объединенный
институт
ядерных
исследований
дубна

E17-94-455

E.A.Kochetov

SU(2) PATH INTEGRAL

Submitted to «Journal of Mathematical Physics»

1994

1 Introduction

Recently, the path integral in the coherent state representation has been considered (Klauder 1978,1979, Berezin 1980, Blaziot and Orland 1981, Negele and Orland 1988). The path integral in the generalized coherent state representation provides a natural and convenient device for quantization of classical systems with nontrivial (nonflat) phase space. Note that the ordinary phase space path integral can be thought of as an integral in the Glauber coherent state representation and is defined on paths in C^1, the ordinary complex plane. It is relevant to boson oscillator-like systems, whereas for spin systems it is more convenient, however, to use a path integral defined on the compact nonflat manifold $\overline{C}^1$, extended complex plane which is homeomorphic to the two-sphere, S^2.

From the group theoretical point of view, coherent states can be associated with unitary irreducible representations of Lie groups (Perelomov 1972, Gilmore 1972). The irreducibility as well as the existence of the group invariant measure ensures that the resolution of unity holds in the coherent state basis (Onofri 1975, Klauder and Skagerstam 1985), which turns out to be the most important property of the coherent states that allows for the path integral construction.

The conventional Glauber coherent states are related to the UIR of the Heisenberg-Weyl group, whereas the spin ($SU(2)$) coherent states— to the UIR' of the $SU(2)$ group (Radcliffe 1971, Arecchi et al 1972, Perelomov 1972). In the $SU(2)$ case integration is carried out over paths in $\overline{C}^1 \simeq S^2$, which can be identified with the $SU(2)$ homogeneous space, $SU(2)/U(1)$ (Perelomov 1985). The path integral in the $SU(2)$ coherent state representation originally appeared in the paper by Klauder 1979 as an integral on S^2. Kuratsuji and Suzuki 1980 have represented it as an integral on paths in $\overline{C}^1$. For recent and comprehensive review on the $SU(2)$ path integral and its applications see Kuratsuji 1992.

As is generally appreciated, a path integral serves as the most effective calculational device either in getting the exact form of a propagator for particular Hamiltonians or in deriving perturbation or semiclassical approximations to the propagator for a general problem. As is known, the path integral in the configuration space has been successfully used for these purposes both in the flat case starting from its invention by Feynman in 1948 and in the spaces with curvature and torsion (For review see, for example, Schulman 1981, Langouhe et al 1988, Kleinert 1992, and Inomata 1992).

There are, however, a number of unsatisfactory points and mysteries when the path integral in the coherent state representation is concerned. By saying this, we do not mean the unsatisfactory points concerning the definition of a path integral itself as the formal limiting form of the time-lattice approximation. These difficulties are the same for both the configuration or coherent state path integrals. In the first place, we mean the so-called overspecification problem (Klauder 1978,1979), i.e., that the semiclassical expansion for a path integral that represents a transition amplitude between coherent states encounters a mystery: the number of boundary conditions turn out to be twice of the equation of motion. The second point is that the coherent state path integral fail to recover the exact spectrum even for the solvable simple models. Restricting ourselves to the $SU(2)$ case we note that the spectrum of the single spin Hamiltonian

$$H = \omega S_z$$

is recovered by the $SU(2)$ path integral calculations with the shift by 1/2 (Kuratsuji and Mizobuchi 1981, Enz and Schilling 1986). More worse, however, is that the semiclassics is not exact in the case of dynamical symmetry as it must be due to the path integral generalization of the Duistermaat-Heckman theorem (Blau et al 1990). All this might obscure the physical interpretation of the coherent state path integral and hinders its usefulness in applications.

On the other hand, when applied to the discrete time formulation the WKB expansion becomes exact and non of the above mentioned difficulties arise (Funahashi et al 1994). Due to this one might suggest that the discretized form of the coherent state path integral is to be used rather than the continuum one. But this in turn would mean that the coherent state path integral fails to put a continuous classical action into correspondence with a quantum system. This would fall in an obvious disagreement with the Berezin quantization scheme developed for homogeneous Kähler manifolds (Berezin 1975) which, for example, $\overline{C}^1$ belongs to.

In the present paper we consider the self-contained $SU(2)$ path integral representation for the matrix element of the quantum propagator in the $SU(2)$ coherent state basis. It is essentially based on the correct continuum expression for the classical action which includes a boundary term. This term arises due to the discontinuity of complex trajectories at the end points. It should be remarked that the existence and importance of boundary terms in the coherent state path integrals have been stated by Faddeev 1975, Berezin 1980, Blazoit and Orland 1981. For a very good review on this topic see Negele and Orland 1988. Nevertheless, till now a very little attention has been paid to the correct continuum form of a path integral and all the difficulties encountered are merely assigned to the fact that a path integral is an ill defined object in itself.

In the present paper, by the simple exampe of the $SU(2)$ coherent state path integral we argue that all the above mentioned difficulties can be safely avoided provided a correct form of a continuous classical action with boundary term is taken into account. This enables one to consider the $SU(2)$ path integral to be a convenient and useful device for quantizing classical systems. This seems to be true for any coherent state path integral, which will be discussed elsewhere. All this indicates that using of the coherent state path integral does not imply encountering of any additional problems as compared to those of the conventional configuration space one. But one should kept in mind, however, that both of these constructions cannot be in general thought of as integrals over certain measures, which means that in case of a trouble the recourse to the time-lattice approximation should be made.

