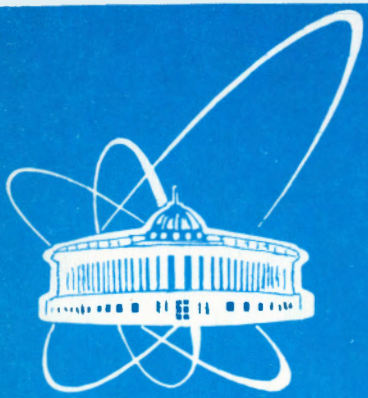

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FERMIONIC STRUCTURE OF TWO-DIMENSIONAL ISING MODEL WITH QUENCHED SITE DILUTION

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Фермионная структура двумерной модели Изинга с вмороженным беспорядком в узлах

Интеграл по антикоммутирующим переменным применен для анализа фермионной структуры двумерной модели Изинга с вмороженным беспорядком (примесным разбавлением) в узлах. В методе реплик задача приведена к теории взаимодействующих фермионов на решетке. Взаимодействие имеет порядок $2N$ по ферми-полям, где N — число реплик, что является несколько необычной чертой возникающей решеточной модели. В непрерывном пределе вблизи критической точки, при слабом разбавлении, возникает, однако, эффективное четырехфермионное взаимодействие. В частности, это приводит к двойной логарифмической зависимости теплоемкости от температуры вблизи точки перехода. Получено также точное значение для параметра наклона кривой $T_c - p$ при слабом беспорядке.

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Fermionic Structure of Two-Dimensional Ising Model with Quenched Site Dilution

We apply a new anticommuting path integral technique to clarify the fermionic structure of the 2D Ising model with quenched site dilution. In the N-replica scheme, the model is explicitly reformulated as a theory of interacting fermions on a lattice. An unusual feature is that the leading term of interaction in the exact lattice theory is of order $2N$ in fermions, where N is the number of replicas. The continuum-limit approximation near T_c for weak dilution produces, however, an effective four-fermion interaction. In particular, this implies the doubled-logarithmic singularity in the specific heat near T_c for weak site dilution. The exact value of the initial slope is also obtained.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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1 Introduction

From the experimental point of view, the site dilution provides, probably, the most simple way to realize quenched disorder in real magnetic materials. In this case, some amount of the magnetic atoms in a sample, chosen at random, are replaced by nonmagnetic impurity atoms. Another sort of disorder is bond dilution, merely preferred in theoretical studies, in which case some of the lattice bonds are assumed to be broken. The two-dimensional Ising model (2DIM) is a natural object to analyze the effects of disorder since it admits the exact analytic solution in the pure case [1]. This model, on the other hand, is too delicate to make definite predictions on common phenomenological grounds. In particular, from the point of view of the Harris criterion [2, 3] (which states that the critical behaviour is modified by quenched disorder if $\alpha > 0$, and is unchanged if $\alpha < 0$, where α is the specific heat exponent for the pure system) the 2D Ising model is marginal, since $\alpha = 0$ in the pure case. The 2D Ising model with quenched dilution has been intensively analyzed during the last decades both by theoretical methods [2]-[16] and in the precise Monte-Carlo experiments [17]-[22]. The theoretical studies here were merely concerned with bond dilution, making use of the fermionic interpretation of the Ising model [4]-[16]. The case of site dilution, however, has not yet been properly analyzed within the fermionic approach. The site dilution can be viewed as locally correlated bond dilution. This makes the traditional transfer-matrix and combinatorial methods of the fermionization, which is the basis for theoretical analysis, less suitable in the case of site disorder [16]. Meanwhile, the Monte-Carlo studies [20] and related analyses [23] give evidence for a different behaviour, at least for moderated and strong site dilution, from that predicted in the theoretical studies of weak bond dilution [6, 14]. The alternative theories for bond dilution are also to be taken into account [11, 12]. A number of questions thus arise about the universality between site and bond dilution, and between weak and strong dilution in both the cases [22]-[25]. Also, see [22] for a critical review.

In this Letter we apply a novel anticommuting path integral technique to clarify the fermionic structure of the two-dimensional Ising model with

quenched site dilution. At the first stage, we transform the partition function of the 2D Ising model with fixed site dilution into a Gaussian fermionic integral. The averaging over the disorder in the N-replica scheme then yields a lattice theory with compact multifermion interaction. An interesting feature is that the term of interaction here is of order $2N$ in fermions, where N is the number of replicas. Notice that the formal limit $N \to 0$ is assumed at final stages. For weak dilution, however, the effective continuum-limit theory responsible for critical behaviour near T_c appears to be the standard $N = 0$ Gross-Neveu model with four-fermion interaction, already well known in the DD-SSL theories for weak bond dilution [6, 14]. In particular, this implies the doubled-logarithmic singularity in the specific heat and log-corrections in other thermodynamic functions near T_c for weak site dilution. An effective four-fermion term arises in the continuum theory, in our case, due to the interplay of the short-wave and long-wave lattice fermionic modes. The parameters of the $N = 0$ Gross-Neveu model for site dilution are then explicitly evaluated. These parameters include some characteristic lattice fermionic averages for the pure Ising model at T_c. The exact value for the initial slope of the $T_c(p)$-p curve is also obtained. The basic results are given in (8), (10), (15), (31), (37). Further comments are added in the following sections.

2 The model

Let us start with the two-dimensional Ising model with fixed distribution of the nonmagnetic sites over the lattice (fixed site dilution). At the first stages, we also assume arbitrary inhomogeneous distribution of the coupling parameters over the lattice bonds since this will not be an obstacle for fermionization. The hamiltonian is:

$$-\beta H = \sum_{mn} [\, b^{(1)}_{m+1n}\, y_{mn} y_{m+1n} \sigma_{mn}\sigma_{m+1n} + b^{(2)}_{mn+1}\, y_{mn} y_{mn+1} \sigma_{mn}\sigma_{mn+1}\,], \quad (1)$$

where the Ising spins, $\sigma_{mn} = \pm 1$, are disposed at the lattice sites, mn, with $m, n = 1, 2, ..., L$, L is the lattice length. To introduce site dilution, we supply each Ising spin by random variable $y_{mn} = 0, 1$ playing the role of the effective magnetic moment of the Ising spin. If $y_{mn} = 1$, the given

site is normal, if $y_{mn} = 0$, the site is dilute. Finally, $b^{(\alpha)}_{mn} = \beta J^{(\alpha)}_{mn}$ are the dimensionless bond couplings, $J^{(\alpha)}_{mn}$ are the exchange energies, $\beta = 1/kT$ is the inverse temperature in the energy units. $J^{(\alpha)}_{mn} > 0$ corresponds to the ferromagnetic bond. For fixed disorder, the partition function and free energy are:

$$Z\{y\} = \sum_{(\sigma)} \mathrm{e}^{-\beta H\{y|\sigma\}} = \mathrm{e}^{-\beta F\{y\}}, \quad (2)$$

