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SUPERFLUIDITY TRANSITION IN STRONGLY COUPLED FERMION SYSTEMS: LOW DENSITY LIMIT

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1 Introduction

The discovery of high-temperature superconductivity in the copper oxides and related materials, which have small coherence lenghts comparable to a few times of the lattice constant, gave a strong impact on investigation of theories of superfluidity and superconductivity for strongly correlated fermion systems beyond the weak coupling limit. A better understanding of quantum liquids of fermions is of relevance not only for strongly correlated electron superfluids such as superconductors, the electron-hole system in semiconductors, liquid Helium, spin-polarized Hydrogen and Cesium, but also for quantum liquids of strongly interacting fermions such as nuclear matter and the quark-gluon system (see [1] for the corresponding references).

The general problem of a unified treatment beyond mean-field theory including the crossover from the Bose-Einstein condensation (BEC) to the Bardeen-Cooper-Schrieffer (BCS) pairing [2] was first attacked by improving the equation of state which expresses the total fermion density n_F as a function of the temperature T and the chemical potential μ. Using a T-matrix approach to the virial expansion, Nozières and Schmitt-Rink [3] have taken into account the contributions of correlations obtained from the ladder diagrams with free internal Green functions. As a consequence, the effect of interaction between the bound states was not included in their formalism. The chemical equilibrium between bound and ionized fermion pairs was studied by Schmitt-Rink et al. [4]. In particular, they pointed out that at positive μ a Bose-type singularity arises in the Beth-Uhlenbeck formula. A generalization of the Beth-Uhlenbeck formula with special emphasis on the

quasiparticle picture was derived by Schmidt et al. [5] which can be applied also to fermion liquids at high densities. It has been shown that the singularity occurring in the Bose distribution is exactly compensated by the Pauli blocking factor $(1 - f_{p_1} - f_{p_2})$ as long as the temperature is above the critical one. Disregarding this compensation due to introducing the quasiparticle picture, the problem of the bosonic singularity at positive μ was discussed later on by Tokumitu et al. [6].

An essential problem is the inclusion of interaction between noncondensed bound states in the equation of state as well as the gap equation, improving the mean-field theory in a consistent way. For a two-dimensional Fermi superfluid, Tokumitu et al. [7] have taken into account the repulsion among two-fermion bound states which arises from the exchange effect of constituent fermions due to the Pauli principle. Haussmann [8] has solved the coupled system of the Dyson equation for the single-particle Green function and the Bethe-Salpeter equation for the two-particle Green function within the T-matrix approximation for the self-energy in a self-consistent way. The results are given for the superfluid transition temperature, the chemical potential, the fermion distribution function and the complex effective mass of the fermion pairs also in the crossover region for dilute fermion liquids. However, to be consistent, the Bethe-Salpeter equation should be improved not only by self-energy corrections but also by vertex corrections [1] in accordance with the Ward identities. The improvement of the self-energy approach by including the corresponding vertex corrections will be denoted as cluster-Hartree-Fock approximation, see Section 2.

Of particular interest is the investigation of thermodynamic phase stability. The influence of bound state formation in fermionic quantum liquids on phase instabilities has extensively been discusses in nuclear matter and in partially ionized plasmas, see [5,9] for further references. On this background, special attention will be given here to the question of phase instability at the Bose condensation of bound pairs.

The fermion system to be considered will be described by the Hamiltonian

$$H = \sum_1 E(1) a_1^+ a_1 + \frac{1}{2} \sum_{121'2'} V(121'2') a_1^+ a_2^+ a_{2'} a_{1'} \qquad (1)$$

with $1 = \vec{p}_1 \alpha_1$; $\vec{p}_1$ denotes the momentum and α_1 – spin and further internal quantum numbers, $E(1) = p_1^2/2m_1$ is the kinetic energy. The interaction potential is assumed to be spin independent and attractive. Furthermore, for strongly coupled systems the solution of the Schrödinger equation for the two-particle problem (s-wave) should give at least one bound state.

For this system, the crossover from the strong coupling case to the weak coupling case can be discussed in two different ways:

(i) A dilute Fermi liquid with Hamiltonian of the type (1), but with variable coupling strength, is considered. At a critical coupling strength of the attractive interaction, bound states are formed. For coupling strengths large compared with the critical one, the low temperature behaviour of the system is dominated by the bound states, and the Bose-Einstein condensation is expected. For coupling strengths small compared with the critical one, Cooper pairing may occur at low temperatures. The crossover between both limiting cases of variable potential strength has been discussed, e.g., in [3,8,10].

(ii) A Hamiltonian with fixed interaction strength is considered. In the zero-density limit, bound state formation is assumed to be possible. With

increasing density, due to the Pauli quenching these bound states will be suppressed and disappear at a critical value of the density (Mott density). For densities small compared with this critical value, at low temperatures a transition to the Bose-Einstein condensation is expected. Above the critical density, at low temperatures a transition into a BCS state can occur. The crossover from strong to weak coupling is controled by the density. In the density (n_F) – temperature (T) plane, a phase diagram for a system with fixed interaction strength can be considered, see [1,5,11].

