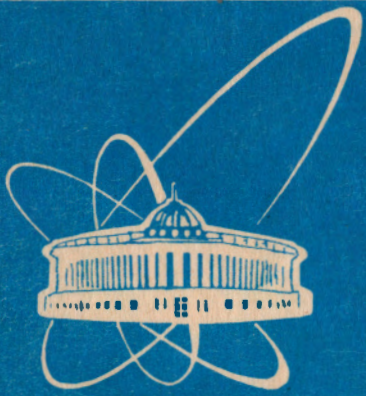

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NON-LINEAR SCHRÖDINGER EQUATION ON A RING

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Нелинейное уравнение Шредингера на кольце

Нелинейное уравнение Шредингера, описывающее разнообразные физические явления, решено для системы, помещенной на тонком кольце. Найдены точные волновые функции и энергетические уровни для основного и возбужденного состояний. Обнаружены критические значения длины окружности кольца, когда меняется структура основного состояния и появляются более высокие возбуждения. Изучены также состояния с конечным угловым моментом, которым соответствуют комплексные волновые функции.

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Non-Linear Schrödinger Equation on a Ring

The non-linear Shrödinger equation which is relevant for different physical phenomena is solved on a thin ring. We obtained the exact real wave functions with their corresponding energies for the ground state and the excited states. Critical values of the ring circumference are found at which the ground state changes its structure and additional higher excited states appear. Also the complex wave functions are studied which correspond to energy levels with finite angular momentum.

The investigation has been performed at the University of Antwerp (UIA), Belgium and Bogoliubov Laboratory of Theoretical Physics, JINR.

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I. INTRODUCTION

Mesoscopic systems are now available in which interference effects become important. Between them one-dimensional ring systems attract considerable attention because of recent experiments and interesting theoretical considerations dealing with systems of nontrivial and rather simple topology. Some new effects have been discovered. Say, energy levels of closed systems are found to be discrete[1] and one obtains energy bands for a one-dimensional metal ring enclosing a magnetic flux (when observing the energy as a function of the flux). The idea of persistent currents that are flux periodic in real normal-metal rings was proposed in Ref. 2 and Ref. 3 for one-dimensional systems. In optics rings lead to many applications too, such as the laser gyroscope[4], bistability effects in ring lasers[5], optical band structures in passive[6] and rotating[7] ring resonators. More information can be found in the book[8] by van Haeringen and Lenstra.

Usually people deal with systems on a ring in the scope of linear physics[9]. On the other hand there exist non-linear phenomena described by the so called non-linear Schrödinger equation in which the potential depends on the wave function. This equation has been used to describe stationary 2D self-focusing of a plane wave, self-trapping phenomena in non-linear optics, propagation of heat pulses in solids, Langmuir waves in plasmas and is related to the Ginzburg-Landau theory of superconductivity (see Ref. 10 and references quoted herein). We approached the same equation from the side of polarons confined in a potential box and considered in the strong-coupling limit[11].

The nonlinear Schrödinger equation can also be solved with the periodical boundary conditions corresponding to an infinitesimal thin ring system in the absence of a magnetic flux but including strong electron-phonon coupling. In the present paper we study how the non-linear phenomena are correlated with the geometry of the system. We consider the non-linear Schrödinger equation on an infinite thin ring with finite radius.

The paper is organized as follows. In Sec. II two sets of real solutions to this equation are derived. The complex solutions are given in Sec. III and the conclusions in Sec. IV.

II. REAL SOLUTIONS

A. Basic Equations and Constant Solution

The Hamiltonian of a polaron confined to move in a one-dimensional structure is of the form

$$H = \frac{p^2}{2m} + \sum_k \hbar\omega_k\, a_k^\dagger a_k + \sum_k \left[V_k a_k e^{ikz} + V_k^* a_k^\dagger e^{-ikz}\right], \tag{2.1}$$

where z and p are the position and momentum operators of the electron, m is the electron band mass and ω_k is the frequency of the phonons with wave vector k. We include no external potential in the Hamiltonian (2.1), so we consider a medium without impurities.

As is well known, a system in the strong coupling limit is described by the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(z)}{\partial z^2} + V_{eff}(z)\phi(z) = E\,\phi(z) \tag{2.2}$$

with the effective potential

$$V_{eff}(z) = \sum_k |V_k|^2 \frac{|\rho_k|^2}{\hbar\omega_k} - \sum_k \frac{|V_k|^2}{\hbar\omega_k}\left[\rho_k^* e^{ikz} + \rho_k e^{-ikz}\right], \tag{2.3}$$

where $\rho_k = \int dz\, |\phi(z)|^2 e^{ikz}$. In the case of an electron interacting with LO-phonons we have $\omega_k = \omega_{LO}$ and

$$V_k = -i\hbar\omega_{LO}\left(\frac{2\alpha'}{L}\sqrt{\frac{\hbar}{2m\omega_{LO}}}\right)^{1/2}, \tag{2.4}$$

where α' is the dimensionless electron-phonon coupling constant[12,13] and L is the circumference of the ring. Performing the scaling $z \to \lambda z$ with $\lambda = \sqrt{\hbar/m\omega_{LO}}/\alpha'$ we arrive at the Schrödinger equation in dimensionless units:

$$-\frac{1}{2}\frac{\partial^2\phi(z)}{\partial z^2} + V_{eff}(z)\phi(z) = \varepsilon\,\phi(z), \tag{2.5}$$

with $\varepsilon = E/\hbar\omega_{LO}\alpha'^2$ and

$$V_{eff}(z) = \sqrt{2}\int_0^a dz\, |\phi(z)|^4 - 2\sqrt{2}\,|\phi(z)|^2, \tag{2.6}$$

where the circumference of the ring can be written as $L = (a/\alpha')\sqrt{\hbar/m\omega_{LO}}$ with a the dimensionless circumference.

When we suppose the wave functions are real, Eq. (2.5) is of the form

$$-\frac{1}{2}\phi'' + \sqrt{2}B\,\phi - 2\sqrt{2}\,\phi^3 - \varepsilon\,\phi = 0, \tag{2.7}$$

where

$$B = \int_0^a dz\, \phi^4(z). \tag{2.8}$$

The description of a system on an infinitesimal thin ring implies that we have to invoke periodical boundary conditions

$$\phi(0) = \phi(a), \quad \phi'(0) = \phi'(a). \tag{2.9}$$

First of all there exists the solution with a constant wave function $\phi(z) = 1/\sqrt{a}$ which is normalized on a ring. From Eq. (2.7) we obtain the corresponding energy level

$$\varepsilon_0^{(0)} = -\frac{\sqrt{2}}{a}, \quad \text{or} \quad E_0^{(0)} = -\hbar\omega_{LO}\frac{\sqrt{2}\alpha'}{L}\sqrt{\frac{\hbar}{m\omega_{LO}}}, \tag{2.10}$$

which is a Coulomb type potential energy in a well generated by the electron in a polar medium.

