mu|$. In this case particle-hole spectral function is peaked at AFM wave vector Q and from Eq.(57) we come to

$$\tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{c-c}} = \frac{-\pi\Lambda Z^2}{W_{\text{coh}}} q^2 I(\tilde{\omega})\theta(\tilde{\omega} - 2\tilde{\mu})\theta(2-\tilde{\omega}), \tag{61}$$

where $\tilde{\omega} = 2\omega/W_{\text{coh}}$ and

$$I(\tilde{\omega}) = \frac{1}{\pi^2}\left[\tilde{\omega}_+ E(\tilde{\omega}_-/\tilde{\omega}_+) - 2\tilde{\omega}K(\tilde{\omega}_-/\tilde{\omega}_+)\right] \tag{62}$$

with $\tilde{\omega}_\pm = 2 \pm \tilde{\omega}$, $K(x)$ and $E(x)$ are the complete elliptical integrals of the first and second kind, respectively. The function $I(x)$ (62) is normalized to 1/4 in the interval $0 < x < 2$ and well approximated by the following linear dependence $I(x) = (2-x)/8$ and we get

$$\tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{c-c}} \simeq \frac{-\pi\Lambda Z^2}{4W_{\text{coh}}^2} f(\omega) q^2$$
$$f(\omega) = (W_{\text{coh}} - \omega)\theta(\omega - 2|\mu|)\theta(W_{\text{coh}} - \omega). \tag{63}$$

That results in the following contribution to λ

$$\lambda_2^{\text{c-c}} \simeq \frac{2(z-1)^2 t J Z^2}{N_1 W_{\text{coh}}^2} I, \tag{64}$$

with

$$I = \int_0^\infty \int_0^x dx dy \frac{\tilde{f}(x-y)}{x^3} \tilde{\chi}(y)\theta(2-y) \tag{65}$$

where the dimensionless function $\tilde{f}(x)$ and $\tilde{\chi}(y)$ stands for $f(\omega)$ (63) and spins susceptibility $\chi(\omega)$ (48) measured in units of J. For $J = 0.4t$, $\delta = 0.1$ we calculated the integral (65) numerically and found $I = 0.5$. Then from Eq.(64) we estimate $\lambda_2^{\text{c-c}} \simeq 0.8$.

As for the remaining contribution $\tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{i-c}}$, with the help of Eqs.(34),(42), and (52) at $T = 0$ we obtain

$$\tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{i-c}} \simeq \frac{-\pi\Lambda Z}{4z\Gamma} q^2$$
$$\times\, \theta(2\omega - W_{\text{coh}} + 2|\mu|)\theta(W_{\text{tot}} - \omega), \tag{66}$$

where $W_{\text{tot}} = W_{\text{coh}} + W_{\text{inc}}$ is the total bandwidth. From Eq.(56) and by using the sum rule (49) the upper value of the incoherent-coherent contributions to $\Pi''_2(\mathbf{q},\omega)^{\text{i-c}}$ is estimated as

$$\Pi''_2(\mathbf{q},\omega)^{\text{i-c}} \simeq \frac{3}{4} n \tilde{\Pi}''_{\mathbf{q-Q}}(\omega)^{\text{i-c}}. \tag{67}$$

That results in

$$\lambda^{i-c} \simeq \frac{3n(z-1)^2 Z t J^2}{16\Gamma N_1 W_{\text{coh}}^2}\left[4 - \frac{W_{\text{coh}}^2}{W_{\text{tot}}^2}\right] \simeq 0.1. \tag{68}$$

The small value of $\lambda^{i-c} \simeq 0.1$ in comparison with $\lambda^{i-c} \simeq 4$ is due to the existence of the large threshold energy ($W_{\text{coh}}/2 - |\mu| \simeq t$) (66) for the incoherent-coherent transitions which are important only in describing the high energy density fluctuations.

Finally, by summing all three contributions (62), (66), and (70) for the velocity renormalization factor we get $\lambda \simeq 5$.