where the sum is taken over all the possible spin configurations provided by $\sigma_{mn} = \pm 1$ at each site. Taking into account the identity for a typical Boltzmann weight: $\exp(b\, yy'\sigma\sigma') = \cosh(b\, yy')(1 + yy'\sigma\sigma' \tanh b)$, which readily follows from $\sigma\sigma' = \pm 1$, $yy' = 0, 1$, the partition function can be written in the form: $Z\{y\} = R\{y\}Q\{y\}$, where $Q\{y\}$ is the reduced partition function and $R\{y\}$ is a nonsingular spin-independent prefactor to be ignored in what follows. The reduced partition function is:

$$Q\{y\} = \mathrm{Sp}_{(\sigma)} \{ \prod_{mn} (1 + t^{(1)}_{m+1n}\, y_{mn} y_{m+1n} \sigma_{mn}\sigma_{m+1n}) \times (1 + t^{(2)}_{mn+1}\, y_{mn} y_{mn+1} \sigma_{mn}\sigma_{mn+1}) \}, \quad (3)$$

where $t^{(\alpha)}_{mn} = \tanh b^{(\alpha)}_{mn}$ and we assume a properly normalized spin averaging such that $\mathrm{Sp}(1) = 1$, $\mathrm{Sp}(\sigma_{mn}) = 0$ at each site.

Since we are interested in quenched disorder, the averaging over the random-site (RS) impurities is to be performed for the free energy rather than for the partition function itself. The standard device to avoid the averaging of the logarithm is the replica trick:

$$\overline{\ln Q\{y\}} = \lim_{N\to 0} \frac{1}{N} \overline{(Q^N\{y\} - 1)}. \quad (4)$$

In this scheme, we take N identical copies of the original partition function and average $Q^N\{y\}$. The formal limit $N \to 0$ is assumed at final stages. We will also assume the following distribution of the RS impurities in the definition of the averaging:

$$W(y_{mn}) = p\,\delta(1 - y_{mn}) + (1 - p)\,\delta(y_{mn}), \quad (5)$$

where $\delta(\,)$ are the correspondent Kronecker's symbols, p is the probability that any given site, chosen at random, is occupied by the normal Ising spin, while $1 - p$ is the probabilty that the given site is dilute.

3 Fermionization

Before averaging over the disorder, we have to transform $Q\{y\}$ into a fermionic Gaussian integral. Let us remember that Grassmann variables (nonquantum fermionic fields) are the purely anticommuting fermionic symbols. Given a set of Grassmann variables $a_1, a_2, \ldots, a_N$, we have $a_i a_j + a_j a_i = 0$, $a_j^2 = 0$. The notion of the integral over Grassmann variables (fermionic path integral) was first introduced by Beresin [26]. The elementary rules of integration for one variable are [26]:

$$\int da_j \cdot a_j = 1\,, \qquad \int da_j \cdot 1 = 0\,. \tag{6}$$

In a multidimensional integral, the differential symbols $da_1, da_2, \ldots, da_N$ are again anticommuting with each other and with the variables. The fermionic exponential is assumed in the sense of its series expansion. For a finite number of the variables, the exponential series definitely terminates at some stage due to the property $a_j^2 = 0$. In the field-theoretical language, the fermionic form in the exponential is called action.

The transformation of $Q\{y\}$ into a fermionic integral can be realized by following literally the fermionization procedure first developed for the pure $2D$ Ising model [27]-[29]. For a recent comment see also [31, 32]. The first step is the fermionic factorization of the local bond Boltzmann weights [27, 28]. For the whole lattice, we introduce a set of the totally anticommuting Grassmann variables, $a_{mn}, a^*_{mn}, b_{mn}, b^*_{mn}$, and write:

$$\begin{aligned} &1 + t^{(1)}_{m+1n} y_{mn} y_{m+1n} \sigma_{mn} \sigma_{m+1n} \\ &= \int da^*_{mn} da_{mn}\, \mathrm{e}^{a_{mn} a^*_{mn}} (1 + a_{mn} y_{mn} \sigma_{mn})\, (1 + t^{(1)}_{m+1n} a^*_{mn} y_{m+1n} \sigma_{m+1n})\,, \\ &1 + t^{(2)}_{mn+1} y_{mn} y_{mn+1} \sigma_{mn} \sigma_{mn+1} \\ &= \int db^*_{mn} db_{mn}\, \mathrm{e}^{b_{mn} b^*_{mn}} (1 + b_{mn} y_{mn} \sigma_{mn})\, (1 + t^{(2)}_{mn+1}\, b^*_{mn} y_{mn+1} \sigma_{mn+1})\,. \end{aligned} \tag{7}$$

These identities can be readily checked using the elementary rules like (6) and taking into account that $\mathrm{e}^{aa^*} = 1 + aa^*$, since $(aa^*)^2 = 0$. Neglecting the sign of the Gaussian fermionic averaging, the weights are now presented as $A_{mn} A^*_{m+1n}$, $B_{mn} B^*_{mn+1}$, where the separable factors (to

be called Grassmann factors) are to be identified from (7). At the next stage we group together, over the whole lattice, the four factors with the same Ising spin (the same index mn) and then sum over $\sigma_{mn} = \pm 1$ in each group of factors independently, thus passing to a purely fermionic expression for $Q\{y\}$. The above four factors with the same σ_{mn} come by factorization of the four bonds attached to the given mn site, they all include also the same variable y_{mn}. There is also the ordering problem, to be solved, for the noncommuting Grassmann factors in their global products. This problem is related to the requirement that we can actually keep nearby the four factors with the same spin when averaging over $\sigma_{mn} = \pm 1$ at each site. In the $2D$ case, the ordering problem is resolved in terms of the 'mirror-ordered factorization procedure' for the global products of weights [27]-[29]. The appearance (as compared with the pure case) of additional parameters y_{mn} in the factors arising in (7) does not matter in this respect. The averaging over $\sigma_{mn} \pm 1$ in the local group of four factors yields an even fermionic polynomial. This polynomial can be represented as an exponential factor associated with the mn lattice site. This is just the exponential factor corresponding to the y^2_{mn} term in (8) below. Thus we come to the result:

$$\begin{aligned} Q\{y\} = \int \prod_{mn} db^*_{mn} db_{mn} da^*_{mn} da_{mn}\, \exp \sum_{mn} \{\, &a_{mn} a^*_{mn} + b_{mn} b^*_{mn} \\ &+ y^2_{mn} [\, a_{mn} b_{mn} + t^{(1)}_{mn} t^{(2)}_{mn} a^*_{m-1n} b^*_{mn-1} \\ &+ (t^{(1)}_{mn} a^*_{m-1n} + t^{(2)}_{mn} b^*_{mn-1})\, (a_{mn} + b_{mn})\,]\, \}\,. \end{aligned} \tag{8}$$

In the RS pure case ($y_{mn} \equiv 1$) we come back to Eq. (11) of Ref. [27] (inhomogeneous distribution of bond couplings was already assumed in [27]). In fact, noting (7), one can guess (8) directly from Eq. (11) of Ref. [27] without special calculation. The partition function with fixed disorder is represented now as a Gaussian fermionic integral.