To obtain other solutions note that the first integral of Eq. (2.7) can be obtained readily (which corresponds to the energy conservation)

$$-\frac{1}{4}\phi'^2 + \frac{1}{\sqrt{2}}\phi^2\left(A - \phi^2\right) = C, \tag{2.11}$$

where we used the notation

$$A = B - \frac{\varepsilon}{\sqrt{2}}. \tag{2.12}$$

Then we proceed by introducing two new parameters Φ_1 and Φ_2

$$\sqrt{2}\,C = \Phi_1\,\Phi_2, \qquad A = \Phi_1 + \Phi_2, \tag{2.13}$$

which, because of symmetry, can be taken $\Phi_2 \geq \Phi_1$. Eq. (2.11) can now be written as

$$\phi'^2(z) = 2\sqrt{2}\left(\phi^2(z) - \Phi_1\right)\left(\Phi_2 - \phi^2(z)\right). \tag{2.14}$$

B. First Set of Solutions

First we consider $\Phi_1 \geq 0$ and from Eq. (2.14) we immediately find that $0 \leq \Phi_1 \leq \phi^2(z) \leq \Phi_2$. The integration of Eq. (2.14) gives us the general solution[11]

$$\phi_1(z) = \sqrt{\Phi_2}\,\mathrm{dn}(2^{3/4}\sqrt{\Phi_2}(z - z_0)\,|m_1), \tag{2.15}$$

with $m_1 = 1 - \Phi_1/\Phi_2$, where $0 \leq m_1 \leq 1$. Here z_0 is an arbitrary constant and the parameter m_1 is the modulus of the Jacobian elliptic function, which is periodical in z with period $2^{1/4}\mathbf{K}(m_1)/\sqrt{\Phi_2}$, where $\mathbf{K}(m_1)$ is the complete elliptic integral of the first kind $\mathbf{K}(m) = \int_0^{\pi/2} d\theta/\sqrt{1 - m_1 \sin^2\theta}$. To satisfy the periodical boundary conditions (2.9) the circumference of the ring should be equal to an integer number n of waves, where from we obtain

$$a\sqrt{\Phi_2} = n2^{1/4}\mathbf{K}(m_1), \quad n = 1, 2, \ldots. \tag{2.16}$$

We also require that the wave functions are normalized $\int_0^a dz\phi^2_{1(2)}(z) = 1$. Using this normalization condition and Eq. (2.16) we arrive at the relation

$$\sqrt{2}\mathbf{K}(m_1)\mathbf{E}(m_1) = \frac{1}{x}, \tag{2.17}$$

where the complete elliptic integral of the second kind with modulus m is defined by $\mathbf{E}(m) = \int_0^{\pi/2} d\theta\sqrt{1 - m\sin^2\theta}$ and where we introduced the notation

$$x = \frac{n^2}{a}. \tag{2.18}$$

Eq. (2.17) determines the parameter m_1 as a function of the parameter x. In order to find the energy [Eq. (2.12)] $\varepsilon^{(1)}_{0,n} = \sqrt{2}(B - A)$ we still need

$$B = \int_0^a dz\,\phi_1^4(z) = \frac{2n^4\mathbf{K}^3(m_1)}{a^3}\int_0^{\mathbf{K}(m_1)} dz\,\mathrm{dn}^4(z|m_1)$$

$$= \frac{2n^4\mathbf{K}^3(m_1)}{3a^3}\left[2(2 - m_1)\mathbf{E}(m_1) - (1 - m_1)\mathbf{K}(m_1)\right], \tag{2.19}$$

while A takes the form

$$A = \Phi_2(1 - m_1) = \frac{\sqrt{2}n^2\mathbf{K}^2(m_1)}{a^2}(2 - m_1). \tag{2.20}$$

With Eqs. (2.19) and (2.20) we arrive at the energy levels

$$\varepsilon^{(1)}_{0,n} = \frac{1}{n^2}F_1(\frac{n^2}{a}), \tag{2.21}$$

where $F_1(x)$ is a universal function for all levels with any value of n

$$F_1(x) = -\frac{2}{3}(x\mathbf{K}(m_1))^2\left[2 - m_1 + \sqrt{2}x\mathbf{K}^2(m_1)(1 - m_1)\right]. \tag{2.22}$$

The dependence of the parameter m_1 and the function F_1 on x is given in Fig. 1. In the limiting case of a large ring $a \gg 1$ ($m_1 \to 1$), we obtain from Eq. (2.17)

$$m_1 = 1 - 16e^{-\sqrt{2}/x} - 128e^{-2\sqrt{2}/x}\frac{1 - x^2}{x^2} + \ldots \tag{2.23}$$

and

$$\varepsilon^{(1)}_{0,n} = -\frac{1}{3n^2}\left(1 + 24e^{-\sqrt{2}a/n^2} + O(e^{-2\sqrt{2}a/n^2})\right). \tag{2.24}$$

In opposite case when the radius of the ring decreases, the l.h.s. of Eq. (2.17) which increases monotonically, reaches its minimum at $m_1 = 0$. The parameter x reaches then its maximal value. It means that for each value of n there exists a critical (minimal) value of the ring circumference $a_c^{(n)} = n^2 a_c$, where

$$a_c = \sqrt{2}\left(\frac{\pi}{2}\right)^2 \tag{2.25}$$

is the critical value for the state with $n = 1$. At the critical value $a_c^{(n)}$ of the ring circumference the energy of the state becomes equal to $\varepsilon_0^{(0)}$ and the wave function takes the constant value $1/\sqrt{a}$. At smaller radii this state stops to exist. In the vicinity of the critical point the energy behaves as follows

$$\varepsilon^{(1)}_{0,n} \approx -\frac{1}{n^2}\left(\frac{2}{\pi}\right)^2\left[1 - \frac{\Delta a}{n^2 a_c} + O\left((\Delta a)^2\right)\right],$$

$$a \approx n^2 a_c + \Delta a. \tag{2.26}$$

It follows from Eq. (2.26) that not only the curves $\varepsilon_0^{(0)}(a)$ and $\varepsilon^{(1)}_{0,n}(a)$ coincide at the critical points but also their slopes. The energy levels $\varepsilon^{(1)}_{0,n}$ vs. the ring circumference a are plotted in Fig. 2. Those levels decrease when a becomes smaller. This we already could expect from the shape of the function $F_1(x)$ in Fig. 1(b) which decreases with increasing x.