It follows that the renormalized sound velocity(56) gets smaller than the Fermi one $\tilde{v}_s = v_s/\sqrt{1+\lambda} < v_F$. Here we notice, that in the Fermi liquid theory, for $v_s > v_F$ the many body corrections leads to the stiffening of the sound [25]. Contrary to this, , in the present case there is a softening of the sound. That is due to the scattering on spin fluctuations given by $\Pi_2(\mathbf{q},\omega)$ term (46). The renormalized sound falls down into particle-hole continuum getting finite damping due to the decay into particle-hole pairs [26]. This process is described by $\Pi_1''(\mathbf{q},\omega)$ (28) . The latter one in the small q,ω limit reads as [see Eqs.(28) and (51)]

$$\Pi_1''(\mathbf{q},\omega) = \frac{-8\pi t^4 Z^2}{N}\omega q^3 \sum_{\mathbf{k}}[2z(\hat{\mathbf{q}}\nabla_{\mathbf{k}}\gamma_{\mathbf{k}})^2 - 1]^2 \times \delta(\epsilon_{\mathbf{k}})\delta(\omega/q - \hat{\mathbf{q}}\mathbf{v}_{\mathbf{k}}), \quad (69)$$

where $\mathbf{v}_{\mathbf{k}} = \nabla_{\mathbf{k}}\epsilon_{\mathbf{k}}$ is the QP velocity. The integration over $\mathbf{k}$ in Eq.(69) results

$$\Pi_1''(\mathbf{q},\omega) = \frac{-\sqrt{2}t^4 Z^2}{\pi t_{\text{eff}}^2|\sin\theta|}\left[1 - \left(\frac{2\omega}{qv_{\text{F}}}\right)^2\right]\omega q^3, \quad (70)$$

where $\cos\theta = [\mu^2 + 2\omega^2/q^2]/v_{\text{F}}^2 - 1$. That results in the following form of the sound damping

$$\Gamma_q \simeq \beta\tilde{v}_s q \quad \beta = \frac{Z^2 t^3}{\pi z t_{\text{eff}} N_1 v_s^2}, \quad (71)$$

for the actual values of the parameters $\beta < 1$, and thus, in accordance with Ref. [6], one obtains that the sound damping is only numerically smaller than its energy.

B. Short–wavelength limit

At large momenta the main spectral weight of density fluctuations is located at high energies, ($\sim t$), near the MF spectrum $\Omega_{\mathbf{q}}$ (38). For instance at $\mathbf{q} = \mathbf{Q}$ we have $\Omega_{\mathbf{Q}} \simeq zt$ while in the exact diagonalization studies [7] the peak is observed at $\omega \simeq 6t$. However, considering a " self-energy" corrections and noting that $\Pi_{\mathbf{q}}'(\omega)$ falls off as $1/\omega^2$ at large frequencies, from Eq.(37) we obtain the renormalized spectrum as $\tilde{\Omega}_{\mathbf{q}} \simeq \sqrt{\Omega_{\mathbf{q}}^2 - \Pi_{\mathbf{q}}'(0)/m_{\mathbf{q}}}$. One can easily show that $\Pi_{\mathbf{q}}'(0) < 0$ and hence the spectrum is shifted to the higher energies. At large momenta the peak is dispersed out from the coherent particle-hole continuum and its broadening is only due to the high energy transitions involving the incoherent band. Since the latter one has been neglected in Ref. [5] the authors observed an infinitely sharp peak. However, as it follows the damping of the high energy mode is comparable to its energy. Near the pole $\tilde{\Omega}_{\mathbf{q}}$ we estimate the damping as

$$\tilde{\Gamma}_{\mathbf{q}} = \frac{-\Pi''(\mathbf{q},\tilde{\Omega}_{\mathbf{q}})}{2m_{\mathbf{q}}\tilde{\Omega}_{\mathbf{q}}} \quad (72)$$

where $\Pi''(\mathbf{q},\omega) = \Pi_1''(\mathbf{q},\omega)^{\text{i-c}} + \Pi_2''(\mathbf{q},\omega)^{\text{i-c}}$ describes the incoherent-coherent transitions. From Eqs.(28) and (42) for $\mathbf{q} = \mathbf{Q}$ we obtain

$$\Pi_1''(\mathbf{Q},\omega)^{\text{i-c}} \simeq \frac{-5\pi(zt)^4 Z}{4\Gamma}. \quad (73)$$

The second contribution $\Pi_2''(\mathbf{q},\omega)^{\text{i-c}}$ from Eqs.(34), (42), and (67) is estimated as

$$\Pi_2''(\mathbf{Q},\omega)^{\text{i-c}} \simeq \frac{-6n\pi(zt)^4 Z}{5\Gamma}. \quad (74)$$

The Eqs.(72),(73), and (74) result in

$$\tilde{\Gamma}_{\mathbf{q}} \simeq \frac{2(25 + 24n)\pi t^3 Z}{5N_1\Gamma\tilde{\Omega}_{\mathbf{q}}} \sim 3t. \quad (75)$$

Thus the peak gets rather broad in accordance with the exact diagonalization results [7].