The plan of this paper is as follows. In the second section we write down the correct continuum $SU(2)$ path integral representation for a transition amplitude (quantum propagator) and show how it can be computed in case of dynamical symmetry. The next section is devoted to the application of the $SU(2)$ path integral to a real spin system, quantum Heisenberg ferromagnet. The main result of the paper constitutes section 4. It concerns the evaluation of the semiclassical $SU(2)$ propagator. We follow the ideas by Levit and Smilansky 1977, Buslaev 1980 and Kuratsuji and Mizobuchi 1981 and express the final result in a closed form in terms of a total classical action and classical trajectories. Section 5 is a conclusion.

2 $SU(2)$ path integral for transition amplitude

At the beginning of this section we recapitulate the necessary ingredients for the $SU(2)$ path integral representation of a transition amplitude.

The starting point is the $SU(2)$ algebra with generators

$$[S_z, S_\pm] = S_\pm, \quad [S_+ S_-] = 2S_z. \tag{1}$$

The $SU(2)$ coherent state is given by

$$|z; S\rangle = (1 + |z|^2)^{-S} e^{zS_+} |S; -S\rangle, \quad S_-|S; -S\rangle = 0, \tag{2}$$

where $S \in N/2$ is the representation index and $z \in SU(2)/U(1) = \overline{C}^1$. The resoluton of unity in the representation S with respect to the $SU(2)$ invariant measure reads

$$\int |z; S\rangle\langle z; S|\, d\mu_S(z) = I_S, \quad d\mu_S = \frac{2S+1}{\pi} \frac{d^2 z}{(1+|z|^2)^2}. \tag{3}$$

Consider the coherent state matrix element

$$G_S(\bar{z}_1, z_2; \tau) = \langle z_1; S| T e^{-i\int_0^\tau H ds} |z_2; S\rangle,$$

where H is an entire function in the $SU(2)$ generators.

For the reader' convenience we give here a brief account of the derivation of the continuum $SU(2)$ path integral representation for a transition amplitude. Our final expression involves boundary term and correct boundary conditions as against that of Kuratsuji 1992. This in turn enables us to properly treat the $SU(2)$ continuum path integral.

In order to express the transition amplitude by path integral, we divide the time interval into N small intervals: $\epsilon = \tau/N$ with $N \to \infty$. Let us define

$$s_k = \epsilon k, \quad \zeta_k = \zeta(s_k), \quad 0 \le k \le N.$$

With the aid of the time discretization together with relation (3) the propagator can be written in the form

$$G_S(z_1, z_2; \tau) = \lim_{N\to\infty} \int \prod_{k=0}^{N} d\mu_k \langle z_1|\zeta_N\rangle\langle\zeta_0|z_2\rangle \prod_{k=1}^{N} \langle\zeta_k| e^{-i\epsilon H(s_k)} |\zeta_{k-1}\rangle. \tag{4}$$

where we have dropped the S label of the coherent state and put $d\mu_k \equiv d\mu_S(\zeta_k)$. Proceeding further we note that

$$\langle\zeta_k| e^{-i\epsilon H(s_k)} |\zeta_{k-1}\rangle = \langle\zeta_k|\zeta_{k-1}\rangle \{\exp\{-i\epsilon H(\zeta_k, \zeta_{k-1}; s_k)\} + O(\epsilon^2)\}.$$

where

$$H(\zeta_k, \zeta_{k-1}; s_k) = \frac{\langle\zeta_k|H(s_k)|\zeta_{k-1}\rangle}{\langle\zeta_k|\zeta_{k-1}\rangle}.$$

To the same order in ϵ

$$G_S(\bar{z}_1, z_2; \tau) = \lim_{N\to\infty} \int \prod_{k=0}^{N} d\mu_k \langle z_1|\zeta_N\rangle\langle\zeta_0|z_2\rangle \prod_{k=1}^{N} \langle\zeta_k|\zeta_{k-1}\rangle \exp\{-i\epsilon H(\zeta_k, \zeta_{k-1}; s_k)\} \tag{5}$$

It can be easily checked that for any element of the Hilbert state-space of a single particle with spin S

$$f(z) = \frac{P_m(z)}{(1+|z|^2)^S}$$

there holds

$$\int \langle\eta|z\rangle f(z)\, d\mu_S(z) = f(\eta), \tag{6}$$

where P_m is an arbitrary polynomial of the degree $m \le 2S$ (Perelomov 1985). This enables one to carry out the integration over $d\mu_N, d\mu_0$ in (5) with the result

$$G_S(z_1, z_2; \tau) = \lim_{N\to\infty} \int_{\zeta_0=z_2}^{\zeta_N=z_1} \prod_{k=1}^{N-1} d\mu_k \langle z_1|\zeta_{N-1}\rangle\langle\zeta_1|z_2\rangle \prod_{k=2}^{N-1} \langle\zeta_k|\zeta_{k-1}\rangle \exp\{-i\epsilon \sum_{k=1}^{N} H(\zeta_k, \zeta_{k-1})\} \tag{7}$$