C. Second Set of Solutions

We have to consider now the case when $\Phi_1 \leq 0$, and to rewrite Eq. (2.14) as follows

$$\phi'^2(z) = 2\sqrt{2}\left(\phi^2(z) + |\Phi_1|\right)\left(\Phi_2 - \phi^2(z)\right), \tag{2.27}$$

from which we obtain $\Phi_2 \geq 0$, $-\Phi_2 \leq \phi^2(z) \leq \Phi_2$. The general solution is of the form[11]

$$\phi_2(z) = \sqrt{\Phi_2}\,\mathrm{cn}(2^{3/4}\sqrt{|\Phi_1| + \Phi_2}\,(z - z_0)\,|m_2) \tag{2.28}$$

with $m_2 = \Phi_2/(|\Phi_1| + \Phi_2)$, where $0 < m_2 \leq 1$. This solution is periodical in z with period $2^{5/4}\mathbf{K}(m_2)/\sqrt{|\Phi_1| + \Phi_2}$. It follows then from the boundary conditions

$$a\sqrt{|\Phi_1| + \Phi_2} = 2n\,2^{1/4}\,\mathbf{K}(m_2), \quad n = 1, 2, \ldots . \tag{2.29}$$

These solutions coincide completely with the antisymmetrical solutions for the same system in an infinitely deep potential well of length a found in Ref. 11. The results are as follows. The equation for the modulus m_2 of the elliptic function (2.28) is of the form

$$\sqrt{2}\,\mathbf{K}^2(m_2)\left[\frac{\mathbf{E}(m_2)}{\mathbf{K}(m_2)} + m_2 - 1\right] = \frac{1}{x}. \tag{2.30}$$

Again the energies can be represented by a function universal for all levels

$$\varepsilon^{(2)}_{0,n} = \frac{1}{n^2} F_2\left(\frac{n^2}{a}\right) \tag{2.31}$$

with $F_2(x)$ being determined as

$$F_2(x) = \frac{\sqrt{2}}{3}\left(x\,\mathbf{K}(m_2)\right)^2 \times \left[2m_2(1 - m_2)\,x\,\mathbf{K}^2(m_2) - \sqrt{2}\,(2m_2 - 1)\right]. \tag{2.32}$$

This universal function $F_2(x)$ and the parameter m_2 as function of x are plotted in Fig. 3. In the limiting case of a large ring $a \gg 1$, or equivalently $x \to 0$ we obtain

$$\varepsilon^{(2)}_{0,n} = -\frac{1}{3(2n)^2}\left[1 - 24e^{-\sqrt{2}a/(2n)^2} + O(e^{-2\sqrt{2}a/n^2})\right]. \tag{2.33}$$

Thus, the energy levels tend to the same values as $\varepsilon^{(1)}_{0,2n}$ (see Fig. 2). Note that for an infinitely deep potential well with length $a \gg 1$ we found in Ref. 11 energy levels $\varepsilon_n = -1/(3n^2)$.

In the opposite case when $a \ll 1$ we have $x \to \infty$ and from Eq. (2.30)

$$\varepsilon^{(2)}_{0,n} = \frac{\pi^2 n^2}{2\,a^2} - \frac{3\sqrt{2}}{2a} - \frac{1}{2\pi^2 n^2} + O(a),$$

$$E^{(2)}_{0,n} = \frac{\pi^2 n^2}{2}\left(\frac{\hbar}{\sqrt{m}L}\right)^2 - \frac{3\sqrt{2}}{2}\frac{\hbar}{\sqrt{m}L}\sqrt{\hbar\omega_{\mathrm{LO}}\alpha'^2} - \frac{1}{2\pi^2 n^2}\hbar\omega_{\mathrm{LO}}\alpha'^2 + O(L). \tag{2.34}$$

The interpretation of this limit is obvious: reducing the circumference of the ring will destroy the polaron because the electron-phonon interaction will be neglectful (even in the strong-coupling limit) in comparison with the interaction due to the confinement into a small ring. The same is also true for higher excited states with large $n \gg 1$.

The energies $\varepsilon^{(2)}_{0,n}$ vs. a are plotted in Fig. 2 as well and the wave functions corresponding to both sets of solutions are given in Fig. 4. In the latter figure the corresponding energies are also indicated. We can see clearly that when the ring circumference a becomes infinitely large both wave functions (corresponding to $\varepsilon^{(1)}_{0,2}$ and $\varepsilon^{(2)}_{0,1}$) will tend to a wave function with infinitely separated peaks. Because both wave functions obtain the same peaks up to the sign and because the different peaks are independent from each other in this limit, the sign of the wave function at such peaks will not influence the energies. Therefore both wave functions will tend to the same energy value (as we found also analytically). For the other energy levels in Fig. 2 a similar behaviour can be found. The only difference will be the number of peaks in the wave function: this number of peaks will be n for the wave function corresponding to $\varepsilon^{(1)}_{0,n}$ and $2n$ for the wave function corresponding to $\varepsilon^{(2)}_{0,n}$.

Note that solutions to the *linear* Schrödinger equation are classified usually by the number of zeroes in the wave function, e.g. a solution ϕ_n has $n-1$ zeroes where $n = 1$ refers to the ground state, $n = 2$ to the first excited state, etc. At the same time the probability distribution $|\phi_n|^2$ of this wave function has n peaks.

From our calculations in this section we can deduce that in *non-linear* physics there exist wave functions without zeropoints [see Fig. 4(a)] or wave functions with large areas which are zero [see Fig. 4(d)]. So, here the classification of excited states can only be done by counting the number of peaks in the probability distribution. Therefore the latter method seems to be more general as the classification by counting the zeropoints.