The integral (8) can be in turn simplified integrating out the a_{mn}, b_{mn} fields by means of the identity $\int db\, da\, \exp(\lambda\, ab + aL + L'b) = \lambda + LL' = \lambda \exp(\lambda^{-1} LL')$, where a, b are Grassmann variables, L, L' are linear fermionic forms independent of a, b, and λ is a parameter [32]. For $\lambda = 0$ the result is LL'. Integrating out the a_{mn}, b_{mn} fields in this way,

we obtain a reduced integral in terms of variables a^*_{mn}, b^*_{mn}. Changing the notation for the fields, for the sake of visual comfort, as follows: $a^*_{mn}, b^*_{mn} \to c_{mn}, -\bar{c}_{mn}$, we find the reduced integral in the form:

$$Q\{y\} = \int \prod_{mn} d\bar{c}_{mn} dc_{mn}\, y^2_{mn}\, \exp \sum_{mn} [\, y^{-2}_{mn} c_{mn}\bar{c}_{mn} + (c_{mn} + \bar{c}_{mn})\,(t^{(1)}_{mn}\, c_{m-1n} - t^{(2)}_{mn}\, \bar{c}_{mn-1}) - y^2_{mn}\, t^{(1)}_{mn} t^{(2)}_{mn}\, c_{m-1n}\bar{c}_{mn-1}\,], \tag{9}$$

where $y^2_{mn} \exp\,(y^{-2}_{mn} c_{mn}\bar{c}_{mn}) = y^2_{mn} + c_{mn}\bar{c}_{mn}$, there is no true singularity at $y^2_{mn} = 0$. The fermionic integrals (8) and (9) are equivalent to each other and to (3). This equivalence holds even independently of the formal interpretation of the variables y_{mn}. However, we have essentially used the property $y_{mn} = 0, 1$ by passing from $Z\{y\}$ to $Q\{y\}$. If $y^2_{mn} = 0, 1$, then one can put y_{mn} instead of y^2_{mn} in the above integrals. In what follows we assume definitely that $y_{mn} = 0, 1$. The structure of the integral can be illuminated then most obviously by expanding the local factor in (9) over the 'eigenstates' with $y_{mn} = 0$ and $y_{mn} = 1$:

$$Q\{y\} = \int \prod_{mn} d\bar{c}_{mn} dc_{mn} \prod_{mn} [\delta(1 - y_{mn}) \exp\,(S_{mn}) + \delta\,(y_{mn}) c_{mn}\bar{c}_{mn}], \tag{10}$$

where $\delta\,(1 - y_{mn}) = y_{mn}$ and $\delta\,(y_{mn}) = 1 - y_{mn}$ are the correspondent Kronecker symbols, $y_{mn} = 0, 1$, and S_{mn} is the local density of action for the RS pure case, see (11).

In the RS pure case, with $y_{mn} \equiv 1$ at all sites, we obtain:

$$Q\{1\} = \int \prod_{mn} D_{mn} \exp \sum_{mn} S_{mn} = \int \prod_{mn} d\bar{c}_{mn} dc_{mn} \exp \sum_{mn} [\, c_{mn}\bar{c}_{mn} + (c_{mn} + \bar{c}_{mn})\,(t^{(1)}_{mn} c_{m-1n} - t^{(2)}_{mn}\bar{c}_{mn-1}) - t^{(1)}_{mn} t^{(2)}_{mn} c_{m-1n}\bar{c}_{mn-1}\,], \tag{11}$$

where D_{mn} and S_{mn} are the abbreviations for the local terms in the fermionic measure and the action, respectively. The inhomogeneous distribution of bond couplings is still preserved. Therefore, integral (11) can be used as a starting point to analyze bond dilution. We can assume as well that random bonds may be antiferromagnetic, thus making connection to spin glasses. The discussion of these subjects, however, is out of the scope of the present exposition.

4 The N-replicas (lattice)

Within the N-replica scheme, see (4), we have to average the replicated partition function. From (10), the replicated partition function is:

$$Q^N\{y\} = \int \prod_{\alpha=1}^{N} \prod_{mn} d\bar{c}^{\alpha}_{mn}\, dc^{\alpha}_{mn} \prod_{mn} [\,\delta\,(1 - y_{mn}) \prod_{\alpha=1}^{N} \mathrm{e}^{S^{\alpha}_{mn}} + \delta\,(y_{mn}) \prod_{\alpha=1}^{N} c^{\alpha}_{mn}\bar{c}^{\alpha}_{mn}\,], \tag{12}$$

where S^{α}_{mn} ($\alpha = 1, \ldots, N$) are the replicas of the RS-pure local action from (11). The averaging over the disorder with distribution (5) then yields:

$$\begin{aligned}
\overline{Q^N\{y\}} &= \int \prod_{\alpha=1}^{N} \prod_{mn} d\bar{c}^{\alpha}_{mn} dc^{\alpha}_{mn} \prod_{mn} [\, p \prod_{\alpha=1}^{N} \mathrm{e}^{S^{\alpha}_{mn}} + (1-p) \prod_{\alpha=1}^{N} c^{\alpha}_{mn}\bar{c}^{\alpha}_{mn}\,] \qquad (13)\\
&= p^{L^2} \int \prod_{\alpha=1}^{N} \prod_{mn} d\bar{c}^{\alpha}_{mn} dc^{\alpha}_{mn} \exp \sum_{mn} [\, \sum_{\alpha=1}^{N} S^{\alpha}_{mn} + \frac{1-p}{p} \prod_{\alpha=1}^{N} c^{\alpha}_{mn}\bar{c}^{\alpha}_{mn}\, \mathrm{e}^{-S^{\alpha}_{mn}}]\\
&= p^{L^2} \int \prod_{\alpha=1}^{N} \prod_{mn} d\bar{c}^{\alpha}_{mn} dc^{\alpha}_{mn} \exp \sum_{mn} [\, \sum_{\alpha=1}^{N} S^{\alpha}_{mn} + \frac{1-p}{p} \prod_{\alpha=1}^{N} c^{\alpha}_{mn}\bar{c}^{\alpha}_{mn}\, \mathrm{e}^{\Delta^{\alpha}_{mn}}].
\end{aligned}$$

We elaborate the product from the first line into a unique exponential making use of the nilpotent property of fermions, $(c^{\alpha}_{mn})^2 = (\bar{c}^{\alpha}_{mn})^2 = 0$. In the final line, referring to the same property, we replace $-S^{\alpha}_{mn}$ by the reduced form, $\Delta^{\alpha}_{mn} = t^{(1)}_{mn} t^{(2)}_{mn}\, c^{\alpha}_{m-1n}\bar{c}^{\alpha}_{mn-1}$, taking into account that prefactor $c^{\alpha}_{mn}\bar{c}^{\alpha}_{mn}$ annihilates the correspondent fermions in $\exp\,(-S^{\alpha}_{mn})$. Yet another possibility, preferably to be used below, is the choice: $\Delta^{\alpha}_{mn} = t^{(1)}_{mn} t^{(2)}_{mn} (\partial_m c^{\alpha}_{mn})\,(\partial_n \bar{c}^{\alpha}_{mn})$, where ∂_m, ∂_n are conventional lattice derivatives: $\partial_m\, c^{\alpha}_{mn} = c^{\alpha}_{mn} - c^{\alpha}_{m-1n}$, $\partial_n\, \bar{c}^{\alpha}_{mn} = \bar{c}^{\alpha}_{mn} - \bar{c}^{\alpha}_{mn-1}$.