Next, by using the difference $\delta_k = \zeta_k - \zeta_{k-1}$ we have

$$\log\langle\zeta_k|\zeta_{k-1}\rangle = S\frac{\bar{\delta}_k\zeta_{k-1} - \bar{\zeta}_k\delta_k}{1+\bar{\zeta}_k\zeta_{k-1}} + O(\delta^2).$$

If δ_k becomes $O(\epsilon)$ in the limit of $\epsilon \to 0$, (7) is reduced to the form

$$\begin{aligned} G_S(\bar{z}_1, z_2; \tau) &= \lim_{N\to\infty}\int_{\zeta_0=z_2}^{\bar{\zeta}_N=\bar{z}_1}\prod_{k=1}^{N-1} d\mu_k \langle z_1|\zeta_{N-1}\rangle\langle\zeta_1|z_2\rangle \\ &\exp\left\{S\sum_{k=2}^{N-1}\frac{\bar{\delta}_k\zeta_{k-1} - \bar{\zeta}_k\delta_k}{1+\bar{\zeta}_k\zeta_{k-1}} - i\epsilon\sum_{k=1}^{N} H(\bar{\zeta}_k, \zeta_{k-1}; s_k)\}\right\} \end{aligned} \quad (8)$$

where we neglect the order $O(\epsilon^2)$ and higher, which is allowed if the limit in eq.(8) exists (for details see, for example, Berezin 1980). It is clear that the variables ζ_N and $\bar{\zeta}_0$ do not enter eq.(8) at all. In other words, The Euler-Lagrange equations for ζ_k and $\bar{\zeta}_k$ are accompanied by the boundary conditions $\zeta_0 = z_2$ and $\bar{\zeta}_N = \bar{z}_1$, respectively. The term

$$\langle z_1|\zeta_{N-1}\rangle\langle\zeta_1|z_2\rangle$$

gives rise to the continuum boundary term to be discussed below.

In the continuum limit (8) gives rise to the formal path integral expression

$$G_S(\bar{z}_1, z_2; \tau) = \int_{z(0)=z_2}^{\bar{z}(\tau)=\bar{z}_1} D\mu_S(z)\exp\Phi, \quad (9)$$

where we have replaced the integration variables by $z(s)$ and $\bar{z}(s)$. The total action Φ includes the boundary term Γ:

$$\Phi = L + \Gamma,$$

where

$$L = S\int_0^\tau \frac{\dot{\bar{z}}z - \bar{z}\dot{z}}{1+|z|^2}ds - i\int_0^\tau H_S(\bar{z}, z)ds, \quad (10)$$

$$\Gamma = S\log\left[\frac{(1+\bar{z}_1 z(\tau))(1+\bar{z}(0)z_2)}{(1+|z_1|^2)(1+|z_2|^2)}\right] \quad (11)$$

and $H_S(\bar{z}, z) = \langle z; S|H|z; S\rangle$. The formal functional measure $D\mu_S$ is to be understood as an infinite pointwise product of measures (3). The path integral is normalized by $G\,|_{H=0} = \langle z_1; S|z_2; S\rangle$. Representation (9-11) is the $SU(2)$ particular case of the general coherent state path integral representation for a transition amplitude which can be found in (Negele and Orland 1988, Kochetov 1994a).

Some general remarks are in order at this stage. The representation (9-11) is to be understood as a formal limiting form of the corresponding time-lattice approximation (8). The justification of the existence of such a limit is a rather nontrivial problem, for details see (Berezin 1971,1980). But even in case this limit does exist this would in no way mean that path integral (9) appears as an integral over a certain measure. This in turn implies that all the formal

manipulations with the path integral, e.g., a change of variables, etc., should be understood as being applied to the discretized pre-limiting form, with the limit being carried out at the end of calculations. We believe following Berezin 1980 that in case the limit of the discretized form of a path integral in which a change of variables has been made coincides exactly with what one would get performing a formal change of variables in a continuum path integral, this change of variables is allowed. However, we would like to remark that in more simple situations as those associated with the Wiener measure a genuine integral calculus over contiuous paths exists. For recent applications to quantum mechanics see, for example, Fischer et al 1992.

The next important point is that variables $\bar{z}(s)$ and $z(s)$ are to be considered to be independent. This stems from the fact that $\bar{z}(s)$ and $z(s)$ are associated with the discretized variables displaced by one time step, $\bar{z}(s_k)$ and $z(s_{k-1})$, respectively (Berezin 1980, Blazoit and Orland 1981). The integration variables $z(\tau)$ and $\bar{z}(0)$ entering boundary term (10) are in fact nothing but $z(\tau - 0)$ and $\bar{z}(+0)$, respectively. The end-points $z(\tau) = z_1$ and $\bar{z}(0) = \bar{z}_2$ do not enter eqs.(10,11) at all. Due to this, the Euler-Lagrange equation $\delta\Phi = 0$ yields the continuous classical trajectories (extremals) $\bar{z}_c(s)$, $0 < s \le \tau$ and $z_c(s)$, $0 \le s < \tau$ specified by the different boundary conditions $\bar{z}_c(\tau) = \bar{z}_1$ and $z_c(0) = z_2$, as it should be. Consequently, the overspecification problem disappears. Note that variation of the Γ-term eliminates the boundary terms coming from the variation of L, which results in the correct form of the equations of motion. In the section 4 we shall perform a regular expansion of Φ around extremals.