III. COMPLEX SOLUTIONS

A. Basic Equations and Constant Solution

Let us search for a solution to Eq. (2.5) of the form

$$\psi(z) = \phi(z)e^{i\chi(z)} \tag{3.1}$$

with an amplitude $\phi(z)$ and a phase $\chi(z)$ being real functions of the coordinate z. Inserting Eq. (3.1) into Eq. (2.5) we arrive at the couple of equations

$$-\frac{1}{2}\phi''(z)+\left[\sqrt{2}B-2\sqrt{2}\phi^2(z)+\frac{1}{2}\chi'^2(z)\right]\phi(z)=0, \tag{3.2}$$

$$\chi''(z)\phi(z)+2\chi'(z)\phi'(z)=0. \tag{3.3}$$

The general solution to Eq. (3.3) can be represented as follows:

$$\chi(z)=\tilde{\chi}(0)+Q\int_0^z\frac{dz'}{\phi^2(z')}, \tag{3.4}$$

where Q is an arbitrary real constant. With this solution being inserted into Eq. (3.2) we arrive for the latter at the nonlinear effective Schrödinger equation for the amplitude $\phi(z)$

$$-\frac{1}{2}\phi''(z)+\sqrt{2}B\phi(z)-2\sqrt{2}\phi^3(z)+\frac{Q^2}{2\phi^3(z)}=0. \tag{3.5}$$

At $Q=0$ we arrive at the set of (real) solutions described in Sec. II. Note that for the external potentials considered in Ref. 11 this set of real solutions are the only existing solutions to Eq. (2.5). For instance, if both the potential and the amplitude $\phi(z)$ vanish at infinity then it follows from Eq. (3.5) that $Q=0$. The same is true if the wave function has zeros on a finite segment. Indeed, if $\phi(z)\approx c\,(z-z_0)^\beta$ at some point z_0 then $\phi''(z)\approx c\,\beta(\beta-1)(z-z_0)^{\beta-2}$. The term with the second derivative has to cancel with the last term of Eq. (3.5) where from we obtain

$$\frac{c\beta(\beta-1)}{(z-z_0)^{2-\beta}}=\frac{Q^2}{c^3(z-z_0)^{3\beta}}. \tag{3.6}$$

It follows then that $\beta=1/2$ and $Q^2=-c^4/4$ while Q and c have to be real parameters.

But one could obtain complex solutions ($Q\neq 0$) when dealing with another potential or another geometry of the system. On a thin ring complex solutions do exist and can be classified by an integer number — a value l_z of the angular momentum. Indeed, periodical boundary conditions imply for the phase $\chi(a)=\chi(0)+2\pi l_z$, $l_z=0,\pm1,\pm2,\ldots$, so that the parameter Q of Eq. (3.4) takes the form

$$Q=2\pi l_z\left(\int_0^a\frac{dz'}{\phi^2(z')}\right)^{-1}. \tag{3.7}$$

We start with the simplest of them when the amplitude takes a constant value $\phi(z)=1/\sqrt{a}$ on the ring. In this case $B=1/a$, $Q=2\pi l_z/a^2$ and for the phase we obtain the conventional form of a quantum-mechanical particle on a ring: $\chi(z)=\chi(0)+2\pi l_z z/a$. The corresponding energy levels are described by the formula

$$\tilde{\varepsilon}_{l_z}^{(0)}=-\frac{\sqrt{2}}{a}+\left(\frac{2\pi}{a}\right)^2\frac{l_z^2}{2},\ \text{or}$$

$$E_{l_z}^{(0)}=-\hbar\omega_{\rm LO}\frac{\sqrt{2}\alpha'}{L}\sqrt{\frac{\hbar}{m\omega_{\rm LO}}}+\frac{2(\pi\hbar l_z)^2}{mL^2}. \tag{3.8}$$

At $l_z=0$ we arrive at the energies (2.10). The meaning of the second term is quite obvious — it is the conventional centrifugal part of the kinetic energy. The energy levels $\tilde{\varepsilon}_{l_z}^{(0)}$ for $l_z=0,1,2,3$ are shown in Fig. 5. The curve for $l_z=0$ which corresponds to the constant wave function of the real solutions (see also Fig. 2) obtains a lower energy for decreasing a. For $l_z\geq 1$ however, we see clearly the increase of the energy for small a due to the centrifugal contribution which becomes more important the larger l_z is and the smaller a is. The corresponding wave functions are of the form $(1/\sqrt{a})e^{2\pi i l_z z/a}$ where we omitted the constant phase factor. Note that energy levels $\tilde{\varepsilon}_{l_z}^{(0)}$, $l_z\geq 1$ tend to their limiting values at large a from *below*: as it follows from Eq. (3.8) there are minima $\tilde{\varepsilon}_{l_z,\min}^{(0)}=-1/(2\pi l_z)^2$ reached at $a_{\min}=2\sqrt{2}\,(\pi l_z)^2$.

B. Other Complex Solutions

To proceed further we use the first integral of Eq. (3.5) which is as follows

$$-\frac{1}{4}\phi'^2(z)+\frac{\phi^2(z)}{\sqrt{2}}\left(A-\phi^2(z)\right)-\frac{Q^2}{4\phi^2(z)}=C, \tag{3.9}$$

where C is an integration constant. From Eq. (3.9) it follows that 1) the solution exists if Q^2 (l_z^2) does not exceed some maximal value, and 2) the function $\phi^2(z)$ varies between a minimal Φ_{min} and maximal Φ_{max} value and does not reach zero when $Q^2\neq 0$ so that $\phi(z)$ has a constant sign and no nodes. Introducing the notations $\Phi(z)=\phi^2(z)$ and

$$A=\Phi_{max}+\Phi_{min}-\Phi_0, \tag{3.10}$$

$$\sqrt{2}C=\Phi_{max}\Phi_{min}-\Phi_0(\Phi_{max}+\Phi_{min}), \tag{3.11}$$

$$Q^2=2\sqrt{2}\Phi_{max}\Phi_{min}\Phi_0, \tag{3.12}$$

where all parameters (Φ_0, Φ_{min}, Φ_{max}) are positive and $\Phi_{min}\leq\Phi_{max}$, we arrive at the equation

$$\Phi'^2(z)=8\sqrt{2}\,(\Phi(z)+\Phi_0)\,(\Phi(z)-\Phi_{min})\,(\Phi_{max}-\Phi(z)). \tag{3.13}$$

This equation can be integrated and results in

$$\phi^2(z) = \frac{\Phi_{min} + m\Phi_0 \mathrm{sn}^2(u|m)}{1 - m\mathrm{sn}^2(u|m)}, \tag{3.14}$$

where

$$u = 2^{3/4}\sqrt{\Phi_{max} + \Phi_0}(z - z_0),$$
$$m = \frac{\Phi_{max} - \Phi_{min}}{\Phi_{max} + \Phi_0}. \tag{3.15}$$

Equivalently, Eq. (3.14) can be written as follows

$$\phi^2(z) = \Phi_{min} + (\Phi_{max} - \Phi_{min})\ \mathrm{cn}^2(u|m), \tag{3.16}$$

where only the constant term z_0 should be changed in the definition (3.15) of u. Note that at $Q = 0$ we arrive either at the solution (2.15) for $\phi_1(z)$ with $\Phi_{max} = \Phi_2$, $\Phi_{min} = \Phi_1$ and $\Phi_0 = 0$ or at the solution (2.28) for $\phi_2(z)$ with $\Phi_{max} = \Phi_2$, $\Phi_{min} = 0$ and $\Phi_0 = |\Phi_1|$.