In general, the macroscopic behaviour of the system described by the Hamiltonian (1) should be obtained from a non-equilibrium approach [1]. We concentrate here on the stationary state with possible singlet (s-wave) pairing at zero momentum. Introducing the single particle density matrix $\langle a_2^+ a_1 \rangle^t = \delta_{12} n(1)$ and the off-diagonal single-particle distribution function (pair amplitude) $\langle a_2 a_1 \rangle^t = F(12,t) = \delta_{\underline{1}2} e^{-2i\mu t/\hbar} F(1)$ with $\underline{1} = -p_1, -\sigma_1$, we find the entropy operator as $S = \hat{H}/T + S_0$ with $\hat{H} = H - \mu N$, $S_0 = \ln \mathrm{Tr} \exp(-\hat{H}/T)$; $N = \Omega n_F$ is the total fermion number, μ the fermion chemical potential, and

$$\begin{aligned} \hat{H} &= \sum_1 (E^{\mathrm{MF}}(1) - \mu) a_1^+ a_1 + \frac{1}{2} \sum_1 \Delta^{\mathrm{MF}}(1) e^{-2i\mu t/\hbar} a_1^+ a_{\underline{1}}^+ + c.c. \\ &+ \frac{1}{2} \sum_{121'2'} V(12,1'2') a_1^+ a_2^+ a_{2'} a_{1'} - (\mathrm{MF}), \end{aligned} \qquad (2)$$

where

$$\begin{aligned} E^{\mathrm{MF}}(1) &= E(1) + \sum_2 V(12,12)_{\mathrm{ex}} n(2), \\ \Delta^{\mathrm{MF}}(1) &= \sum_2 V(1\underline{1},2\underline{2}) F(2). \end{aligned} \qquad (3)$$

The term (MF) denotes the subtraction of the mean-field terms already contained in E^{MF}, Δ^{MF}, and the index ex denotes the antisymmetrized expression $V(12,12) - V(12,21)$.

Evaluating mean values with the statistical operator $\rho = \exp(-S)$, we have to specify approximations to be performed within many-particle theory. Neglecting two-particle correlations, the entropy operator is diagonalized using a Bogoliubov canonical transformation. In equilibrium, the BCS solution for the thermodynamic properties is immediately obtained.

The neglect of fluctuations is not justified especially in the case where bound states are formed. To improve the mean-field approximation, one has to include also the two-particle correlation function

$$\left\langle a_1^+ a_2^- a_{2'} a_{1'} \right\rangle = \delta_{11'}\delta_{22'} n(1) n(2)_{\mathrm{ex}} + F^*(1) F(1') \delta_{12} \delta_{1'2'} + c(a_1^+ a_2^+ a_{2'} a_{1'}) \quad (4)$$

and the corresponding higher correlation functions.

A self-consistent treatment of correlations can be performed by using the formalism of many-particle field theory with thermodynamic Green functions, see Fetter and Walecka [12]. In shorthand notation, the single-particle Green function is given by the Dyson equation, $G^{-1}(1,z) = z - E(1) - \Sigma(1,z)$. The self-energy can be represented by a cluster decomposition, $\Sigma = T_2 G + T_3 GG + \ldots$, whereas the n-particle T-matrices are given by the Bethe-Salpeter equations such as $T_2 = V + VGGT_2$, see [9]. This means that G has to be determined self-consistently. The solution of the n-particle T-matrix gives a continuum of scattering states but may also lead to bound states.

A commonly used approximation [3-7,10,11] to obtain a closed system of equations is to truncate the cluster decomposition of Σ. Dropping all T-matrices with particle number higher than 2, an equation of state will be obtained [13] which reproduces the correct second virial coefficient (Beth-Uhlenbeck formula).

However, the truncation of the cluster expansion of Σ at T_2 does not provide us with a consistent treatment of the two-particle properties. In the Bethe-Salpeter equation, besides improving the single-particle propagator introducing the self-energy, we have also to improve the interaction kernel by vertex corrections to be consistent in the same order of density. Looking for all first-order terms in the cluster-cluster interaction, the cluster-Hartree-Fock approximation has been given in [14]. In particular, the correct low-density limit of bound-bound interaction in the Born approximation including all exchange terms is obtained [1]. The problem to find consistent mean-field approximations from the exact static (energy-independent) part of the mass operator has also been discussed by Schuck, see [15] and further references given there.

Instead of using the Matsubara Green function technique, an alternative approach can be given from a functional integral representation. Using this approach for a system of interacting fermions, Drechsler and Zwerger [16] have introduced the order parameter Δ via a Hubbard-Stratonovich transformation. Integrating out the fermion degrees of freedom and expanding in powers of Δ they obtained a Ginzburg-Landau theory. The improvement of the ordinary mean-field theory by including fluctuations in evaluating functional integrals is the subject of recent investigations, see [17] for the Nambu–Jona-Lasinio model.

In the present paper, we apply the thermodynamic Green function approach to a zero–range interaction. Basic relations collected in Section 2 are evaluated in Section 3 with special emphasis on the disappearence of bound states at high densities due to the Pauli quenching (Mott effect). The low-density strong coupling limit, particularly the thermodynamic stability, is considered in Section 4. The results are discussed and compared with other approaches in Section 5.

2 Basic equations

We briefly describe some relevant relations which are derived in a more general form in Ref. [1]. Applying the Bogoliubov canonical transformation $a_1 = u_1 b_1 + v_1 b_{\underline{1}}^+, a_{\underline{1}} = u_1 b_{\underline{1}} - v_1 b_1^+$, with $1 \pm (1+\vartheta_1^2)^{-1/2}$ equal to $2|u_1|^2$ (upper sign) or $2|v_1|^2$ (lower sign), respectively,

$$\vartheta_1 = \frac{2F(1)}{1-2n(1)}, \tag{5}$$

the mean values $\langle b_2 b_1 \rangle^t$ vanish. This way no off-diagonal single particle distributions occur in the b representation, similarly to the case of the normal state.