For large momenta but low energy, charge excitation spectrum should show some low energy structure related to the contribution from particle-hole continuum to $N_{\mathbf{q}}''(\omega)$. Since $\Omega_{\mathbf{q}}$ is larger in (ξ,ξ) direction than in $(\xi,0)$, the low energy structure should be less pronounced in the latter case. The same anisotropy has been observed in the exact diagonalization studies [7].

V. OPTICAL CONDUCTIVITY

In this section we discuss the optical conductivity $\sigma(\omega)$. In the linear response theory of Kubo [27] the frequency dependent conductivity is given by the relaxation function for currents

$$\sigma_{zz}(\omega) = \frac{ie^2}{V}((J_z|J_z))_\omega. \quad (76)$$

By using the continuity equation and equation of motion for the GF's, one can easily relate the longitudinal conductivity to the dynamical charge susceptibility (5)

$$\sigma_{xx}(\omega) = -\frac{ie^2}{V} \lim_{q\to 0} \frac{\omega N_{\mathbf{q}}(\omega)}{q^2}, \tag{77}$$

where $q = q_x$. From Eqs. (37) and (77) we express conductivity in terms of the memory function

$$\sigma_{xx}(\omega) = \frac{ie^2}{V} \frac{D}{\omega - M(\omega)} \tag{78}$$

where $D = \lim_{q\to 0} m_{\mathbf{q}}/q^2 = ztN_1$ is the Drude weight which is given by the half of the averaged kinetic energy $D = -\langle H_t \rangle/2$, $m_{\mathbf{q}}$ and N_1 are defined in Eqs. (35) and (36), respectively. The memory function $M(\omega)$ reads as

$$M(\omega) = \frac{\Pi(\omega) - \Pi(0)}{\omega}, \tag{79}$$

where

$$\Pi(\omega) = \lim_{q\to 0} \frac{\Pi(\mathbf{q}, \omega)}{Dq^2}, \tag{80}$$

with $\Pi(\mathbf{q}, \omega)$ defined in Eq. (27). Since for small q, $\Pi_1(\mathbf{q}, \omega) \sim q^4$ (28),(51) and $\Pi_2(\mathbf{q}, \omega) \sim q^2$ (29),(52) only the second one contributes to $\Pi(\omega)$. The latter is given by Eqs.(33)and (34) at $q = 0$ with $M_{\mathbf{k},\mathbf{q},\mathbf{p}}$ replaced by the transport vertex given by

$$M_{\mathbf{k},\mathbf{p}} = \lim_{q\to 0} \frac{M_{\mathbf{k},\mathbf{q},\mathbf{p}}}{q^2} = [t_{\mathbf{k}-\mathbf{p}} v_{\mathbf{k}} - t_{\mathbf{k}} v_{\mathbf{k}-\mathbf{p}} - 2tv_{\mathbf{p}}] + \frac{1}{2}[J + J_{\mathbf{p}}][v_{\mathbf{k}-\mathbf{p}} - v_{\mathbf{k}}], \tag{81}$$

where $t_{\mathbf{k}} = zt\gamma_{\mathbf{k}}$, $J_{\mathbf{k}} = zJ\gamma_{\mathbf{k}}$, and $v_{\mathbf{k}} = \partial t_{\mathbf{k}}/\partial k_x$.

We rewrite conductivity (78) in the form of the generalized Drude law as follows

$$\sigma_{xx}(\omega) = \frac{e^2}{V} \frac{\tilde{D}(\omega)}{1/\tilde{\tau}(\omega) - i\omega}, \tag{82}$$

where an effective Drude weight and the relaxation time are given by

$$\tilde{D}(\omega) = \frac{D}{1 + \lambda(\omega)} \quad \tilde{\tau}(\omega) = \frac{\tau(\omega)}{1 + \lambda(\omega)}, \tag{83}$$

with

$$\lambda(\omega) = -\frac{M(\omega)'}{\omega} \quad \frac{1}{\tau(\omega)} = -M(\omega)'', \tag{84}$$

and $1 + \lambda(\omega)$ is the interaction induced mass enhancement factor. The latter one in the static limit is calculated in the proceeding section and is estimated to be of order 6. That is in a good agreement with the optical measurement data [22]. Here we mainly focus on the analysis of the low frequency behavior of the relaxation rate

$$\Gamma(\omega) = \frac{1}{\tau(\omega)} = -\frac{\Pi''(\omega)}{D\omega}. \tag{85}$$