The periodicity in u on the ring is expressed by the condition that there should be an integer number (n) of waves on the circumference of the ring. Because the period in u equals $2\mathbf{K}(m)$ one has

$$2^{3/4}\sqrt{\Phi_{max} + \Phi_0}\, a = 2n\mathbf{K}(m). \tag{3.17}$$

From the periodicity condition (3.17), the normalization condition and the definition of m in Eq. (3.15) all the parameters (Φ_0, Φ_{min} and Φ_{max}) can be determined. Indeed, the normalization condition $\int_0^a dz\, \phi^2(z) = 1$ leads to

$$\Phi_{min} = \frac{1}{a}\left\{1 - \sqrt{2}x\mathbf{K}(m)\left[\mathbf{E}(m) - (1-m)\mathbf{K}(m)\right]\right\}, \tag{3.18}$$

the definition of m in Eq. (3.15) gives

$$\Phi_{max} = \frac{1}{a}\left\{1 + \sqrt{2}x\mathbf{K}(m)\left[\mathbf{K}(m) - \mathbf{E}(m)\right]\right\}, \tag{3.19}$$

and finally from the periodicity condition (3.17) we obtain

$$\Phi_0 = \frac{1}{a}\left\{-1 + \sqrt{2}x\mathbf{K}(m)\mathbf{E}(m)\right\}. \tag{3.20}$$

In this way, all those parameters are determined as a function of the parameter m.

The parameter m itself can be obtained combining Eq. (3.7) with Eq. (3.12) which leads to the equation

$$2\pi\sqrt{y} = 2^{3/4}\sqrt{(\Phi_0 a)(1-\gamma)}\frac{\mathbf{\Pi}(\gamma, m)}{\mathbf{K}(m)} \tag{3.21}$$

where

$$\mathbf{\Pi}(\gamma, m) = \int_0^{\pi/2} \frac{d\phi}{[1 - \gamma\sin^2\phi]\sqrt{1 - m\sin^2\phi}} \tag{3.22}$$

which is the complete elliptic function of the third kind,

$$y = \frac{l_z^2}{a} \tag{3.23}$$

and

$$\gamma = \frac{\sqrt{2}xm\mathbf{K}^2(m)}{1 + \sqrt{2}x\mathbf{K}(m)\left[\mathbf{K}(m) - \mathbf{E}(m)\right]}. \tag{3.24}$$

From the equation Eq. (3.21) one can determine for given quantum numbers n and l_z the corresponding m.

Also the energy $\varepsilon = \sqrt{2}(B - A)$ can be determined as a function of this m. The parameter A defined in Eq. (3.10) is given by

$$A = \frac{1}{a}\left\{3 - 3\sqrt{2}x\mathbf{K}(m)\mathbf{E}(m) + \sqrt{2}(2-m)x\mathbf{K}^2(m)\right\}, \tag{3.25}$$

and $B = \int_0^a dz\, \phi^4(z)$ has the form

$$B = \frac{1}{a}\left\{1 + \frac{2}{3}x^2\mathbf{K}^2(m)\left[-3\mathbf{E}^2(m) + 2(2-m)\mathbf{E}(m)\mathbf{K}(m) - (1-m)\mathbf{K}^2(m)\right]\right\}. \tag{3.26}$$

From Eqs. (3.25) and (3.26) the energy levels corresponding to the complex solutions can be calculated

$$\varepsilon_{l_z,n} = \frac{1}{n^2}F_{12}(x, y) \tag{3.27}$$

with

$$F_{12}(x,y) = \sqrt{2}x\left\{-2 + \sqrt{2}x\mathbf{K}(m)\left[3\mathbf{E}(m) - (2-m)\mathbf{K}(m)\right] + \frac{2}{3}x^2\mathbf{K}^2(m)\left[-3\mathbf{E}^2(m) + 2(2-m)\mathbf{E}(m)\mathbf{K}(m) - (1-m)\mathbf{K}^2(m)\right]\right\}. \tag{3.28}$$

where $m = m(y)$ is obtained by solving Eq. (3.21).

Numerical calculations indicated however that Eq. (3.21) has no solutions for $l_z \geq 1$. The only complex solutions are therefore of the form $(1/\sqrt{a})e^{2\pi i l_z z/a}$.

IV. CONCLUSIONS

In this paper the nonlinear effective Schrödinger equation was solved on a thin ring. Both, real and complex solutions are found. We obtained the exact real wave functions with their corresponding energies for the ground state and the excited states, and found critical values of the ring circumference at which the highest excited state, or the ground state if no excited states exist yet, changes its structure and an additional excited state appears. The complex wave functions correspond to energy levels with finite angular momentum l_z and are of the form $\Psi(z) = (1/\sqrt{a})e^{2\pi i l_z z/a}$.

With the above calculations we found that for linear and nonlinear problems (as above) we have the feature that the number of peaks in the wave function of an excited state can be used to obtain a classification or the quantum number of the excited state.

If we compare those solutions with the states used, e.g., by Büttiker et al.[14], we see that additional real solutions are found here which are not present in their linear theory. Therefore, the nonlinear effective Schrödinger equation should be used, e.g. by applying flux-modified boundary conditions (for more details about this method see Refs. 1, 15, 16).