The entropy operator is transformed to

$$S = \tilde{S}_0 + \sum_1 s(1) b_1^+ b_1 + \frac{1}{2} \sum_{121'2'} W(12,1'2') b_1^+ b_2^+ b_{2'} b_{1'} - (\mathrm{MF}) + S^{\mathrm{off}}. \tag{6}$$

The off-diagonal term S^{off} contains noncompensating numbers of the operators b, b^+. It can consistently be dropped as long as we are considering the cluster-Hartree-Fock approximation.

Within the Matsubara-Green function technique, the cluster-Hartree-Fock approximation [14] is given by the T_2-matrix approximation for the self-energy as well as the corresponding terms for the interaction kernel of the two-particle Bethe-Salpeter equation. The latter can be related to the two-particle wave equation determining the eigenstates with total momentum P and internal quantum number n, see Ref. [9].

$$(s(1)+s(2)-S_{nP})\phi_{nP}(12) + \sum_{1'2'} [W(12,1'2')(1- < b_1^- b_1 > - < b_2^+ b_2 >)$$
$$+ \sum_{343'4'} W(1234,1'2'3'4') C(3'4',34)] \phi_{nP}(1'2') = 0 \tag{7}$$

with

$$\begin{aligned} W(1234,1'2'3'4') &= W(13,1'3')_{\mathrm{ex}} \delta_{4'2} \delta_{42'} + W(14,3'2') \delta_{4'2} \delta_{31'} \\ &- W(14,3'4') \delta_{2'2} \delta_{31'} + (1 \to 2, 3 \to 4). \end{aligned} \tag{8}$$

and the correlation function

$$\begin{aligned} C(12,1'2') &= \mathrm{Tr}(e^{-S}\, b_1^+ b_2^+ b_{2'} b_{1'}) - < b_1^+ b_{1'} >< b_2^+ b_{2'} >_{\mathrm{ex}} \\ &= \sum_{nP} g(S_{nP}) [\phi_{nP}^*(1'2') \phi_{nP}(12) - \delta_{nP}(12) \delta_{nP}(1'2')_{\mathrm{ex}}]. \end{aligned} \tag{9}$$

Here $\delta_{nP}(12)$ denotes the free-particle solution $\delta_{p_1+p_2,P} \delta_{n,p}$; and $g(S_{nP}) = [\exp(S_{nP})-1]^{-1}$, the Bose distribution function.

With the self-energy ($f(s) = (\exp s + 1)^{-1}$)

$$\Sigma(1,z) = \sum_{2,nP} [g(S_{nP}) + f(s(2))] \frac{(s(1)+s(2)-S_{nP})^2}{z+s(2)-S_{nP}} |\phi_{nP}(12)|^2, \tag{10}$$

the single-particle occupation $< b_1^+ b_1 >$ is evaluated with the help of the spectral function, see [5], where a generalized Beth-Uhlenbeck formula is derived. The diagonal single-particle density is given by

$$n(1) = \frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{1+\vartheta_1^2}}(1 - 2 < b_1^+ b_1 >). \qquad (11)$$

The equation for the pair amplitude F including the effects of two-particle correlations is found as

$$2(E^{\mathrm{MF}}(1) - \mu)F(1) + \sum_{1'} V(1\underline{1}, 1'\underline{1}')(1 - 2n(1))F(1')$$
$$= \sum_{33'4'} V(31, 3'4')c(a_3^+ a_{3'} a_{4'} a_{\underline{1}}, t) - (1 - \underline{1}) \qquad (12)$$

with

$$c(a_3^+ a_{3'} a_{4'} a_1, t) = \mathrm{Tr}[e^{-S(t)} a_3^+ a_{3'} a_{4'} a_1] - n(3)F(14', t)\delta_{33'}$$
$$+ \; n(3)F(13', t)\delta_{34'} - n(1)F(4'3', t)\delta_{13}. \qquad (13)$$

Applying the Bogoliubov transformation, the r.h.s. of eq. (12) is expressed by the two-particle correlation function as

$$\sum_{1'33'}[Y(13, 1'3')C(1'3', 13) - \tilde{Y}(13, 1'3')C^*(1'3', 13)] - (1 \leftrightarrow \underline{1}) \qquad (14)$$

Explicit expressions for $s(1)$, W, Y are given in Ref. [1].

The two-particle wave equation (7), the single-particle density (11) and the equation for the pair amplitude (12) should be solved in a self-consistent way to describe the stationary state for a superfluid system including two-particle correlations at given temperature T and total fermion density $n_F = \Omega^{-1}\sum_1 n(1)$. Neglecting all two-particle correlations, $C(12, 1'2') = 0$, we obtain the BCS approximation.

3 Model calculation

We consider a quantum liquid of fermions (1) with spin 1/2 which interact via a spin independent potential. The evaluation of eqs. (7), (11), (12) is simplified for a zero-range interaction potential $V(12, 1'2') =$ const. To avoid not well defined divergent expressions, we take the zero-range limit of a finit range interaction.