Following the Sec. IV we approximate $\Pi''(\omega)$ as

$$\Pi''(\omega) \simeq \int_0^{\omega} d\omega_1 \chi''(\omega_1) \times \begin{cases} \tilde{\Pi}''_{\mathbf{Q}\cdot}(\omega - \omega_1) & \omega < 2|\mu| \\ \tilde{\Pi}''_{\mathbf{Q}}(\omega - \omega_1) & \omega > 2|\mu| \,. \end{cases} \tag{86}$$

In the case $\omega < 2|\mu|$ the effective spectral function of particle-hole excitations $\tilde{\Pi}''_{\mathbf{Q}\cdot}(\omega)$ (59) is given by

$$\tilde{\Pi}''_{\mathbf{Q}\cdot}(\omega) \simeq \frac{-4\pi\Lambda t Z^2}{W_{\text{coh}} D} \sqrt{\frac{\omega}{|\mu|}}. \tag{87}$$

We remark the square root behavior of particle-hole spectral function $\Pi''_{\mathbf{Q}\cdot}(\omega) \sim \sqrt{\omega}$ instead of the conventual linear ω-dependence [25]. This behavior results in the square root singularity of the structure factor that is known as $2k_F$ anomaly familiar for the electron system in low dimension [23]. It also leads to the deviation from the conventional square law resulting in the following form of relaxation rate

$$\Gamma(\omega) \simeq \frac{16\Lambda Z^2 t\chi}{15\pi D W_{\text{coh}}^2 \sqrt{|\mu|}\omega_0} \omega^{3/2}. \tag{88}$$

Here we note, that ω-dependence ($\omega^{3/2}$) of inverse life time for electrons close the saddle points has been obtained in Refs. [28,29]. In the present case Van Hove singularity plays no role. The obtained $\omega^{3/2}$-dependence of the relaxation time is rather due to the coexistence of the peak in the spin fluctuation spectrum and the $2k_F$ "anomaly" in the particle-hole spectral function at $q \sim Q$. We notice that the former one ,$2k_F$ "anomaly", is not related to the FS topology and is inherent to the low dimensional electron system.

Now we consider the region $\omega > 2|\mu|$. In this case the particle-hole spectral function is peaked at AFM wave vector and is almost ω independent for low frequencies $\omega \ll W_{\rm coh}$ (63), that results in

$$\Gamma(\omega) \simeq \frac{\pi \Lambda Z^2 \chi}{8 W_{\rm coh} D \omega_0} \omega. \tag{89}$$

Unlike to the previous case, now the electron band structure is mainly responsible for obtained behavior. Of course the AFM character of spin fluctuations favors the scatter process with momentum transfer Q and thus enhance its contribution to the relaxation rate.

To summarize the low energy behavior of optical conductivity, we have shown that the relaxation rate due to the electron scattering on spin fluctuations exhibits the crossover from $\Gamma(\omega) \sim \omega^{3/2}$ behavior at low frequencies $\omega \ll 2|\mu|$ to linear ω-dependence at $\omega > 2|\mu|$.

Now we discuss the conductivity at intermediate frequencies. The exact diagonalization studies of the latter quantity have suggested a possible explanation of the MIR absorption within the one band model [17,30]. For instance, as it was observed in Ref. [30], the finite frequency part of $\sigma(\omega)$ is dominated by a single excitation which scales with J in the underdoped regime [30]. The origin of this excitation was ascribed to transitions in which an internal degrees of freedom of the spin-bag QP are excited. The presence of the extra absorption ranging from MIR frequency to ~ 1 eV was also observed in $1/N$ expansion study of the $t-J$ model [31]. The authors of Ref. [31] interpreted this feature as due to the incoherent motion of charge carriers. Below we also support the latter point.