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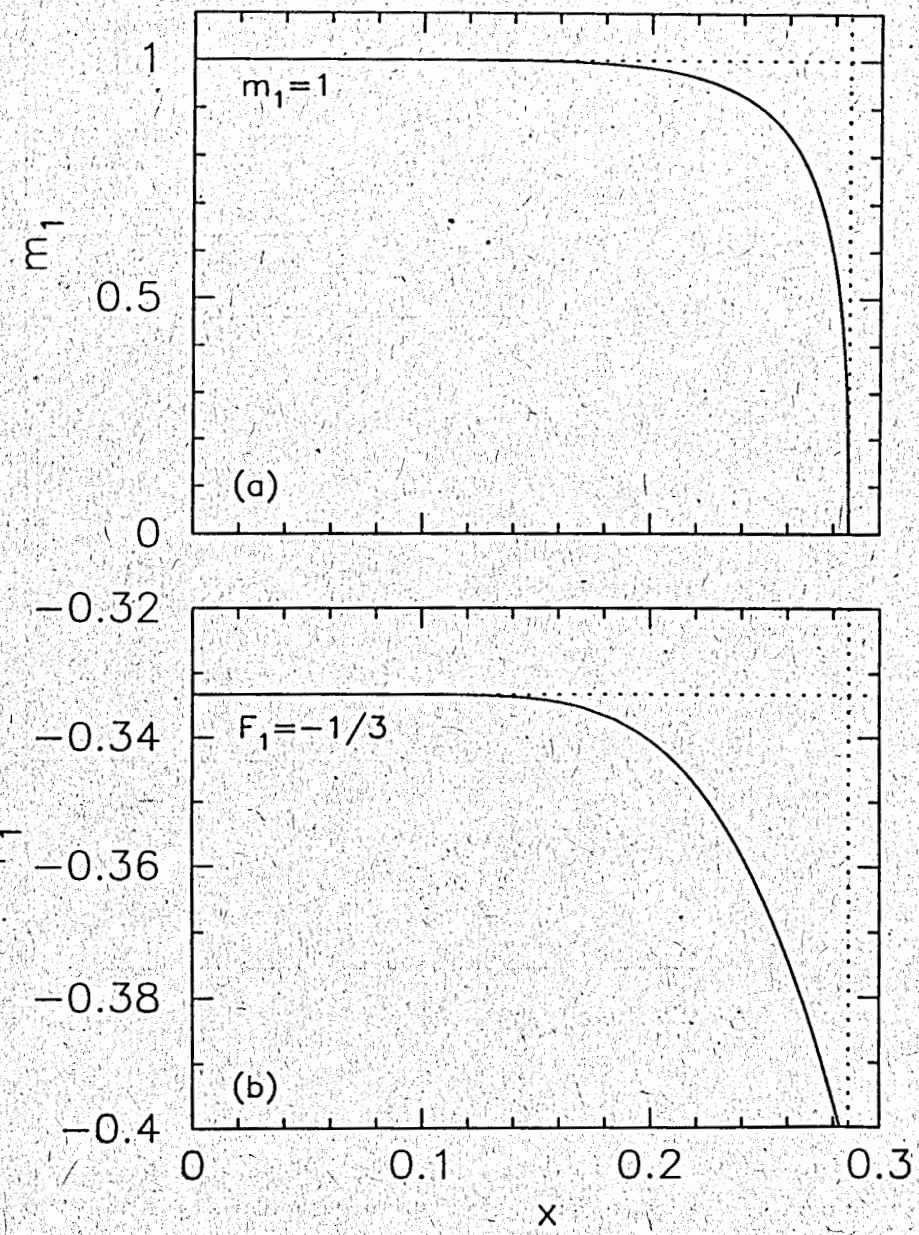

FIG. 1. The dependence of the parameter m_1 and the function F_1 on x.

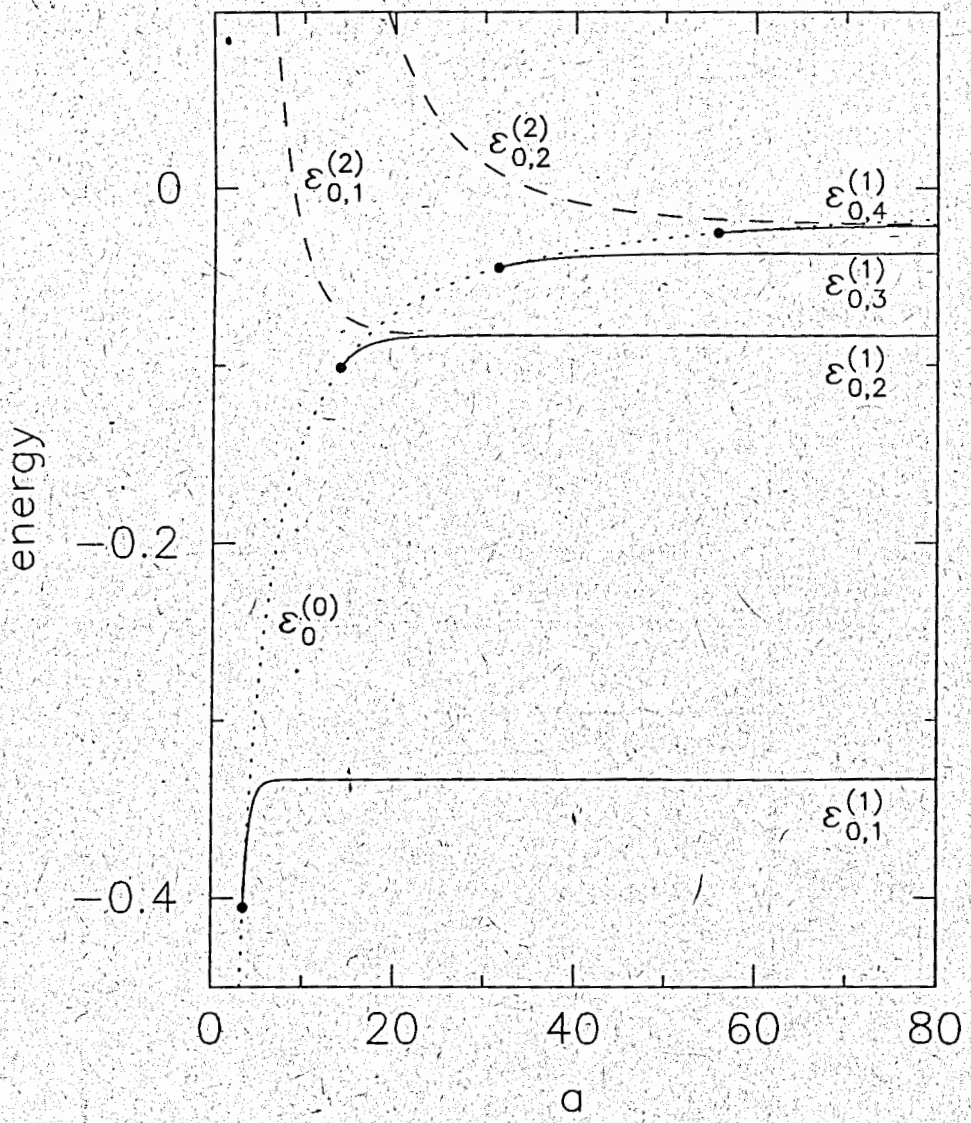

FIG. 2. Energy levels of a polaron on a ring vs. the scaled ring circumference for the states with real wave functions. The energy of the constant solution corresponds to the dotted line. The solid lines indicate the energy levels of the first set of solutions and the dashed lines the second set.