We have thus formulated the exact lattice fermionic theory for the 2D Ising model with quenched site dilution. The integral resulting in (13) can be written shortly in the form:

$$\overline{Q^N\{y\}} = \int D \exp\,(S_0 + \lambda_0\, S_{\mathrm{int}}), \qquad \lambda_0 = \frac{1-p}{p}, \tag{14}$$

where D is the replicated fermionic measure, S_0 is the Gaussian part of the action, which is the replicated action for the RS pure case, eq. (11),

while the non-Gaussian term of interaction can be written, in particular, as follows:

$$\lambda_0 S_{\rm int} = \lambda_0 \sum_{mn} \{ \prod_{\alpha=1}^{N} c^{\alpha}_{mn} \bar{c}^{\alpha}_{mn} \exp\left(t^{(1)}_{mn} t^{(2)}_{mn} \left(\partial_m c^{\alpha}_{mn}\right)\left(\partial_n \bar{c}^{\alpha}_{mn}\right)\right)\}$$
$$= \lambda_0 \sum_{mn} \{ \prod_{\alpha=1}^{N} c^{\alpha}_{mn} \bar{c}^{\alpha}_{mn} \left[1 + t^{(1)}_{mn} t^{(2)}_{mn} \left(\partial_m c^{\alpha}_{mn}\right)\left(\partial_n \bar{c}^{\alpha}_{mn}\right)\right]\} . \qquad (15)$$

The interaction appears to be of order $2N$ in fermions, where N is the number of replicas, with $N \to 0$ at final stages. This is a somewhat unusual feature of the resulting model. In a sense, in (15), at each lattice site we have simply the product of virtual dilute sites ('holes') over the replicas. Respectively, the coupling constant is of order $1-p$, which is a fraction of the dilute sites over the lattice. Despite some unexpected features, the interaction (15) is of a relatively simple and compact form. The resulting fermionic theory can be used, probably, in future research, for the analysis of strong and moderated dilution (it might be important to take properly into account short lattice distances). For weak dilution near T_c, where the continuum-limit approximation can be applied, the standard four-fermion term in fact arises in the effective action due to the interplay of higher- and low-momentum lattice fermionic modes. This is discussed in section 7. Before, we comment shortly on the lattice analytics and continuum limit for the pure 2D Ising model. In what follows we assume ferromagnetic case with the homogeneous distribution of bond coupling parameters over the lattice.

5 The pure case (lattice)

From (11) specified to the homogeneous distribution of bond couplings, we obtain the Gaussian integral for the partition function of the regular $2D$ Ising model:

$$Q = \int \prod_{mn} d\bar{c}_{mn} dc_{mn} \exp \sum_{mn} \left[c_{mn}\bar{c}_{mn} - t_1\, c_{m-1n}\bar{c}_{mn} - t_2\, c_{mn}\bar{c}_{mn-1} \right.$$
$$\left. - t_1 t_2\, c_{m-1n}\bar{c}_{mn-1} + t_1\, c_{mn}c_{m-1n} + t_2\, \bar{c}_{mn-1}\bar{c}_{mn} \right], \qquad (16)$$

where $t_{1,2} = \tanh b_{1,2}$, $b_{1,2} = J_{1,2}/kT$. Let us pass to the momentum space by Fourier substitution:

$$c_{mn} = \frac{1}{\sqrt{L^2}} \sum_{pq} c_{pq}\, \mathrm{e}^{-i\frac{2\pi p}{L}m + i\frac{2\pi q}{L}n}, \quad \bar{c}_{mn} = \frac{1}{\sqrt{L^2}} \sum_{pq} \bar{c}_{pq}\, \mathrm{e}^{+i\frac{2\pi p}{L}m - i\frac{2\pi q}{L}n}, \qquad (17)$$

with $-L/2 \le p, q \le L/2$, where L is lattice length. The partition function becomes:

$$Q = \int \prod_{pq} d\bar{c}_{pq} dc_{pq} \exp \sum_{pq} [(1 - t_1 \mathrm{e}^{i\frac{2\pi p}{L}} - t_2 \mathrm{e}^{i\frac{2\pi q}{L}} - t_1 t_2 \mathrm{e}^{i\frac{2\pi p}{L} + i\frac{2\pi q}{L}}) c_{pq}\bar{c}_{pq}$$
$$+ t_1 \mathrm{e}^{i\frac{2\pi p}{L}} c_{L-pL-q} c_{pq} + t_2 \mathrm{e}^{i\frac{2\pi q}{L}} \bar{c}_{pq}\bar{c}_{L-pL-q}]. \qquad (18)$$

Making the $pq \leftrightarrow L - pL - q$ symmetrization of the sum (needed since variables with momenta pq and $L - pL - q$ interact) we find the integral (18) to be decoupled into a product of simple low-dimensional integrals, each over $d\bar{c}_{pq} dc_{pq} d\bar{c}_{L-pL-q} dc_{L-pL-q}$. The pq-integral factor can be readily evaluated making use of the elementary rules, see (6), which results in the explicit solution for Q^2. For similar momentum-space calculations see also [29]. Taking then the limit of infinite lattice, for the free energy per site we obtain:

$$-\beta f_Q = \frac{1}{2} \int_{-\pi-\pi}^{\pi\,\pi} \frac{dp\, dq}{(2\pi)^2} \ln [(1 + t_1^2)(1 + t_2^2) - 2t_1(1 - t_2^2)\cos p$$
$$- 2t_2(1 - t_1^2)\cos q], \qquad (19)$$

which is equivalent to the Onsager solution [1]. As it follows from the analysis of the free-energy form, in the ferromagnetic case, $t_1, t_2 > 0$, the critical point is given by the condition:

$$1 - t_1 - t_2 - t_1\, t_2 = 0. \qquad (20)$$

The specific heat exhibits the logarithmic singularity near the critical point, $C \simeq A_c |\ln|\tau||$, $\tau \sim |T - T_c| \to 0$, where A_c is the critical amplitude. For the symmetric lattice ($t_1 = t_2$) the amplitude is: $A_c = (8/\pi)\, b_c^2$, where $b_c = \frac{1}{2}\ln(1 + \sqrt{2}) = (J/kT_c)$ is the dimensionless inverse critical temperature.