Particularly, we consider a separable interaction where the solution of the T_2-matrix for the isolated two-particle system can immediately be given. Having introducing the total ($P = p_1 + p_2$) and the relative ($p = (p_1 - p_2)/2$) momentum, the Yamaguchi-type interaction

$$V(12, 1'2') = -\lambda w(p)w(p')\delta_{P,P'}\delta_{\sigma_1,\sigma_{1'}}\delta_{\sigma_2,\sigma_{2'}}, \qquad (15)$$

with $w(p) = (\hbar^2 p^2/m\epsilon + \gamma^2)^{-1}$ and $\lambda = 8\pi\gamma(1+\gamma)^2\Omega^{-1}(\hbar^2/m\epsilon)^{3/2}\epsilon$ yields at zero density a bound state, the binding energy being $E_{0P} = -\epsilon + \hbar^2 P^2/4m$ and the wave function

$$\phi_0(p) = c(x^2+1)^{-1}(x^2+\gamma^2)^{-1}, \qquad (16)$$

where $c^2 = 8\pi\gamma(\gamma+1)^3\Omega^{-1}(\hbar^2/m\epsilon)^{3/2}$ and $x = (\hbar^2/m\epsilon)^{1/2}p$. With $e = E/\epsilon$, the scattering phase shifts result as

$$\tan\delta(E) = 2\sqrt{e}\gamma(1+\gamma)^2/[(\gamma^2+e)^2 - (1+\gamma)^2(\gamma^2-e)]. \qquad (17)$$

The limiting case of a zero-range interaction to be considered here is obtained in the limit $\gamma \to \infty$, but $\epsilon = \hbar^2/ma_F^2$ taken as a fixed value, where a_F denotes the scattering length for the two-particle interaction (cf. [1] for arbitrary γ). In this limiting case, the Hartree-Fock shifts of the single-particle states vanish, $E^{\rm MF}(1) = E(1)$, whereas $\Delta^{\rm MF}(1) = \Delta$ is momentum independent. For the sake of simplicity, in the following we will use a system of units where $\hbar, m, \epsilon$ are equal to unity.

Neglecting all correlations, we obtain the mean field solution where the superfluidity transition is found from the equation ($\mathcal{P}$ - principal value)

$$1 = {\rm Re}\sqrt{-2\mu_{\rm BCS}} + \frac{4}{\pi}\int_0^\infty p^2 dp \frac{\mathcal{P}}{p^2 - 2\mu_{\rm BCS}} \frac{1}{e^{(p^2/2-\mu_{\rm BCS})/T}+1}. \qquad (18)$$

Together with the fermion density in Hartree-Fock approximation

$$n_{\rm F,BCS} = \frac{1}{\pi^2}\int_0^\infty \frac{p^2 dp}{e^{(p^2/2-\mu_{\rm BCS})/T}+1} \qquad (19)$$

we find the critical values $\mu_{\rm BCS}(T)$, $n_{\rm F,BCS}(T)$ where the transition to the superfluid state in the mean field approximation occurs, see Fig. 1.

Eqs. (18), (19) represent the zero-gap solutions of the well-known mean-field relations

$$n_{\rm F,MF}(\mu, T) = \frac{1}{\pi^2}\int_0^\infty p^2 dp [\frac{1}{2} - \frac{1}{2}\frac{p^2/2-\mu}{E(p)}\{1 - 2f(E(p))\}] \qquad (20)$$

where $E(p) = [(p^2/2-\mu)^2 + \Delta^2]^{1/2}$ contains the gap parameter $\Delta(\mu, T)$ to be determined from

$$1 = -\frac{2}{\pi}\int_0^\infty p^2 dp [\frac{1}{2E(p)}\tanh(\frac{E(p)}{2T}) - \frac{1}{p^2}]. \qquad (21)$$

However, the mean-field approximation (19), (20) to the equation of state which relates the density to the chemical potential and the temperature is not always applicable. At low temperatures ($T < 1$) and not too high densities, the formation of bound states becomes of importance. Two-particle

contributions are obtained by solving the Bethe-Salpeter equation. In the quasiparticle approximation, i.e. by neglecting the correlation function C in the wave equation (7), the two-particle T-matrix in the normal phase has the form

$$T(P, z) = \frac{4\pi}{\Omega}[1 - \sqrt{-z + P^2/4} - \frac{1}{2\pi^2}\int d^3p \frac{1}{-z + P^2/4 + p^2} \times (\frac{1}{e^{((P/2-p)^2/2-\mu)/T}+1} + \frac{1}{e^{((P/2-p)^2/2-\mu)/T}+1})]^{-1} \qquad (22)$$

Poles at $z = E_b^f(P, \mu, T) < P^2/4$ describe bound states. In particular, the shift of the bound state, with total momentum $P = 0$, due to the Pauli blocking by free quasiparticles is obtained from the solution of

$$1 = {\rm Re}\sqrt{-E_b^f} + \frac{4}{\pi}\int_0^\infty p^2 dp \frac{\mathcal{P}}{p^2 - E_b^f} \frac{1}{e^{(p^2/2-\mu)/T}+1} \qquad (23)$$

From the solution of eq.(21) with $E_b = 0$,

$$1 = \frac{4}{\pi}\int_0^\infty dp \frac{1}{e^{(p^2/2-\mu_{\rm Mott})/T}+1}. \qquad (24)$$

we find a quasiparticle density $n_{\rm F,Mott}(T)$ corresponding to relation (19) above which, due to the Pauli blocking, no bound state with $P = 0$ is formed. This Mott line is also shown in Fig.1. It separates a low-density region where bound states can exist from a high density region where they are partially blocked out, depending on their total momentum, see also [11].