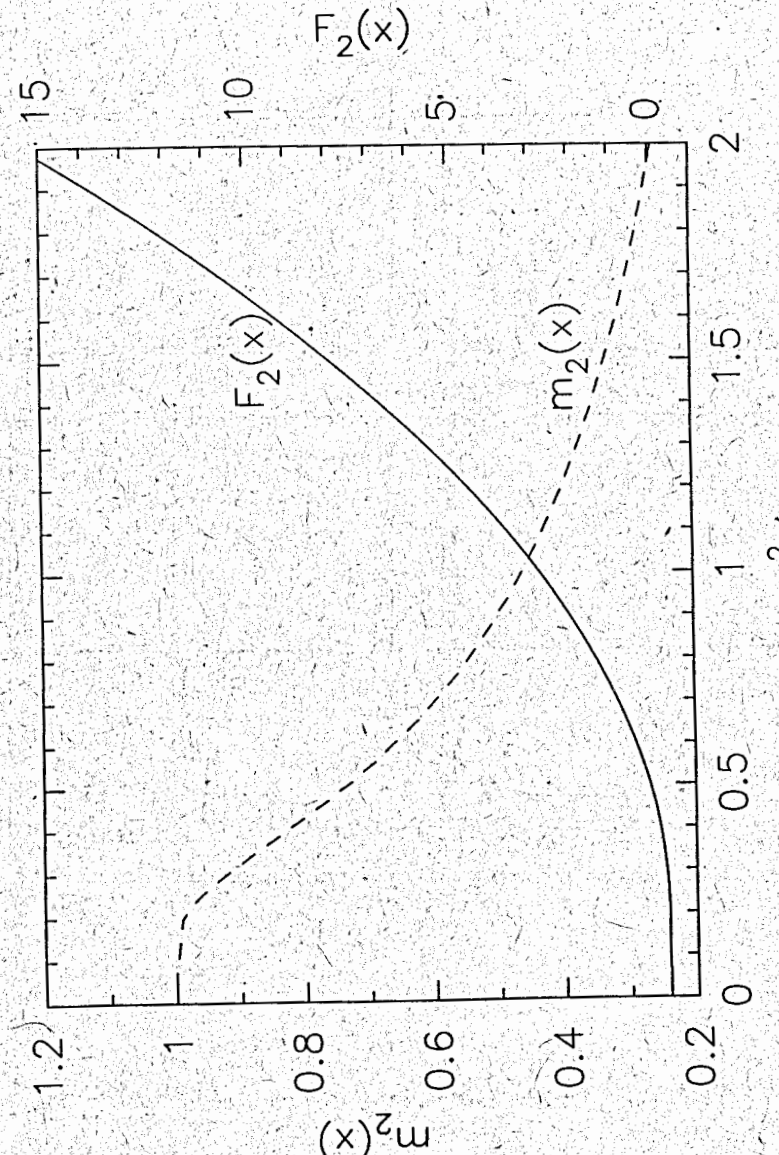

FIG. 3. The dependencies of the parameter m_2 and the function F_2 on x (taken from Ref. 11).

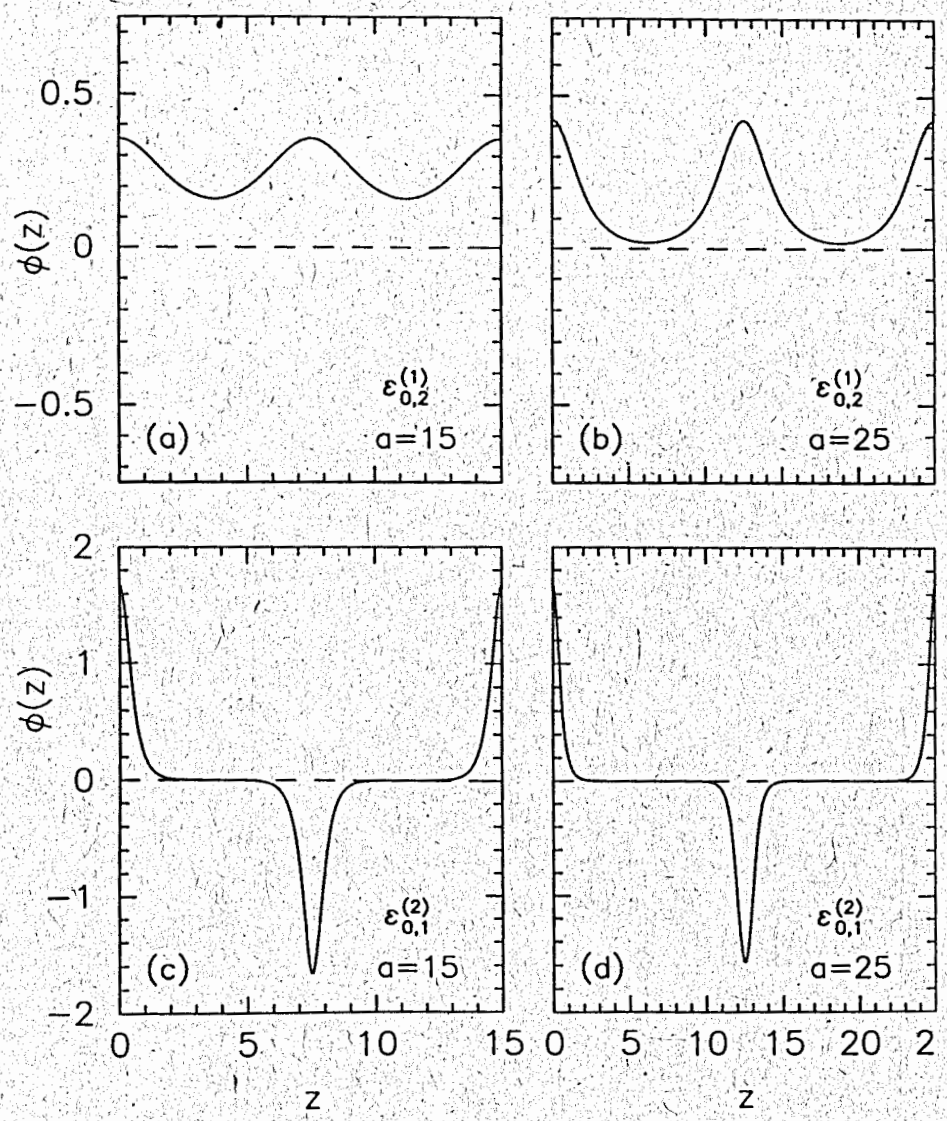

FIG. 4. The wave functions corresponding to the energies $\varepsilon^{(1)}_{0,2}$ and $\varepsilon^{(2)}_{0,1}$. For large circumferences a the wave functions will have peaks between which no tunneling is possible. The energy will tend then to the same limiting value.