Actually, with increasing energy an extra channel of optical transitions opens. These are the transitions which involve an incoherent band of the single–particle spectral function. As we have already discussed the incoherent–coherent transitions are characterized by the energy scale $\Delta = W_{\rm coh} - 2|\mu|$ being a threshold energy for creating " particle-hole" pairs with a "hole" in the incoherent band. Due to this extra channel at $\omega > \Delta$ the real part of $\sigma(\omega)$ starts to increase. Since $\sigma(\omega)$ vanishes in the limit $\omega \to \infty$ there should be a peak in conductivity at energies of order Δ. Since the coherent bandwidth $W_{\rm coh}$ (and hence Δ) scales with J it follows that the typical energy of the peak is also J. That is in accordance with the exact diagonalization results [30].

VI. CONCLUSION

To summarize, we have developed a self-consistent theory for both the dynamical charge susceptibility and the optical conductivity within the memory function formalism in terms of the Hubbard operators.

We have found that in the long-wavelength limit the charge fluctuation spectrum is mainly governed by the sound mode (55). Although unrenormalized sound velocity is larger than the Fermi velocity the "self-energy" corrections leads to the softening of the sound, it falls down into the the particle-hole continuum and thus acquires a finite damping due to the decay into pair excitations. The sound damping (71) is only numerically smaller than its energy and hence there is no well–defined sound mode.

At large momenta the density fluctuation spectrum mainly consists of a broad high-energy peak which nearly follows MF dispersion (38). At large enough wave vectors the peak is dispersed out from the coherent particle-hole continuum and its broadening (72) is due to the high energy $\sim t$ transitions involving the incoherent band of the single–particle excitations. Contribution from the particle-hole continuum results in some low energy structure of the charge fluctuation spectrum. Since the high energy peak is situated at higher frequencies in (ξ, ξ) direction than in the $(\xi, 0)$ one a low energy structure should be less pronounced in the former case.

We have also discussed the optical conductivity. At low frequencies we analyzed $\sigma(\omega)$ in terms of the generalized Drude law. We have shown that there is large mass enhancement of order $m^*/m \simeq 6$, due to the electron scattering on spin fluctuations. This scattering process also leads to the non-Drude fall off of the low energy part of $\sigma'(\omega)$. Namely, the relaxation rate shows a power law ω–dependence with the exponent 3/2 at low frequencies $\omega < 2|\mu|$ and is linear in ω at frequencies $\omega > 2|\mu|$. As for the intermediate frequency conductivity, we have pointed out the existence of a characteristic energy Δ (of order J) above which an extra channel of the optical transitions opens. These are the transitions in which a particle-hole pairs with a hole in the incoherent band are excited and they might be responsible for the experimentally observed MIR absorption.

Obtained results are in a good agreement with the exact diagonalization studies of small clusters [7].

ACKNOWLEDGMENTS

We would like to thank V. Kabanov, V. Yushankhai, V. Oudovenko and N. Perkins for useful discussions and comments. Financial support by the the INTAS-RFBR Program Grant No 95-591 is acknowledged. One of the authors (N. M.) acknowledges also the support by NREL in the framework of Subcontract No AAX-6-16763-01.

REFERENCES

[1] W. Brening, Phys. Rep. **251**, 153 (1995).

[2] G. A. Thomas *et al.*, Phys. Rev. Lett. **61**, 1313 (1988).

[3] S. Uchida *et al.*, Phys. Rev. B **43**, 7942 (1991).

[4] Z. Wang, Y. Bang, and G. Kotliar, Phys. Rev. Lett. **67**, 2773 (1991).

[5] R. Zeyher and M. L. Kulić, Phys. Rev. B **54**, 8985 (1996).

[6] G. Khaliullin and P. Horsch, Phys. Rev. B **54**, R9600 (1996).

[7] T. Thoyama, P. Horsch, and S. Maekawa, Phys. Rev. Lett. **74**, 980 (1995); R. Eder, Y. Ohta, and S. Maekawa, *ibid.* **74**, 5124 (1995).

[8] J. Hubbard, Proc. Roy. Soc. A **281**, 401 (1964).

[9] D. Pines, in *High Temperature Superconductors an the C^{60} systems*, ed. H. C. Ren, p. 1 (Gordon and Breach, 1995).

[10] D. N. Zubarev, Usp. Fiz. Nauk. **71**, 71, (1972).

[11] D. Forster, *Hydrodynamic fluctuations, Broken Symmetry and Correlation Functions* (Benjamin, New York, 1975).