Together with the in-medium scattering phase shifts $\delta_P(E; \mu, T)$ obtained from the T-matrix (22), in the normal phase an equation of state $n_{\rm F}(\mu, T)$ can be derived in the form of a generalized Beth-Uhlenbeck formula [5]

$$n_{\rm F} = \frac{1}{\Omega}\sum_1 f((E_1 - \mu)/T)$$

$$+ \frac{2}{\Omega} \sum_{P>P_{\rm Mott}} (g((E_b^f(P)-2\mu)/T) - g((P^2/4-2\mu)/T)) \tag{25}$$
$$- \frac{2}{\Omega}\sum_P \int_0^\infty \frac{dE}{\pi}\frac{d}{dE} g((E+P^2/4-2\mu)/T)[\delta_P(E) - \frac{1}{2}\sin(2\delta_P(E))],$$

$E_1 = E(1) - \sum_{21'2'} V(12,1'2')C(1'2',12)$ denotes the quasiparticle energy as obtained from (10).

With the mean-field result $\mu_{\rm BCS}(T)$ from eq. (18), according to Nozières and Schmitt-Rink [3] the improved value $n_{\rm F,NSR}(T) = n_F(\mu_{\rm BCS}, T)$ for the transition to superfluidity can be given which is also shown in Fig. 1. In contrast with [3], where a simple Beth-Uhlenbeck formula was used, we take the generalized Beth-Uhlenbeck formula (25) derived by Zimmermann and Stolz [13] which can also be used at high densities where the chemical potential becomes positive [5].

The improvement in evaluating the equation of state (25) instead of (19) gives a modification for the phase transition temperature to superfluidity particularly below the Mott line. The low-density limit is correctly described where at low temperatures the system consists of bound pairs. These nearly free bosons undergo a Bose-Einstein condensation if the chemical potential takes the value $\mu = -1/2$, what also follows from the condition (18) for the transition to the superfluid state. According to Nozières and Schmitt-Rink, a crossover from the Bose-Einstein condensation of noninteracting bosons with binding energy -1 ($n_{\rm F,BEC}(T) = 2\zeta(3/2)(T/\pi)^{3/2}$ in Fig.1) to the Bardeen-Cooper-Schrieffer pairing is obtained, see also [5,10].

An open problem, however, is the correct low density behaviour where the interaction between the bound states is the dominant process. Obviously, in

the BCS equation for the critical value of the chemical potential (18) as well as in the T-matrix (22) or the equation (23) for the bound state energies, the influence of the medium is considered in the quasiparticle picture as seen from the occurrence of Fermi functions. However, the inclusion of correlations is of importance at least in the strong coupling limit. We will evaluate the influence of correlations in the following Section for the low-density limit of a strongly coupled system.

4 The low-density strong coupling limit

The solution of the self-consistent system of equations (7), (11), (12) that contain the effect of in-medium correlations on the formation of bound states as well as the transition to the superfluid state is simplified in the limit of low temperatures ($T \ll 1$) and low densities ($n_F \ll n_{\rm F,Mott}$). In this case, the dissociation of bound states is low, so that the contribution of free quasiparticles to the density can be neglected. The two-particle correlation function is given by the ground state contribution

$$C(12,1'2') = \sum_P g((E_b(P)-2\mu)/T)\phi_P^*(12)\phi_P(1'2'). \tag{26}$$

The binding energy $E_b(P)$ and the normalized wave function $\phi_P(12)$ have to be determined from the two-particle wave equation (7) as performed below.

Furthermore, in the low density limit we assume that $n(1) \ll 1$. Particularly, only the lowest orders with respect to the correlated densities will be considered. With $n(1) \ll 1$, for the coefficients of the Bogoliubov transformation we have $|u_1| \approx 1 - F^2(1)/2$, $|v_1| \approx F(1)$.

The two important equations for the pair amplitude (12) and for the two-particle states (7), respectively, can now be given in the form

$$2(E(1)-\mu)F(1)+\sum_{1'}V(1\underline{1},1'\underline{1}')(1-2n(1))F(1')$$
$$=2\sum_{343'4'}V(34,3'4')C(3'4',34)(\delta_{3,1}+\delta_{4,\underline{1}})F(1), \quad (27)$$

$$(E(1)+E(2)-E_b(P))\phi_P(12)+\sum_{1'2'}V(12,1'2')(1-n(1)-n(2))\phi_P(1'2')$$
$$=\sum_{343'4'}V(34,3'4')\tilde{C}(3'4',34)(\delta_{3,1}+\delta_{4,2})\phi_P(12). \quad (28)$$

Here $\tilde{C}(12,1'2')=C(12,1'2')+F^*(12,t)F(1'2',t)$ contains the correlations of the two-particle states with total momentum $P\neq 0$ and the possible condensate at $P=0$.