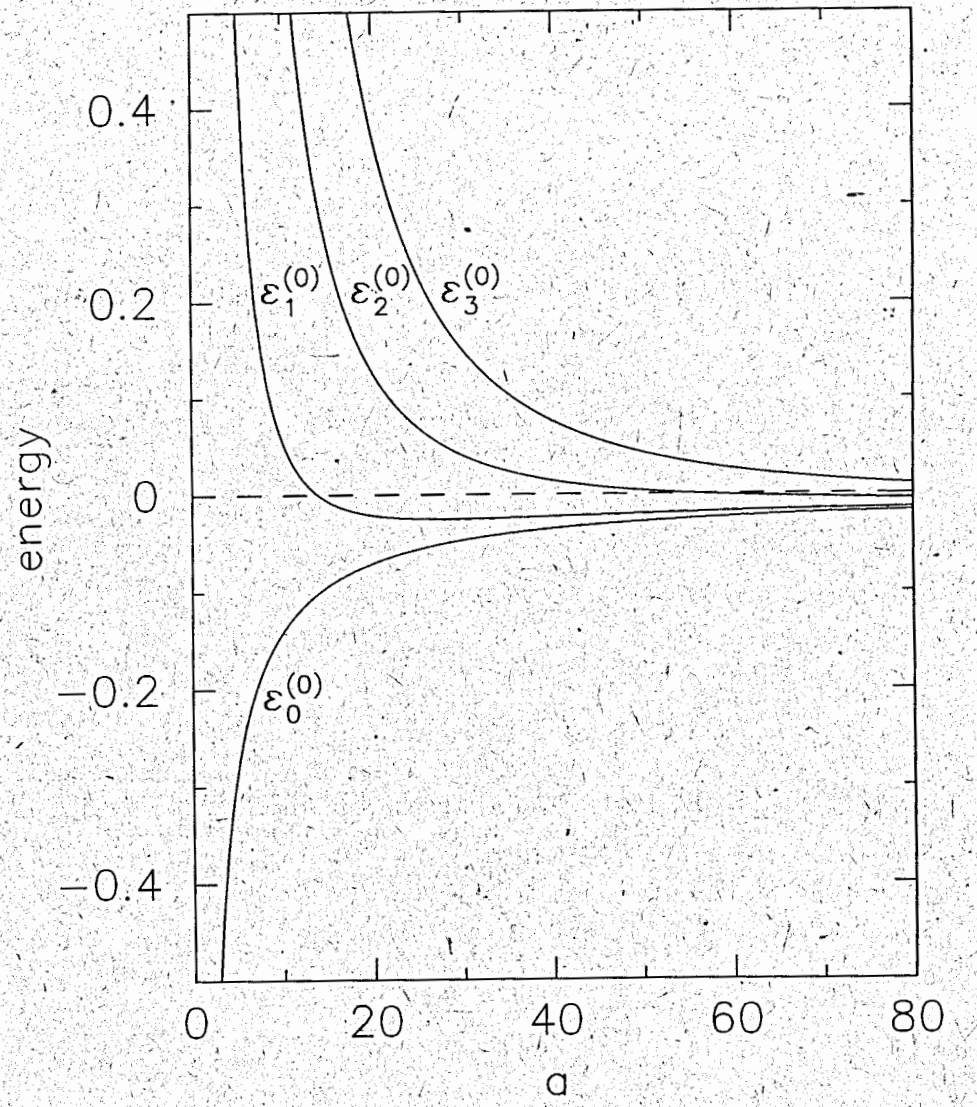

FIG. 5. Energies of a polaron on a ring vs. the scaled ring circumference for the states with complex wave functions. The corresponding wave functions are of the form $(1/\sqrt{a})e^{2\pi i l_z z/a}$ where we omitted the constant phase factor.

REFERENCES

[1] H. F. Cheung, Y. Gefen, E. K. Riedel, and W. H. Shih. Phys. Rev. B **37**, 6050 (1988).

[2] M. Büttiker, Y. Imry, and R. Landauer. Phys. Lett. **96A**, 365 (1983).

[3] M. Büttiker. Phys. Rev. B **32**, 1846 (1985).

[4] Various contributions in: *"Physics of optical ring gyros"*, Eds. S. F. Jacobs, M. Sargent III, M. O. Scully, J. Simpson, V. Sanders, and J. E. Killpatrick. Proc. SPIE , 487 (1984).

[5] P. Lett, W. Christian, S. Singh, and L. Mandel. Phys. Rev. Lett. **47**, 1892 (1981); L. Mandel, R. Roy, and S. Singh. In: *"Optical bistability"*, Eds. C. M. Bowden, M. Ciftan and Th. R. Roble (Plenum, New York, 1981).

[6] R. J. C. Spreeuw, J. P. Woerdman, and D. Lenstra. Phys. Rev. Lett. **61**, 318 (1988).

[7] D. Lenstra and S. H. M. Geurten. Optics Commun. **75**, 63 (1990).

[8] *"Analogies in Optics and Micro-Electronics"*, Eds. W. van Haeringen , D. Lenstra (North-Holland, Amsterdam, 1991).

[9] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).

[10] A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin. Proc. of IEEE **61**, 1443 (1973).

[11] P. Vansant, M. A. Smondyrev, F. M. Peeters, and J. T. Devreese. Commun. JINR E17-94-29, Dubna, 1994 (subm. to Z. Phys. B).

[12] F. M. Peeters, X. Wu, and J. T. Devreese, Phys. Rev. B **33**, 3926 (1986).

[13] F. M. Peeters and M. A. Smondyrev. Phys. Rev. B **43**, 4920 (1991).

[14] M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas. Phys. Rev. B **31**, 6207 (1985).

[15] N. Byers and C. N. Yang. Phys. Rev. Lett. **7**, 46 (1961).

[16] F. Bloch. Phys. Rev. B **2**, 109 (1970).

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