6 Majorana and Dirac Fields

Let us make a comment on the continuum-limit interpretation of the pure 2D Ising model near the critical point, also see [30, 31]. It is interesting that already the exact lattice action from (16) can be written in the field-theoretical like form, cf. [31]. In terms of the lattice derivatives $\partial_m c_{mn} = c_{mn} - c_{m-1n}$, $\partial_n \bar{c}_{mn} = \bar{c}_{mn} - \bar{c}_{mn-1}$, this action becomes:

$$S\,(c,\bar{c}\,)_{\,\mathrm{pure}} = \sum_{mn} [\,\underline{m}\,c_{mn}\bar{c}_{mn} + \lambda_1\,(\partial_m c_{mn})\bar{c}_{mn} + \lambda_2\,c_{mn}(\partial_n \bar{c}_{mn}) + t_1\,(\partial_m c_{mn})c_{mn} + t_2\,\bar{c}_{mn}(\partial_n \bar{c}_{mn}) - t_1 t_2\,(\partial_m c_{mn})\,(\partial_n \bar{c}_{mn})\,], \qquad (21)$$

with $\underline{m} = 1 - t_1 - t_2 - t_1\,t_2$, $\lambda_1 = t_1 + t_1\,t_2$, $\lambda_2 = t_2 + t_1\,t_2$. Evidently, $\underline{m}$ plays the role of mass, the critical point corresponds to $\underline{m} = 0$. Taking the formal limit to the continuum space: $mn \to (x_1, x_2) = x$, $\partial_m, \partial_n \to \partial_1, \partial_2$, and changing the notion for the fields: $c_{mn}, \bar{c}_{mn} \to \psi(x), \bar{\psi}(x)$, we obtain the continuum-limit counterpart of the exact lattice action (21) of the same formal structure. The second order kinetic term with $\partial_1\partial_2$ can be ignored. The remaining action is of the Majorana type and can be transformed into a canonical form by a suitable rotation and rescaling of the fields, $\psi, \bar{\psi} \to \psi_1, \psi_2$, cf. [30, 31]. In this way we obtain the action in a canonical Majorana form:

$$S\,(\psi_1,\psi_2) = \int d^2x\,[\,\overline{m}\,\psi_1\psi_2 + \psi_1\,\tfrac{1}{2}\,(\partial_1 + i\partial_2)\psi_1 + \psi_2\,\tfrac{1}{2}\,(-\partial_1 + i\partial_2)\psi_2]\,, \qquad (22)$$

with the rescaled mass:

$$\overline{m} = \frac{1 - t_1 - t_2 - t_1\,t_2}{\sqrt{2\,(t_1 t_2)_c}}\,, \qquad (23)$$

where $(\;)_c$ stands for the criticality condition, eq. (20).

The 2D Majorana action (22) can be rewritten also in the matrix notation. Assuming $\sigma_1, \sigma_2, \sigma_3$ to be the standard Pauli matrices, we find:

$$S_{\mathrm{major}} = \frac{1}{2}\int d^2x\;\tilde{\Psi}\,(x)\,[\,\overline{m} + \hat{\partial}\,]\,\Psi\,(x)\,, \qquad (24)$$
$$\tilde{\Psi}\,(x) = \Psi^{\,T}\,(x)\,(i\,\sigma_2)\,, \quad \hat{\partial} = \gamma_1\partial_1 + \gamma_2\partial_2\,,$$

where $\tilde{\Psi}$ and Ψ are the two-component left and right Majorana spinors, $\Psi = (\psi_1, \psi_2)$, and $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$ are the 2D gamma-matrices. Finally, we can pass to the Dirac theory of charged fermions by doubling the number of fermions in the Majorana representation [14, 31]. Taking two identical copies S' and S'' of the Majorana action, eq. (24), we consider the combined action $S_{\mathrm{dirac}} = (S' + S'')_{\mathrm{major}}$. Introducing new Dirac fields by substitution $\Psi = \frac{1}{\sqrt{2}}(\Psi' + i\,\Psi'')$, $\bar{\Psi} = \frac{1}{\sqrt{2}}(\tilde{\Psi}' - i\,\tilde{\Psi}'')$, we find the combined action in the form:

$$S_{\mathrm{dirac}} = \int d^2x\;\bar{\Psi}\,(x)\,[\,\overline{m} + \hat{\partial}\,]\,\Psi\,(x)\,, \qquad (25)$$

where $\overline{m}$ and $\hat{\partial}$ are the same as in (24), while $\bar{\Psi}$ and Ψ are now charged Dirac spinors with four independent components.

The formal meaning of the transformation to the continuum limit can be most readily understood in the momentum space. The change $\partial_m, \partial_n \to \partial_1, \partial_2$ in the momentum space corresponds to the approximation:

$$e^{ip} - 1 \simeq ip\,, \quad e^{iq} - 1 \simeq iq\,, \qquad (26)$$

where p, q are the quasidiscrete Fourier momenta, $p, q \leftrightarrow \frac{2\pi p}{L}, \frac{2\pi q}{L}$, $-\pi \leq p, q \leq \pi$, see (17), and $|p|$ below is modulus of (p, q). The approximation like (26) is legal only in the neighborhood of the origin in the momentum space, $0 \leq |p| \leq k_0$, where $k_0 \ll 1$ is some fixed small cut-off momentum. There is also an effective infrared bound for $|p|$ introduced by the mass, which implies that $\overline{m}$ is small and we are near T_c. The continuum-limit formulation thus captures the essential features of the exact lattice theory in the low-momentum sector responsible for the singularities in the thermodynamic functions and the large-distance behaviour of the correlations as $T \to T_c$ ($\overline{m} \to 0$). In this region, the approximation is exact. In particular, the logarithmic singularity in the specific heat can still be recovered from (24) or (25), with the exact value of the critical amplitude.

7 Gross–Neveu model ($N \to 0$)

To extract the continuum-limit theory for the model with interaction (13)-(15), we shall distinguish explicitly the Fourier harmonics (cf. (17)) with

higher and low fermionic momenta, $|p| > k_0$ and $|p| < k_0$, where $k_0 \ll 1$ is some small fixed cut-off. Introducing separable notation, a, ψ, for the corespondent fields, in the coordinate space we write:

$$c_{mn} = (a_{mn})_{|p|>k_0} + (\psi_{mn})_{|p|<k_0}\,, \quad \bar{c}_{mn} = (\bar{a}_{mn})_{|p|>k_0} + (\bar{\psi}_{mn})_{|p|<k_0}\,. \tag{27}$$

The Gaussian part of the exact lattice action (14) is additive in the momentum space, so $S_0 = S_a + S_\psi$, where S_a and S_ψ are independent. However, the non-Gaussian term of interaction (15) is non-additive with respect to $c \to a + \psi$. We have thus to substitute explicitly $c \to a + \psi$ into this term, and then integrate over the higher-momentum lattice modes $(a_{mn})_{|p|>k_0}$. This is done below for weak dilution, in the first order of perturbation in $\lambda_0 \sim 1 - p \ll 1$.