The average occupation $n(1)$ of single-particle states is obtained from the entropy operator (6). Considering $\Delta_b(1)$ (r.h.s. of eq. (27)) as a small quantity, according to (3) and using (27) we have

$$\Delta=-2(E(1)-\mu)F(1)/(1-2n(1))+\Delta_b(1), \quad (29)$$

so that (5) $\vartheta_1=(\Delta-\Delta_b(1))/(E(1)-\mu)$ and

$$s(1)=\sqrt{(E(1)-\mu)^2+\Delta^2}+0(\Delta_b^2(1)). \quad (30)$$

The evaluation of $<b_1^+b_1>$ can be performed by standard methods [5] evaluating the spectral function from the self-energy (10). Up to terms of higher order in the densities of the bound states and the condensate, we have by analogy with (25)

$$<b_1^+b_1>=f(s(1))+\sum_{P,2}g((E_b(P)-2\mu)/T)|\phi_P(12)|^2 \quad (31)$$

and from (11)

$$n(1)=\frac{1}{2}-\frac{1}{2}\frac{1}{\sqrt{1+\Delta^2/(E(1)-\mu)^2}}(1-2f(s(1)))+\sum_{P,2}g((E_b(P)-2\mu)/T)|\phi_P(12)|^2. \quad (32)$$

Expanding for $F(1)\ll 1$, we have the equation of state

$$\begin{aligned} n_F(\mu,T) &= 2\Omega^{-1}F^2\delta_{E_b(0),2\mu}+2\Omega^{-1}\sum_{nP}g((E_{nP}-2\mu)/T) \\ &= 2\Omega^{-1}\sum_{12}\tilde{C}(12,12). \end{aligned} \quad (33)$$

where in the superfluid state $F(1\underline{1})=F\varphi_0(1\underline{1})$ is related to the order parameter Δ according to $\Delta=\sqrt{8\pi/\Omega}F$. Eq. (33) coincides in the low density limit with the results known from the theory of weakly interacting Bose gas, see [8,12], where the noncondensed density in the mean-field approximation is given by

$$n_{F,n}=\frac{2}{\pi}\int P^2dP[\frac{\epsilon_P}{E_P}\frac{1}{\exp(E_P/T)-1}+\frac{\epsilon_P-E_P}{2E_P}] \quad (34)$$

with $E_P=[(P^2/4)^2+2\pi P^2F^2/\Omega]^{1/2}$, $\epsilon_P=P^2/4+4\pi F^2/\Omega$. Expressions (32), (33) for the density can be used to show explicitly the modification of the two-particle wave equation (28) due to the density effects in first order,

$$(E(1)+E(2)-E_b(P))\phi_P(12)+\sum_{1'2'}V(12,1'2')\phi_P(1'2')$$
$$+\sum_{34P'}[g((E_b(P')-2\mu)/T)+F^2\delta_{P',0}]|\phi_{P'}(34)|^2(\delta_{3,1}+\delta_{4,2})$$
$$\times[E_b(P)+E_b(P')-E(1)-E(2)-E(3)-E(4)]\phi_P(12)=0. \quad (35)$$

where the interaction potential has been eliminated by applying the Schrödinger equation.

For the zero-range interaction. the energy shift of bound states due to the interaction with bound states in the medium is given in the cluster-Hartree-Fock approximation by the Pauli blocking as seen from the overlap integral of the wave functions. With the ground-state wave function (16), for the shift $\Delta E_b(P) = E_b(P) + 1 - P^2/4$ of the bound state energy perturbation theory gives

$$\Delta E_b(P) = \Omega^{-1}\sum_{P'}[g((E_b(P') - 2\mu)/T) + F^2\delta_{P',0}]\frac{8\pi}{1+(P-P')^2/16}. \quad (36)$$

In the low-temperature limit ($P' \approx 2\sqrt{T}$), we expand the last term in (35) with respect to P'. The summation over P' gives the fermion density,

$$\Delta E_b(P) \approx n_F \frac{4\pi}{1+P^2/16}. \quad (37)$$

The behaviour at small P-values can be accounted for by introducing an effective mass m^* according to

$$\frac{P^2}{4m^*/m} = \frac{P^2}{4} - n_F\frac{\pi P^2}{4} \quad (38)$$

so that $m^* = m(1 + \pi n_F + ...)$ in accordance with [8].

A similar consequent low-density expansion for the gap equation (27) yields

$$\begin{aligned} &2(E(1) - \mu)F(1) + \sum_{1'} V(1\underline{1}, 1'\underline{1}')F(1') \\ &- \sum_{34P'}[g((E_b(P') - 2\mu)/T) + F^2\delta_{P',0}]|\phi_{P'}(34)|^2)(\delta_{3,1} + \delta_{4,\underline{1}})/(1+\delta_{P',0}) \\ &\times[E_b(P) + E_b(P') - E(1) - E(\underline{1}) - E(3) - E(4)]F(1) = 0, \end{aligned} \quad (39)$$

The perturbative expansion yields a shift in the chemical potential in the superfluid state according to

$$\mu_{sf} = -1/2 + 2\pi n_F - 2\pi F^2/\Omega. \quad (40)$$

Within a more sophisticated approach, expanding near the mean-field result with respect to small fluctuations due to correlations, the chemical potential follows as

$$\mu = \mu_{MF} + 4\pi\Omega^{-1}\sum_P g((E_b(P) - 2\mu)/T) \quad (41)$$

what coincides with (40) in the limiting case under consideration.