The relative shift of the arguments of $\bar{z}$ and z functions means nothing but a particular definition of the time-ordering product of operators at coinciding arguments as a normal product. Actually, this prescription follows from the chosen quantization scheme (Negele and Orland 1988). This turns out to be important in calculations. For example, even in the trace calculations of the coherent-state path integral where no boundary terms appear the shift of arguments should be kept in mind to ensure the correct result.

We are now ready to consider the first important application of the $SU(2)$ path integral, the exact evaluation of the propagator for a system governed by a Hamiltonian belonging to the $SU(2)$ algebra. This is referred to as the case of a dynamical symmetry. As is known the propagator in a closed form could be obtained by using the Wie-Norman disentangling procedure (Wie and Norman 1963) which is to be applied directly to the operator

$$T\,e^{-i\int_0^\tau H ds}.$$

This results in the representation of the T-exponent by a finite product of exponential operators whose number is equal to $3 = dim\,SU(2)$. This is sufficient for explicit computation of the propagator. Being a purely algebraic procedure, the Wei-Norman method, however, requires a lot of tedious calculations. One might hope to reach the goal in a simpler way by making an appropriate change of variables in a path integral. Let us show how it goes in the $SU(2)$ case.

Consider the Hamiltonian

$$H = 2A(t)S_z + f(t)S_+ + \bar{f}(t)S_-, \quad (12)$$

which is relevant to the system of a single spin in a fluctuating magnetic field. It may also represent some kind of an effective interaction, the c-functions A and $f, \bar{f}$ being the dynamical variables. This is the case in the next section.

By using the explicit form of the $SU(2)$ coherent states (2) we get

$$H_S(\bar{z}, z) = -2SA(t)\frac{1-|z|^2}{1+|z|^2} + 2Sf(t)\frac{\bar{z}}{1+|z|^2} + 2S\bar{f}(t)\frac{z}{1+|z|^2}. \quad (13)$$

Let us take a general $SU(2)$ element in the form

$$g = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in SU(2).$$

The group $SU(2)$ acts in $\overline{C}^1$ by the following canonical (projective) transformation

$$z \to \frac{uz + v}{-\bar{v}z + \bar{u}}. \tag{14}$$

We then change the integration variables in (9) by means of the substitution (14), parameters u and v being time-dependent. The integration measure in (9) remains invariant. The $SU(2)$ parameters are fixed by demanding the terms linear in f and $\bar{f}$ in Φ vanish after the substitution has been made. As a result G_S is simplified to

$$G_S(\bar{z}_1, z_2; \tau) = \frac{[\bar{a}(\tau) - \bar{b}(\tau)z_2 + b(\tau)\bar{z}_1 + a(\tau)\bar{z}_1 z_2]^{2S}}{(1 + |z_1|^2)^S(1 + |z_2|^2)^S}, \tag{15}$$

where functions

$$a(t) = u(t)\bar{u}(0) + v(t)\bar{v}(0), \; b(t) = v(t)u(0) - u(t)v(0)$$

are the solutions to the first-oder differential equations

$$\begin{aligned} \dot{a} &= -iAa + if\bar{b} \\ \dot{b} &= -iAb - if\bar{a} \end{aligned} \tag{16}$$

with the boundary conditions $a(0) = 1, \quad b(0) = 0$. This coincides exactly with what has been obtained by Ellinas 1992 in the time-lattice approach, which justifies the change (14) in the continuum $SU(2)$ path integral. It is quite clear that in general case the G-motion of a phase space is to be used as an appropriate change of integration variables. Concluding this section, we would like to point out that the group theoretical approach is not only useful for coherent state path integrals but is also successfully used in Feynman's path integral (Junker 1989, Inomata 1992).

3 $SU(2)$ path integral for Heisenberg model

As an instructive example of the $SU(2)$ path integral's applications, we consider the quantum Heisenberg ferromagnet. The partition function for the quantum Heisenberg ferromagnet

$$Z = Tr\,\exp(-\beta H), \quad H = -\frac{1}{2}J_{ij}\vec{S}_i\vec{S}_j,$$

where the summation over repeated indices is assumed and J_{ij} is the positive definite symmetric matrix of the exchange interaction ($J_{ii} = 0$), can be written as (Liebler and Orland 1981)

$$\begin{aligned} Z &= \int \prod_i D\vec{\psi}_i(t) \exp\left(-\frac{1}{2}\int_0^\beta \vec{\psi}_i(t)J_{ij}^{-1}\vec{\psi}_j(t)dt\right) \\ &\quad \times Tr\left[T\exp\left(-\int_0^\beta \vec{\psi}_i(t)\vec{S}_i(t)dt\right)\right], \end{aligned} \tag{17}$$

In deriving (17), the well-known Hubbard-Stratonovich identity has been used.