[12] W. Götze and P. Wölfle, J. Low Temp. Phys. **5**, 575 (1971); Phys. Rev. B **6**, 1226 (1972).

[13] Yu. A. Tserkovnikov, Theor. Math. Fiz. **50**, 261 (1982).

[14] The irreducibility means that only the part of "current" operator orthogonal to the density one is retained in the memory function. For the rigorous definition see Ref. [13].

[15] G. Jackeli and N. M. Plakida, Theor. Mat. Fiz. **114**, 426 (1998).

[16] N. M. Plakida and V.S. Oudovenko (unpublished).

[17] For a review see E. Dagotto, Rev. Mod. Phys. **66**, 763 (1994).

[18] W. Stephan and P. Horsch, Phys. Rev. Lett. **66**, 2258 (1991).

[19] D. Vollhardt, Rev. Mod. Phys.**56**, 99 (1984).

[20] Although experiment suggests that the spin susceptibility is peaked at the incommensurate wave vectors, we do not discuss this case since it does not introduce qualitatively new physics.

[21] J. Jaklič and P. Prelovšek, Phys. Rev. Lett. **75**, 1340 (1995).

[22] A. V. Puchkov, D. N. Basov, and T. Timsuk, J. Phys.: Condens. Matter **8**, 10049 (1996).

[23] P. Benard, L. Chen, and A.-M. S. Tremblay, Phys. Rev. B **47**, 15217 (1993) and refernces therein.

[24] A. Virosztek and J. Ruvalds, Phys. Rev. B **42**, 4064, (1990).

[25] D. Pines and P. Nozieres, *Theory of Quantum Liquids* (Benjamin, New York, 1966).

[26] Since $\Pi''_2(\mathbf{q},\omega)$ (29) involves a higher order scattering processes it results in a next order in small q corrections to the sound damping.

[27] R. Kubo, J. Phys.Soc. Jpn. **12**, 570 (1957).

Phys. Rev. B **46**, 11 798, (1992).

[28] S. Gopalan, O. Gunnarsson, and O.K. Andersen,

[29] R. A. Hlubina, T. M. Rice, Phys. Rev. B **51**, 9523, (1992).

[30] R. Eder, P. Wróbel, and Y. Ohta, Phys. Rev. B **54**, R11034 (1996).

[31] Y. Bang and G. Kotliar, Phys. Rev. B **48**, 9898 (1993).

Received by Publishing Department
on September 3, 1998.

Джакели Г., Плакида Н.М. E17-98-251

Зарядовая динамика и оптическая проводимость в $t-J$ модели

Динамическая зарядовая восприимчивость и оптическая проводимость вычисляются для $t-J$ модели в парамагнитной фазе на основе формализма функции памяти в терминах операторов Хаббарда. Получена самосогласованная система уравнений для функций памяти в рамках приближения взаимодействующих мод. Показано, что в длинноволновом пределе спектр зарядовых флуктуаций дается затухающей звуковой модой, в то время как в коротковолновом пределе существует широкий максимум с характерной энергией ($\sim t$). Исследование оптической проводимости показывает, что зависимость функций релаксации от частоты имеет вид $\omega^{3/2}$ при $\omega < 2|\mu|$ и описывается линейным законом при $\omega > 2|\mu|$. Полученные результаты хорошо согласуются с результатами точной диагонализации.

Работа выполнена в Лаборатории теоретической физики им. Н.Н.Боголюбова ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1998

Jackeli G., Plakida N.M. E17-98-251

Charge Dynamics and Optical Conductivity of the $t-J$ Model

The dynamical charge susceptibility and the optical conductivity are calculated in the planar $t-J$ model within the memory function method, working directly in terms of Hubbard operators. The density fluctuation spectrum consists of a damped sound-like mode for small wave vectors and a broad high energy peak ($\sim t$) for large momenta. The study of the optical conductivity shows that electron scattering from spin fluctuations leads to the Drude frequency dependent relaxation rate which exhibits a crossover from $\omega^{3/2}$ behaviour at low frequencies ($\omega < 2|\mu|$), to a linear ω-dependence for frequencies large than $2|\mu|$. Due to the spin-polaron nature of charge carriers extra absorptions starting at frequency $\omega \lesssim J$ arise. Obtained results are in good agreement with exact diagonalization studies as well as with experimental results for copper oxides.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1998