Let us write the action (14) in the form $S = S_a + S_\psi + \lambda_0 S_{int}(a, \psi)$, $\lambda_0 = \frac{1-p}{p}$. In the first order in λ_0 we obtain:

$$\begin{aligned}
\overline{Q^N} &\simeq \int D_a D_\psi \, \mathrm{e}^{S_a + S_\psi} \left(1 + \lambda_0 S_{int}(a, \psi)\right) \\
&\simeq \int D_a \, \mathrm{e}^{S(a)} \int D_\psi \, \mathrm{e}^{S_\psi} \left\{ \frac{\int D_a \mathrm{e}^{S_a} (1 + \lambda_0 S_{int}(a, \psi))}{\int D_a \mathrm{e}^{S_a}} \right\} \\
&\simeq \int D_a \, \mathrm{e}^{S_a} \int D_\psi \, \mathrm{e}^{S_\psi + \lambda_0 \langle S_{int}(a,\psi)\rangle_a}\,.
\end{aligned} \tag{28}$$

Substituting (27) into $\langle S_{int}(a, \psi)\rangle_a$, we follow the standard field-theoretical prescriptions for critical fluctuations that only the most important quartic and quadratic terms in the $\psi, \bar{\psi}$ fields are actually to be taken into account [14, 15]. In terms of the characteristic fermionic averages $\langle A\rangle$ and $\langle B\rangle$ presenting the contribution of the higher-momenta lattice modes, we then find:

$$\begin{aligned}
\langle S_{int}(a, \psi)\rangle_a &= \sum_{mn} \prod_{\alpha=1}^{N} \left[\langle A\rangle + \langle B\rangle \, \psi^\alpha_{mn} \bar{\psi}^\alpha_{mn} + \ldots \right] \qquad (29) \\
&= \langle A\rangle^N \sum_{mn} \Big[1 + \frac{\langle B\rangle}{\langle A\rangle} \sum_{\alpha=1}^{N} \psi^\alpha_{mn} \bar{\psi}^\alpha_{mn} + \frac{1}{2} \frac{\langle B\rangle^2}{\langle A\rangle^2} \Big(\sum_{\alpha=1}^{N} \psi^\alpha_{mn} \bar{\psi}^\alpha_{mn} \Big)^2 + \ldots \Big]\,,
\end{aligned}$$

where

$$\begin{aligned}
\langle A\rangle &= \langle a\bar{a} \, \exp\left(t_1 t_2 \, \partial_1 a \, \partial_2 \bar{a}\right)\rangle_a = \langle a\bar{a}\rangle_a + t_1 t_2 \, \langle a\bar{a} \, \partial_1 a \, \partial_2 \bar{a})\rangle_a\,, \\
\langle B\rangle &= \langle \exp\left(t_1 t_2 \, \partial_1 a \, \partial_2 \bar{a}\right)\rangle_a = 1 + t_1 t_2 \, \langle \partial_1 a \, \partial_2 \bar{a})\rangle_a\,.
\end{aligned} \tag{30}$$

These are the lattice averages for pure model (16) with infrared truncation at k_0. Here we assume the evident abbreviation with dropped $^\alpha_{mn}$ index and $\partial_1, \partial_2 \leftrightarrow \partial_m, \partial_n$ correspondence. For instance, $\langle \partial_1 a \partial_2 \bar{a}\rangle_a$ means $\langle \partial_m a^\alpha_{mn} \partial_n \bar{a}^\alpha_{mn}\rangle_a$, etc. The dependence on small momentum cut-off k_0 can be in fact ignored (put $k_0 = 0$). The averages are then to be evaluated with the pure lattice action, eqs. (16)-(18), and are $^\alpha_{mn}$ independent.

In (29), in the first line (...) is the abbreviation for the 'gaussian kinetic terms', which can be ignored,. and in the second line (...) stands for the total sum of all irrelevant corrections, including the gaussian kinetic terms, quartic kinetic terms, and higher-order corrections in the $\psi, \bar{\psi}$ fields of any kind. The effective action for the $\psi, \bar{\psi}$ fields is $S_{\text{eff}} = (S_\psi)_{\text{free}} + \lambda_0 \langle S_{\text{int}}(a, \psi)\rangle_a$. The terms preserved explicitly in (29) contribute to the Gaussian part of this action (modification of mass) and generate the four-fermion interaction. A trivial constant term may be ignored. The Gaussian part of the action S_{eff} appears in the form of the replicated continuum-limit version of the action (21), cf. (22), but with a modified mass. After uniformization of the Gaussian part and the correspondent rescaling of the fields, $\psi, \bar{\psi} \to \psi_1, \psi_2$, we obtain the effective continuum-limit action in the form of the N-colored Gross-Neveu model [6, 14]:

$$\begin{aligned}
S_{\text{G-N}} = \int d^2x \{ \sum_{\alpha=1}^{N} [\overline{m}_N \, \psi^\alpha_1 \psi^\alpha_2 + \frac{1}{2} \psi^\alpha_1 \left(\partial_1 + i\partial_2\right) \psi^\alpha_1 \\
+ \frac{1}{2} \psi^\alpha_2 \left(-\partial_1 + i\partial_2\right) \psi^\alpha_2] + g_N [\sum_{\alpha=1}^{N} \psi^\alpha_1 \psi^\alpha_2]^2 \}\,,
\end{aligned} \tag{31}$$

with the effective mass and coupling constant given as follows:

$$\overline{m}_N = \frac{1 - t_1 - t_2 - t_1 t_2}{\sqrt{2(t_1 t_2)_c}} + \langle A\rangle^N \, \frac{1-p}{p} \frac{\langle B\rangle}{\langle A\rangle} \frac{1}{\sqrt{2\,(t_1 t_2)_c}}\,, \tag{32}$$

$$g_N = \langle A\rangle^N \, \frac{1-p}{p} \frac{\langle B\rangle^2}{\langle A\rangle^2} \frac{1}{4\,(t_1 t_2)_c}\,. \tag{33}$$

Since the replica limit $N \to 0$ is assumed at final stages, one can put $N = 0$ and $\langle A\rangle^N = 1$ already in (32) and (33). In the matrix notation, the action (31) becomes:

$$S_{\text{G-N}} = \frac{1}{2} \sum_{\alpha=1}^{N} \int d^2x \, \tilde{\Psi}^\alpha(x) \left(\overline{m}_0 + \hat{\partial}\right) \Psi^\alpha(x)$$

$$+ g_0 \int d^2x \left[\frac{1}{2}\sum_{\alpha=1}^{N} \bar{\Psi}^\alpha(x)\Psi^\alpha(x)\right]^2 . \qquad (34)$$

The Gaussian part here is the replicated Majorana action (24) with modified mass (32). By analogy with the pure case, we can pass as well to the Dirac field interpretation. The doubling of Majorana fermions can be realized simply by doubling the number of replicas, $N \to 2N$, and we find:

$$S_{\text{G-N}} = \sum_{\alpha=1}^{N} \int d^2x\, \bar{\Psi}^\alpha(x)\,(\,\overline{m}_0 + \hat{\partial}\,)\,\Psi^\alpha(x)$$
$$+ g_0 \int d^2x\,[\sum_{\alpha=1}^{N} \bar{\Psi}^\alpha(x)\Psi^\alpha(x)\,]^2 , \qquad (35)$$

where $\bar{\Psi}^\alpha, \Psi^\alpha$ are charged Dirac spinors, cf. (25). In (34), (35) we assume the mass and charge already at $N = 0$.