Looking for the critical density where the pair amplitude vanishes, $F \to 0$, the shift in the chemical potential is compensated by the shift of the ground-state energy $E_b(0)$ at zero momentum. Compared with the Bose - Einstein condensation of a noninteracting gas, the only effect we obtain is a negative shift of the ideal Bose-Einstein condensation temperature at fixed fermion density due to the effective mass

$$T_{\Delta\to 0}(n_F) = T_{BEC}(n_F)(1 - \pi n_F) \quad (42)$$

in accordance with Haussmann [8], see Fig. 2.

Having the shifts of the bound states (36) and of the chemical potential (40) at our disposal, we solve the equation of state (33) to find the chemical potential for given T and n_F. In the normal phase we have

$$n_{F,n}(\mu, T) = \frac{1}{\pi^2}\int_0^\infty \frac{P^2 dP}{\exp[(-1 + 64\pi n_{F,n}/(16+P^2) + P^2/4 - 2\mu)/T] - 1}. \quad (43)$$

as shown in Fig. 3. Above the critical density $n_{F,\Delta\to 0}$ we find solutions for n_F, μ at given $T, \Delta^2 = 8\pi F^2/\Omega$, according to (40) and

$$n_F = n_{F,n}(\mu_{sf}, T) + 2F^2/\Omega, \quad (44)$$

where now

$$n_{\mathrm{F,n}}(\mu_{\mathrm{sf}},T)=\frac{1}{\pi^2}\int_0^\infty\frac{P^2dP}{\exp[(-4\pi n_{\mathrm{F}}P^2/(16+P^2)+P^2/4+4\pi F^2/\Omega)/T]-1}. \tag{45}$$

An improved equation of state which can be applied at low temperatures also at higher densities is obtained by using the mean-field result $n_{\mathrm{F}}=n_{\mathrm{F,n}}(\mu_{\mathrm{sf}},T)+n_{\mathrm{MF}}(\Delta,T)$, where

$$n_{\mathrm{F,n}}(\mu_{\mathrm{sf}},T)=\frac{1}{\pi^2}\int_0^\infty P^2dP \tag{46}$$
$$\times\{\exp[(-1-2\mu_{\mathrm{MF}}(\Delta,T)+\frac{P^2}{4}(1-\frac{16\pi n_{\mathrm{F,n}}}{16+P^2})+\frac{16\Delta^2}{16+P^2})/T]-1\}^{-1}.$$

In the low-density case under consideration, eqs. (45) and (46) coincide.

Whereas in the normal phase ($\Delta=0$) we can take T and μ to evaluate the density $n_{\mathrm{F}}=n_{\mathrm{F,n}}(\mu,T)+n_{\mathrm{F,MF}}(\mu,T)$ (43), (20), in the superfluid phase we can start with Δ,T, find the solution μ_{sf} from eq. (41) with $\mu_{\mathrm{MF}}(\Delta,T)$, eq. (21), and then evaluate the densities (46). This procedure is simplified in the low density limit where, after solving eq. (45), the density and the chemical potential are obtained from (44) and (40), respectively. The normal phase is described by (43).

In Fig.3, the resulting equation of state is shown for $T=0.05$. There are regions of density where the solution for the chemical potential as a function of the total fermion density at fixed temperature is not unique. Furthermore, the condition $\partial\mu/\partial n_B|_T>0$ for thermodynamic stability is not always fulfilled. This indicates instability of the homogeneous solution with respect to phase separation. Applying the Maxwell construction, the densities $n_{\mathrm{F,1}}$, $n_{\mathrm{F,2}}$ of equilibrium phases at the first order phase transition are also

shown in Fig. 3 (notice that the criterium of equal areas holds not in the logarithmic representation). The instability region in the density-temperature plane where in the low density region the first order phase transition from the normal state to the superfluid state occurs is also shown in Fig.2.

5 Discussion

The inclusion of interacting bound states already given in Ref. [1] is treated for a zero-range interaction in the low-density limit where their influence on the equation of state is dominant. Compared with Haussmann [8], the improvement obtained in the general formalism by including in addition to the self-energy also vertex corrections, see Sec. 2, becomes inoperative in the limit of the zero-range interaction. Indeed, only exchange terms (Pauli exclusion) survive where the interaction can be eliminated applying the Schrödinger equation. In the low-density strong coupling limit, the results given in [8] can be reproduced within the cluster-Hartree-Fock approximation. In general, however, considering the low-density limit, the self-energy and the vertex corrections contribute to the two-particle properties in the same order [9,14] corresponding to the Ward identities.

We emphasize that the self-consistent T-matrix treatment given by Haussmann [12] as well as the cluster-Hartree-Fock approximation given here do not represent the exact solution of the interaction between two bound states. Instead, on the level of bound states the interaction is treated only in the Born approximation. For the full solution, we have to iterate the Born approximation which corresponds to the full solution of the four-particle Green function or of the four-particle T-matrix. Only in this way the correct second virial coefficient on the level of bound states will be obtained which corresponds to the fourth virial coefficient on the level of the elementary fermions.

The thermodynamic stability is inspected. As a new result, a first order phase transition from the normal state to the superfluid state is found in the low-density strong coupling limit . Instead of applying the Maxwell construction to the equation of state $\mu(n_F, T)$, we have also considered the free energy, which leads to identical results for the region of instability. In contrast to [8], we find an extension of the region of superfluidity to temperatures above T_{BEC}. However, as is well-known, the Hartree-Fock type approximations overestimate the instability region. The full treatment of the cluster-cluster interaction would improve also the results for thermodynamic stability.