The partition function of noninteracting spins in an external fluctuating field $\psi(t)$ can be written as the $SU(2)$ path integral"

$$\begin{aligned} Z_0(\vec{\psi}) &\equiv Tr\left[T\exp\left(-\int_0^\beta \vec{\psi}_i(t)\vec{S}_i(t)dt\right)\right] \\ &= \int_{\alpha_i(0)=\alpha_i(\beta)} \prod_i D\mu_S(\alpha_i)\exp\left(S\int_0^\beta \frac{\dot{\bar{\alpha}}_i\alpha_i - \bar{\alpha}_i\dot{\alpha}_i}{1+|\alpha_i|^2}dt\right. \\ &\quad \left. + S\int_0^\beta \psi_i^{(0)}\frac{1-|\alpha_i|^2}{1+|\alpha_i|^2}dt - S\int_0^\beta \frac{\psi_i\alpha_i}{1+|\alpha_i|^2}dt - S\int_0^\beta \frac{\bar{\psi}_i\bar{\alpha}_i}{1+|\alpha_i|^2}dt\right). \end{aligned} \tag{18}$$

where we have put $\psi = \psi_x + i\psi_y$, $\bar{\psi} = \psi_x - i\psi_y$ and $\psi^{(0)} = \psi_z$. Note that α stands here for the integration variable with the periodic boundary conditions, which results in no boundary term in the action. This should come as no surprise, for one should clearly discriminate between the cases of a transition amplitude and of a density matrix and its trace. Statistical mechanics is different from quantum dynamics.

Integral (18) can be evaluated by the change (14) with the result

$$Z_0 = \exp L(\vec{\psi}), \quad L(\vec{\psi}) = \sum_i \log\frac{\sinh(S+\frac{1}{2})\int_0^\beta(\psi_i^{(0)} + \bar{\psi}_i z_i)dt}{\sinh\frac{1}{2}\int_0^\beta(\psi_i^{(0)} + \bar{\psi}_i z_i)dt}. \tag{19}$$

where the function $z = u/v$ depends upon ψ via the Riccati equation

$$\dot{z} - \psi^{(0)}z - (\bar{\psi}/2)z^2 + \psi/2 = 0, \quad z(0) = z(\beta). \tag{20}$$

The partition function thus becomes

$$Z = \int D\vec{\psi}\exp(-S(\vec{\psi})) \tag{21}$$

$$S(\vec{\psi}) = -\frac{1}{2}\int \vec{\psi}_i(t)J_{ij}^{-1}\vec{\psi}_j(t)dt + L(\vec{\psi}) \tag{22}$$

Representation (21,22) is a starting point for the systematic mean field expansion.

Let the stationary mean field $\vec{\Phi}$ be chosen in the z direction: $\vec{\Phi}_i = (0, 0, \Phi_i)$. The saddle-point equation

$$\frac{\delta S}{\delta\vec{\psi}_j(t)}\,|_0 = 0$$

(the subscript $|_0$ denotes a quantity at the stationary point) reads

$$\Phi = J_0\, b(\beta\Phi), \tag{23}$$

where $b(x) = SB_S(Sx)$ and $B_S(x)$ is the Brillouin function. To derive (23), we have put $J_0 = \sum_j J_{ij}$ and $\Phi_i = \Phi$ assuming translation invariance for the system. Expanding $S(\vec{\psi})$ around Φ up to the second order, one arrives at ($\vec{\eta} = \vec{\psi} - \vec{\psi}\,|_0$)

$$\begin{aligned} Z = \; & \exp\{-S(\Phi)\}\int D\vec{\eta}\exp\left\{-\frac{1}{2}\int_0^\beta \eta_i^{(0)}(t)(J^{-1}_{eff;ln})_{ij}(t,s)\eta_j^{(0)}(s)dtds\right. \\ & \left. -\frac{1}{2}\int_0^\beta \bar{\eta}_i(t)(J^{-1}_{eff;tr})_{ij}(t,s)\eta_j(s)dtds\right\}. \end{aligned} \tag{24}$$

the effective inverse longitudinal and transverse interactions being given by

$$(J^{-1}_{eff;ln})_{ij}(t,s) = J^{-1}_{ij}\delta(t-s) - \frac{\delta^2 L}{\delta\psi_i^{(0)}(t)\delta\psi_j^{(0)}(s)}\Big|_0 = J^{-1}_{ij}\delta(t-s) - b'(\beta\Phi)\delta_{ij}, \quad (25)$$

$$(J^{-1}_{eff;tr})_{ij}(t,s) = J^{-1}_{ij}\delta(t-s) - 2\frac{\delta^2 L}{\delta\psi_i(t)\delta\psi_j(s)}\Big|_0 = J^{-1}_{ij}\delta(t-s) - 2b(\beta\Phi)\frac{\delta z_i(t)}{\delta\psi_j(s)}\Big|_0 . \quad (26)$$

By taking the functional derivative of eq. (20) one finds

$$(\frac{d}{dt} - \Phi)\frac{\delta z_i(t)}{\delta\psi_j(s)}\Big|_0 = -\frac{1}{2}\delta_{ij}\delta(t-s), \quad \frac{\delta z_i(t=0)}{\delta\psi_j(s)}\Big|_0 = \frac{\delta z_i(t=\beta)}{\delta\psi_j(s)}\Big|_0,$$

which results in

$$2b(\beta\Phi)\frac{\delta z_i(t)}{\delta\psi_j(s)}\Big|_0 = \frac{1}{2}\langle TS_i^-(t)S_j^+(s)\rangle_0$$
$$= \delta_{ij} b \exp(\Phi(t-s))\{n_\Phi\theta(t-s) + (1+n_\Phi)\theta(s-t)\} \equiv G_{ij}^{(0)}(t-s|\Phi), \quad (27)$$

the temperature Green function for noninteracting spins with the Hamiltonian $H = H_0 = \Phi\sum S_i^{(z)}$; $n_\Phi = (e^{\beta\Phi}-1)^{-1}$.