The $N = 0$ Gross-Neveu model was first introduced as an effective field theory for weak *bond* dilution in the pioneering papers by Dotsenko and Dotsenko (DD) [4]-[6]. Their theory has been reanalyzed later on as regards to the behaviour of correlations by Shalaev, Shankar, and Ludwig (SSL) [7]-[9], see [14] for a comprehensive review. An advanced analysis of correlations in the $N = 0$ G-N model in terms of the perturbed conformal field theories has been performed quite recently by Dotsenko, Picco and Pujol [33, 34]. We thus expect the same changes in critical behaviour inspired by weak *site* dilution as those predicted in the DD-SSL theories for weak bond dilution. From the renormalization-group (RG) analysis of the $N = 0$ G-N model it follows that in a closed vicinity of the critical point the singularity in the specific heat is doubled-logarithmic [6, 15]:

$$C \propto \frac{1}{2g_0} \ln\,(1 + \frac{2g_0}{\pi} \ln \frac{1}{|\tau|}) . \qquad (36)$$

The RG calculation also shows [6] that the correlation radius gains the logarithmic correction, $R_c \sim \tau^{-1} |\ln \tau|^{1/2}$ (cf. $R_c \sim \tau^{-1}$ in the pure case, where $\tau \sim |T - T_c|$). The two-spin correlation function $G(R) = \langle \sigma(0)\sigma(R) \rangle$ taken exactly at T_c remains to be the same as for the pure system [14]: $G(R) \sim R^{-1/4}$, $R \to \infty$. [That no logarithmic or other

corrections appear due to the disorder in $G(R)$ in the $N = 0$ G-N theory [7]-[9] can now be assumed to be definitely established [14, 33, 34], this question was a subject of some debate in previous decade [14]]. Assuming the standard scaling arguments to be true, this then implies that the magnetization and magnetic susceptibility gain the logarithmic corrections to the pure-case power singularities at criticality [14, 22]: $M\,(\tau\,|\,0) \sim |\tau|^{1/8} |\ln \tau|^{-1/16}$, $\chi\,(\tau\,|\,0) \sim |\tau|^{-7/4} |\ln \tau|^{7/8}$, $M\,(0\,|\,h) \sim h^{1/15}$, where $\tau \sim |T - T_c|$ and h is a small magnetic field. These predictions have been checked also, for weak dilution, in a series of the precise Monte-Carlo experiments [17, 18, 21, 22]. The question about the effects produced by strong and moderated dilution is less evident, however, both on theoretical [12, 14] and experimental (Monte-Carlo) [20, 22] sides, this holds for both bond and site dilution.

In conclusion to this section, let us specify in more detail the parameters of the $N = 0$ G-N model for site dilution for the isotropic lattice ($t_1 = t_2$, or $J_1 = J_2$). This will yield also the exact value for the initial slope of the $T_c(p)$ curve. The effective mass and coupling constant include the lattice constants $\langle A \rangle$ and $\langle B \rangle$. Since the dilution is weak, and we are near T_c of the pure system, it is reasonable to specify these parameters exactly at T_c of the pure system. Also, the dependence on small cut-off momenta, $k_0 \ll 1$, can be ignored. The averages $\langle A \rangle$ and $\langle B \rangle$ are then to be taken for the pure lattice system at T_c. If $t_1 \neq t_2$, however, these parameters still remain be functions of lattice anisotropy. They become definite numbers in the isotropic case. By a straightforward though somewhat lengthy calculation we have obtained the following values for the isotropic lattice ($J_1 = J_2$, $T = T_c$, $k_0 = 0$):

$$\langle A \rangle_c = \frac{1}{2}(\frac{1}{\sqrt{2}} + \frac{1}{\pi})(1 + \frac{1}{\sqrt{2}} - \frac{1}{\pi}) = 0.712048\,,$$
$$\langle B \rangle_c = 2\,t_c\,(1 + \frac{1}{\sqrt{2}} - \frac{1}{\pi}) = 1.150517\,, \quad t_c = \sqrt{2} - 1\,,$$
$$\frac{\langle B \rangle_c}{\langle A \rangle_c} = 4\sqrt{2}\,t_c\,(1 + \frac{\sqrt{2}}{\pi})^{-1} = 1.615786\,. \qquad (37)$$

For the isotropic lattice, the critical point is given by condition $(1 - 2t - t^2)_c = 0$, with the solution $t_c = \mathrm{e}^{-2b_c} = \sqrt{2} - 1$, $b_c = \frac{1}{2}\log(1 + \sqrt{2})$.

The pure-case mass $\overline{m} = \overline{m}(1)$ (23) then appears in the form: $\overline{m}(1) \simeq 2(t_c - t)/t_c \simeq 4b_c(T - T_c)/T_c$. Taking into account (37), the effective mass ($N = 0$) of the G-N model becomes:

$$\overline{m}_0 = 4\, b_c \frac{T - T_c}{T_c} + \frac{1-p}{p} \frac{4}{1 + \sqrt{2}/\pi}$$
$$= (1.763) \frac{T - T_c}{T_c} + (2.758) \frac{1-p}{p}, \qquad (38)$$

while the coupling constant ($N = 0$) appears in the form:

$$g_0 = \frac{1-p}{p} \frac{8}{(1 + \sqrt{2}/\pi)^2} \simeq (3.804) \frac{1-p}{p}. \qquad (39)$$

These are the values of mass and charge of the $N = 0$ Gross-Neveu model (31)-(35) for weak site dilution near T_c in the isotropic case. These values are exact in the linear orders in $1 - p$ and $T - T_c$.

The expression for mass (38) enables one to define the initial slope for the $T_c(p) - p$ curve at $p = 1$. The initial slope is the coefficient in the expansion $T_c(p) = T_c(1)\,[1 - S_c \frac{1-p}{p} + \ldots]$. Dividing both sides of the equation $\overline{m}_0(p) = 0$ by $1 - p$ and taking the limit $p \to 1$, we find:

$$S_c = \frac{1}{T_c} \frac{dT_c}{dp}\bigg|_{p=1} = \frac{(b_c)^{-1}}{1 + \sqrt{2}/\pi} = 1.564785\,, \qquad (40)$$

which is in agreement with the earlier calculation of S_c [35]. As distinct from a direct spin-lattice perturbative analysis in [35], the present calculation includes the N-replica trick with fermions.