Going over to higher densities, the systematic expansion with respect to small fluctuations near the mean-field solution would be of interest. In particular, the derivation of a Ginzburg-Landau equation should be possible as done, e.g., by Drechsler and Zwerger [16] using the functional integral representation of interacting fermions. Furthermore, the self-consistent treatment [8] including the contribution of free particles will be the subject of further work to solve the equation of state in the entire temperature-density plane. Exploratory calculations have been done in [1] by using simplifying assumptions such as the rigid shift approximation for the single-particle energies and the neglect of the motion of two-particle states. Nevertheless, in contrast with the strange behaviour of the NSR solution where near the Mott density different critical temperatures are possible, see also Ref. [1],the inclusion of

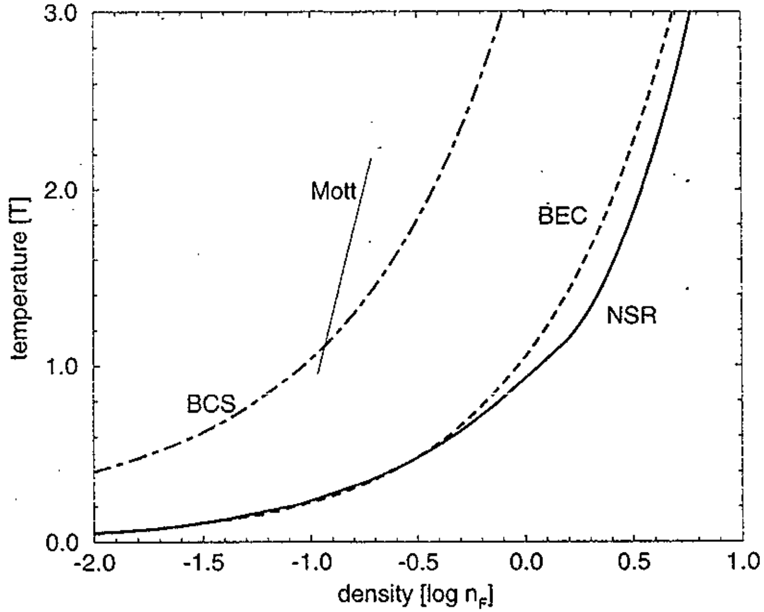

Fig. 1. Phase diagram for a fermionic model system with zero-range interaction (15), $\gamma \to \infty$, binding energy ϵ. Temperature and total particle density are given in reduced units $T^* = T/\epsilon, n^* = n(\hbar^2/m\epsilon)^{3/2}$. Bound states are blocked out for densities above the Mott line. The critical temperature for the transition to the superfluid state according to Nozières and Schmitt-Rink (NSR) [3] neglecting the interaction with correlations in the medium is shown. Furthermore, the solution of the Gorkov equation for vanishing gap (BCS) and the Bose-Einstein condensation of noninteracting bound states (BEC) are presented.

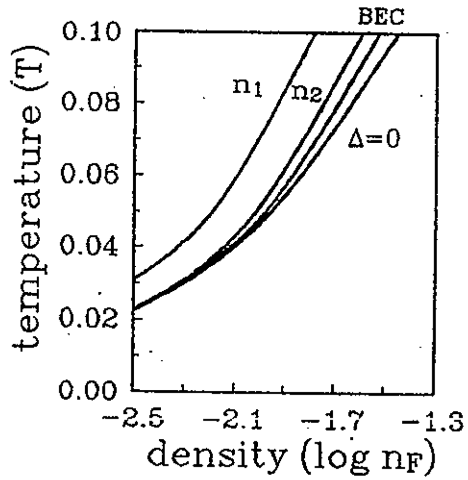

Fig. 2. Phase transition in the low-density strong coupling region. The Bose-Einstein condensation (BEC) for noninteracting bound states is compared with the critical line ($\Delta = 0$) at which the gap for the interacting fermion system vanishes, eq. (45). Furthermore, the critical densities n_1, n_2 for the first order phase transition to the superfluid phase are shown.

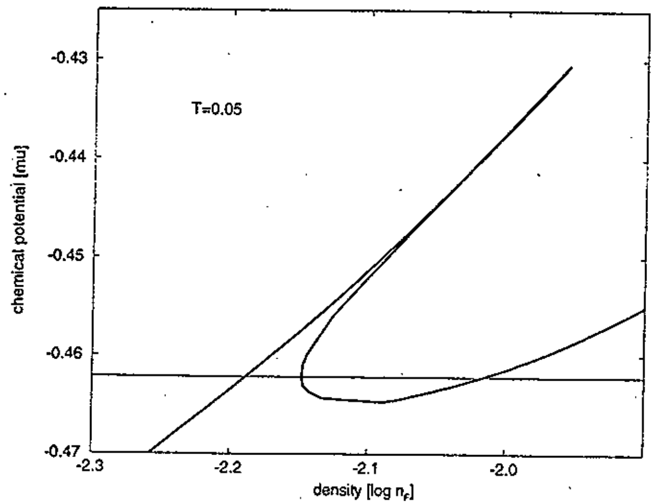

Fig. 3. Equation of state $\mu(n_F, T)$ for the fermionic model system with zero-range interaction showing the region of thermodynamic instability. The first order phase transition is determined by the Maxwell construction.

correlations in the medium seems to give a smooth critical temperature as presented in [1].

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