Turning back to eq.(26), one finds

$$(J^{-1}_{eff;tr})_{ij}(t-s) = J^{-1}_{ij}\delta(t-s) - G_{ij}^{(0)}(t-s|\Phi). \quad (28)$$

In the energy-momentum representation this reads

$$J_{eff}(\omega_n,\vec{q}) = \frac{J(\vec{q})}{1 - J(\vec{q})G^{(0)}(\omega_n)}, \quad (29)$$

In view of eq.(28), the Dyson equation for the whole propagator

$$G = G_0 + G_0 J_{eff} G_0 \quad (30)$$

gives ($G_0(\omega_n) = b/(i\omega_n + \Phi)$; $\omega_n = 2\pi n/\beta$)

$$G(\omega_n,\vec{q}) = \frac{b(\beta\Phi)}{\epsilon_{\vec{q}} + i\omega_n}, \quad (31)$$

where

$$\epsilon_{\vec{q}} = b(\beta\Phi)(J(0) - J(\vec{q}))$$

is the temperature-dependent energy of the spin wave excitation. Path integral (24) is easily calculated to yield a contribution to the partition function coming from the Gaussian fluctuations around the mean field. The result coincides exactly with that of Liebler and Orland 1981. What is important is that our approach does not invoke any Fermi (Bose) oscillator-like representations for the spin operators which would perturb the original problem and complicate

calculations (Manousakis 1991). A further point is that this formalism provides convenient expansions for the Green functions around the mean field in powers of the effective interactions $J_{ln;eff}$ and $J_{tr;eff}$ (Kochetov 1994b).

As has been shown in this section, the $SU(2)$ path integral provides us with a new method for studying the thermodynamics of the quantum Heisenberg model. The very same technique for the path integral over coherent states associated with more complicated groups could be easily developed. The $U(2|1)$ supergroup relevant for the Hubbard $(t-J)$ model provides quite a nontrivial example of this kind (Wiegmann 1988).

4 Stationary phase expansion

This section constitutes the main result of the present paper, the derivation of the $SU(2)$ propagator in the quasiclassical region $S \gg 1$ in a closed form. Our final result is expressed in terms of classical trajectories and provides a convenient formula for the quasiclassical propagator of spin systems. It is in a full accordance with the path integral generalization of the Diustermaat-Heckman theorem which states that WKB is exact when dynamics leaves a metric tensor invariant (Blau et al 1990).

The semiclassical motion of quantum system is described by the approximation

$$\Phi \cong \Phi\,|_c + \frac{1}{2}\delta^2\Phi\,|_c, \quad \delta\Phi\,|_c = 0 \quad (32)$$

with the boundary conditions $z(0) = z_2$ and $\bar{z}(\tau) = \bar{z}_1$. The subscript "c" denotes quantity along the extremals

$$2S\dot{z} = -i(1+|z|^2)^2\partial_{\bar{z}}H, \quad z(0) = z_2,$$
$$2S\dot{\bar{z}} = i(1+|z|^2)^2\partial_z H, \quad \bar{z}(\tau) = \bar{z}_1. \quad (33)$$

We introduce variations

$$\eta = z - z_0, \quad \bar{\eta} = \bar{z} - \bar{z}_0,$$

which satisfy

$$\eta(0) = 0, \quad \bar{\eta}(\tau) = 0.$$

The phase function Φ is then expanded up to second order in η and $\bar{\eta}$, which results in

$$G_S(\bar{z}_1, z_2; \tau) \simeq G_{red}\exp\Phi_c, \quad (34)$$

The reduced propagator is given by ($\hbar = 1$)

$$G_{red} = \int D\eta D\bar{\eta}\exp\left[\frac{1}{2}\delta^2\Phi_c(\bar{\eta},\eta)\right]$$
$$\delta^2\Phi_c = \int_0^\tau(\dot{\bar{\eta}}\eta - \dot{\eta}\bar{\eta})ds - i\int_0^\tau(\eta^2 A + \bar{\eta}^2 C + 2\bar{\eta}\eta B)ds, \quad (35)$$

where

$$A = \partial_z\left[\frac{(1+|z|^2)^2}{2S}\partial_z H\right]_c, \quad C = \partial_{\bar{z}}\left[\frac{(1+|z|^2)^2}{2S}\partial_{\bar{z}} H\right]_c$$

and

$$B = \frac{1}{2}\left[\partial_{\bar{z}}\frac{(1+|z|^2)^2}{2S}\partial_z H + z \leftrightarrow \bar{z}\right]_c .$$

In view of eq.(46) we arrive at

$$Det\,\frac{\tilde{K}}{K_0} = \frac{1+|z_c(0)|^2}{1+|z_c(\tau|^2}\exp(-i\int_0^\tau Bdt)\left(\frac{\partial z_c(0)}{\partial\bar{z}_1}\right)^{-1} = \frac{1+|z_c(\tau)|^2}{1+|z_c(0)|^2}\exp(-i\int_0^\tau Bdt)\left(\frac{\partial \bar{z}_c(\tau)}{\partial z_2}\right)^{-1}. \quad (50)$$