8 Concluding discussion

We have applied a new noncombinatorial method of fermionization to clarify the fermionic structure of the 2D Ising model with quenched site dilution. In fact, the method works equally well for both site and bond dilution. We have specified the discussion merely to the case of site dilution as being less studied theoretically. For the first step, the partition function of the 2DIM with fixed site dilution and arbitrary inhomogeneous

distribution of the bond coupling parameters over the lattice was transformed into a fermionic Gaussian integral, eq. (8). The resulting integral was then simplified integrating out extra fermionic degrees of freedom. This yields the Gaussian like representation with two fermionic variables per site for fixed site dilution, eqs. (9)-(11). Even for the pure case, this reduction of extra fermionic degrees of freedom essentially simplifies the analytics of any kind and illuminates Majorana-Dirac structure of the 2DIM already at the lattice level. The quenched averaging over the disorder in the N-replica scheme results, in turn, in the exact lattice theory of interacting fermions for the case of site dilution, eqs. (13)-(15). The interaction in the lattice theory appears to be of order $2N$ in fermions, where N is the number of replicas. However, the continuum-limit approximation for weak dilution near criticality results in the standard $N = 0$ Gross-Neveu model with four-fermion interaction. The $N = 0$ G-N model has been already intensively analyzed as an effective field theory for bond dilution [6, 14]. This then implies similar behaviour of the thermodynamic functions and correlations in the disordered $2D$ Ising model both for site and bond dilution near T_c, at least for small amount of impurities. The predictions of the $N = 0$ G-N model for weak dilution are the log-log singularity in the specific heat and log-corrections in M and χ, see section 7 for more comments. We have also specified in detail the parameters (mass and charge) of the $N = 0$ G-N model for site dilution in terms of the lattice fermion Green's functions and evaluated the initial slope of the $T_c - p$ curve. The present analysis thus supports the hypothesis about the universality in critical behaviour produced by different sorts of disorder for a small fraction of impurities in the ferromagnetic 2D Ising model. The question about the universality between weak and strong dilution remains to be open, in essence. A more straightforward analysis, directly in lattice interpretation, seems to be needed. A compact form of interaction in the exact lattice theory (13)-(15) provides grounds to suppose further progress in this respect. Finally, we note that the Gaussian fermionic representations with fixed disorder like (8), (9), (11) yet preserve all the information about the original spin-variable model (3) that is included in the hamiltonian (1). There are no also any restrictions, at

this stage, on the sign of the bond coupling parameters. Therefore, these representations can be used as the starting point to try other problems with disorder in the $2D$ Ising model, beyond site or bond dilution (spin glasses, models with annealed or regular inhomogeneities, etc). As well, one can try to apply other methods of quenched averaging, beyond the N-replica scheme. The later possibility may be favorable, in particular, concerning the problem of strong and moderated dilution.

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References

[1] L. Onsager, Phys. Rev. 65 (1944) 117.

[2] A.B. Harris, J. Phys. C 7 (1974) 1671.

[3] R.B. Stinchcombe, in Phase Transitions and Critical Phenomena, Vol. 7, C. Domb and J.L. Lebowitz, eds. (Academic Press, London, 1983) p. 151.

[4] Vik.S. Dotsenko and Vl.S. Dotsenko, Sov. Phys. JETP Lett. 33 (1981) 37.

[5] Vik.S. Dotsenko and Vl.S. Dotsenko, J. Phys. C 15 (1982) 495.

[6] V.S. Dotsenko and V.S. Dotsenko, Adv. Phys. 32 (1983) 129.

[7] B.N. Shalaev, Sov. Phys. Solid State, 26 (1984) 1811.

[8] R. Shankar, Phys. Rev. Lett. 58 (1987) 2466.

[9] A.W.W. Ludwig, Nucl. Phys. B 330 (1990) 639.

[10] G. Jug, Phys. Rev. Lett. 53 (1984) 9.

[11] P.N. Timonin, Sov. Phys. JETP 68 (1989) 512.

[12] K. Ziegler, Nucl. Phys. B 334 (1990) 499.

[13] R. Shankar and G. Murthy, Phys. Rev. B 36 (1987) 536.

[14] B.N. Shalaev, Phys. Rep. 237 (1994) 129.

[15] Vik. S. Dotsenko, Usp. Fiz. Nauk 165 (1995) 481.

[16] G. Jug and B.N. Shalaev, Phys. Rev. B 54 (1996) 3442.

[17] J.-S. Wang, W. Selke, Vl.S. Dotsenko and V.B. Andreichenko, Physica A 164 (1990) 221.

[18] V.B. Andreichenko, Vl.S. Dotsenko, W. Selke and J.-S. Wang, Nucl. Phys. B 334 (1990) 531.

[19] H.-O. Heuer, Phys. Rev. B 45 (1992) 5691.

[20] J.-K. Kim and A. Patrascioiu, Phys. Rev. Lett. 72 (1994) 2785.

[21] A.L. Talapov and L.N. Shchur, Europhys. Lett. 27 (1994) 193.

[22] W. Selke, L.N. Shchur and A.L. Talapov, Monte Carlo Simulations of Dilute Ising Models. In: Annual Reports of Computational Physics, Ed. by D. Stauffer (World Scientific, Singapore, 1994), p. 17.

[23] R. Kühn, Phys. Rev. Lett. 73 (1994) 2268.

[24] F.D.A. Aarao Reis, S.L.A. de Queiroz and R.R. dos Santos, Phys. Rev. B 54 (1996) R9616.

[25] J.J. Ruiz-Lorenzo, J. Phys. A 30 (1997) 485.

[26] F. A. Berezin, The Method of Second Quantization (Academic Press, New York, 1966).

[27] V.N. Plechko, Sov. Phys. Doklady, 30 (1985) 271.

[28] V.N. Plechko, Sov. Phys. Theor. Math. Phys. 64 (1985) 748.

[29] V. N. Plechko, Physica A 152 (1988) 51.

[30] C. Itzykson and J.M. Drouffe, Statistical Field Theory (Cambridge Univ. Press, Cambridge, 1989) Vol. 1.

[31] V.N. Plechko, In: Proceedings of the VII International Conference on Symmetry Methods in Physics, Dubna, Russia, July 10–16, 1995. Ed. by A.N. Sissakian and G.S. Pogosyan (JINR 96-224, Dubna, 1996) Vol. 2, p. 443-450. hep-th/9607053.

[32] V.N. Plechko, In: Proceedings of the V International Conference on Path Integrals from meV to MeV, Dubna, Russia, May 27–31, 1996. Ed. by V.S. Yarunin and M.A. Smondyrev (JINR 96-321, Dubna, 1996) p. 295-299. hep-th/9609044.

[33] Vl. Dotsenko, M. Picco and P. Pujol. Phys. Lett. B 347 (1995) 113.

[34] Vl.S. Dotsenko, M. Picco and P. Pujol, Nucl. Phys. B (Proc. Suppl.) 45A (1996) 145.

[35] H. Au-Yang, M.E. Fisher and A.E. Ferdinand, Phys. Rev. B 13 (1976) 1238.

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