To end up the derivation, let us compute

$$\frac{\partial^2\Phi_c}{\partial\bar{z}_1\partial z_2}.$$

It is worth noting that we take the derivative of the total classical action including boundary term. It is the total action Φ_c that enters the final answer. The calculation is straightforward and yields

$$\frac{\partial^2\Phi_c}{\partial\bar{z}_1\partial z_2} = \frac{1}{2}\left[\frac{2S}{(1+|z_c(\tau)|^2)^2}\frac{\partial \bar{z}_c(\tau)}{\partial z_2} + \frac{2S}{(1+|z_c(0)|^2)^2}\frac{\partial z_c(0)}{\partial\bar{z}_1}\right] \quad (51)$$

Combining eqs.(50) and (51) one finally ends up with

$$\left(Det\,\frac{\tilde{K}}{K_0}\right)^{-1} = \frac{(1+|z_c(\tau)|^2)(1+|z_c(0)|^2)}{2S}\frac{\partial^2\Phi_c}{\partial\bar{z}_1\partial z_2}\exp(i\int_0^\tau Bdt), \quad (52)$$

which yields for the $SU(2)$ semiclassical propagator

$$G_S(\bar{z}_1,z_2;\tau) = \exp(\Phi_c + \frac{i}{2}\int_0^\tau Bdt)\left[\frac{(1+|z_c(\tau)|^2)(1+|z_c(0)|^2)}{2S}\frac{\partial^2\Phi_c}{\partial\bar{z}_1\partial z_2}\right]^{1/2} \quad (53)$$

Thus we have expressed the $SU(2)$ semiclassical propagator in terms of the total classical action and classical orbitals, the factor

$$\frac{(1+|z_c(\tau)|^2)(1+|z_c(0)|^2)}{2S}$$

revealing nothing but the curved nature of the phase space. Note that the B-term dependence is purely of a kinematic nature and simply plays the role of a normalization. This term with the help of the Euler-Lagrange equations can also be expressed through the extremals:

$$B = \frac{i}{2}(\partial_z\dot{z} - \partial_{\bar{z}}\dot{\bar{z}})\,|_c\,.$$

Here we remark that Kuratsuji and Mizobuchi 1981 have made an earlier attempt to derive the WKB formula for the $SU(2)$ propagator in the $SU(2)$ path integral technique. Unfortunately, they have not taken account of the boundary term, whereas it is the total action Φ that enters the final result (53). This as well as using of an overspecified set of the Euler-Lagrange equations have made them impossible derive a correct final formula.

As a simple test example consider

$$H = \omega S_z.$$

As is easily seen $B=\omega$ and the solutions to eqs.(33) read

$$z_c(s) = z_2\exp(-i\omega s) \quad \bar{z}_c(s) = \bar{z}_1\exp(-i\omega(\tau-s)), \quad (54)$$

which in turn results in

$$\Phi_c = iS\omega\tau + 2S\log(1+\bar{z}_1z_2e^{-i\omega\tau}) - S\log(1+|z_1|^2) - S\log(1+|z_2|^2)$$

$$\frac{\partial^2\Phi_c}{\partial\bar{z}_1\partial z_2} = \frac{2Se^{-i\omega\tau}}{(1+\bar{z}_1z_2e^{-i\omega\tau})^2}.$$

Formula (53) yields

$$G_S(\bar{z}_1z_2;\tau) = \exp\Phi_c. \quad (55)$$

According to the path integral generalization of the Diustermaat-Heckman theorem the WKB expansion is exact, provided a metric of the underlying phase space is invariant under classical dynamics (Blau 1990). This is just the case for $H=\omega S_z$. The same conclusion follows directly from the time-lattice calculations (Funahashi et al 1994). All this means that (55) is exact, which is nothing but the consequence of the $SU(2)$ dynamical invariance of H. Moreover, the dynamical invariance, i. e., the fact that H belongs to the $SU(2)$ algebra, results in

$$G_{red} = 1,$$

which is of importance in deriving the generalized Bohr-Sommerfeld quantization condition.

In the end of this section, let us briefly comment on the semiclassical quantization condition for the $SU(2)$ path integral. In the case of stationary bound states, quantum system evolves along the closed path which lies on the constant energy surface $H(\bar{z},z)=E$. Single-valuedness of the semiclassical wave function implies (Keller 1958, Kuratsuji and Mizobuchi 1981)

$$-iS\int_0^\tau\frac{\dot{\bar{z}}z-\bar{z}\dot{z}}{1+|z|^2}ds = (2n+\nu/2)\pi, \quad (56)$$

where z and $\bar{z}$ are solutions to eqs.(33) and $n=0,1,...2S$. The integration in (56) is carried out over the period of motion τ and the index ν stands for the number of singularities of the semiclassical propagator G_{red} along the classical orbit.

In the case

$$H = \omega S_z$$

we have $\tau = 2\pi/\omega$ and $z(s) = ze^{-i\omega s}, \bar{z}(s) = \overline{z(s)}$. Equation (56) takes the form

$$S_z = S - n,$$

which means that the energy of the system is given by

$$E \equiv \omega S_z = \omega m, \quad m = -S, -S+1, ..., S-1, S.$$

Note that Kuratsuji and Mizobuchi 1981 have obtained the above expression with the value of m being shifted by 1/2. The difference originates from the fact that $G_{red}=1$ implies that $\nu=0$, whereas Kuratsuji and Mizobuchi derived that $\nu=2$. Note also that their result is not WKB exact, which contradicts the Diustermaat-Heckman theorem.

5 Conclusion

We have presented the $SU(2)$ coherent state path integral representation for a transition amplitude which involves boundary term and correct boundary conditions. This enables us to develop self-consistent calculational schemes both in getting the exact form of a propagator for particular Hamiltonians as well as in deriving semiclassical approximations to the propagator for a general problem. By comparing our calculations with those in the time-lattice approach we have argued that the $SU(2)$ motion provides a change of variables that compute path integral in the case of the $SU(2)$ dynamical symmetry. Being formally equivalent to the Wie-Norman disentangling procedure, this method, however, turns out to be much more effective and convenient when applied to spin systems in a fluctuating external field. This has been demonstrated by the example of the Heisenberg ferromagnet.

We have also succeeded in deriving a closed formula for a quasiclassical $SU(2)$ propagator starting from the $SU(2)$ coherent state path integral. The result is expressed in terms of classical trajectories and a total action only. It agrees with the Distermaat-Heckman theorem and recovers the spectrum of the WKB exact models.

6 Acknowledgment

The author thanks the University of Erlangen-Nürnberg for the hospitality during his visit to Germany where this paper was completed. He is grateful to Prof. Hajo Leschke and Dr. Georg Junker for discussions which are helpful for this work.

References

[1] Arecchi F T Courtens E Gilmore R and Thomas H 1972 Phys. Rev. **6A** 2211

[2] Berezin F A 1971 Theor. Math. Phys. **6** 194

[3] Berezin F A 1975 Commun. Math. Phys. **40** 153

[4] Berezin F A 1980 Usp. Fiz. Nauk **132** 497 (in Russian)

[5] Blaizot J P and Orland H 1981 Phys. Rev. **24C** 1740

[6] Blau M Keski-Vakkuri E and Niemi A J 1990 Phys. Lett. **246B** 92

[7] Buslaev V S 1980 "Variational Calculus" (Leningrad 1980) (in Russian)

[8] Ellinas D 1992 Phys. Rev. **45A** 1822

[9] Enz M and Schilling R 1986 J. Phys. **19C** 1765

[10] Faddeev L D 1975 in "Les Houches 1975" Proceedings of the Methods in Field Theory edited by R Balian and J Zinn-Justin (North-Holland Amsterdam 1975)

[11] Fischer W Leschke H and Müller P 1992 J. Phys. **A25** 3835

[12] Funahashi K Kashiwa T Sakoda S and Fujii K 1994 "Coherent States, Path Integral, and Semiclassical Approximation" Preprint HEP-TH/9409166

[13] Gilmore R 1972 Ann. Phys. (NY) **74** 391

[14] Inomata A "Path Integrals for $SU(2)$ and $SU(1,1)$" in Inomata A Kuratsuji H and Gerry C C 1992 "Path Integrals and Coherent States of $SU(2)$ and $SU(1,1)$" (World Scientific Singapore 1992)

[15] Junker G 1989 J Phys **A22** L587

[16] Keller J B 1958 Ann. Phys. **4** 180

[17] Klauder J R 1978 in "Path Integrals", Proceedings of the NATO Advanced Summer Institute, edited by G Papadopoulos and J Devreese (Plenum, New York, 1978)

[18] Klauder J R 1979 Phys. Rev. **19D** 2349

[19] Klauder J P and Skagerstam Bo-S 1985 "Coherent States" (World Scientific Singapore 1985)

[20] Kleinert H 1990 "Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics" (World Scientific Singapore 1990)

[21] Kochetov E A 1994a to appear in J. Math. Phys.

[22] Kochetov E A 1994b to be published

[23] Kuratsuji H and Suzuki T 1980 J. Math. Phys. **21** 472

[24] Kuratsuji H and Mizobuchi Y 1981 J. Math. Phys. **22** 757

[25] Kuratsuji H 1992 "Path Integrals in the $SU(2)$ Coherent State Representation and Related Topics" in Inomata A Kuratsuji H and Gerry C C 1992 "Path Integrals and Coherent States of $SU(2)$ and $SU(1,1)$" (World Scientific Singapore 1992)

[26] Langouche F Roekaerts D and Tirapegui E 1982 "Functional Integration and Semiclassical Expansion" (Reidel Publishing Company 1982)

[27] Leibler S and Orland H 1981 Ann. Phys. (NY) **132** 277

[28] Levit S and Smilansky U 1977 Ann. Phys. (NY) **108** 165

[29] Levitan B M and Sargsian I S 1970 "Introduction to the Spectral Theory" (Moscow 1970)

[30] Manousakis E 1991 Rev. Mod. Phys. **63** 1

[31] Negele J W and Orland H 1988 "Quantum Many-Particle Systems" (Addison-Wesley Publishing Company 1988)

[32] Perelomov A M 1972 Comm. math. Phys. **26** 222

[33] Perelomov A M 1985 "Generalizes Coherent States" (Springer-Verlag Berlin 1985)

[34] Radcliffe J M 1971 J. Phys **4A** 313

[35] Schulman L 1982 "Techniques in Path Integration" (Jhon Wiley 1982)

[36] Onofri M J 1975 J. Math. Phys. **16** 1087

[37] Wei J and Norman E 1963 J. Math. Phys. **4** 575

[38] Wiegmann P B 1988 Phys. Rev. Lett. **60** 821

Received by Publishing Department
on November 25, 1994.