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SU(2) PATH INTEGRAL FOR HEISENBERG FERROMAGNET

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1 Introduction

The partition function for the quantum Heisenberg ferromagnet

$$Z = Tr\exp(-\beta H)\,, \quad H = -\frac{1}{2}\sum_{i,j} J_{ij}\vec{S}_i\vec{S}_j\,,$$

where $\vec{S}_i$ are $SU(2)$ generators of dimension $2S+1$ and J_{ij} is the positive definite symmetric matrix of the exchange interaction ($J_{ii}=0$) can be written as [1]

$$\begin{aligned} Z &= \int \prod_i D\vec{\psi}_i(t)\exp\left(-\frac{1}{2}\int\limits_0^\beta \vec{\psi}_i(t)J_{ij}^{-1}\vec{\psi}_j(t)dt\right) \\ &\times Tr\left[T\exp\left(-\int\limits_0^\beta \vec{\psi}_i(t)\vec{S}_i(t)dt\right)\right], \end{aligned} \qquad (1)$$

where the symbol T stands for the chronological product, and summation over repeated indices is assumed. In deriving (1), the well-known Hubbard-Stratonovich identity has been used. The partition function of noninteracting spins in an external fluctuating field $\psi(t)$ can be written as the $SU(2)$ path integral

$$\begin{aligned} Z_0(\vec{\psi}) &\equiv Tr\left[T\exp\left(-\int\limits_0^\beta \vec{\psi}_i(t)\vec{S}_i(t)dt\right)\right] \\ &= \int\limits_{\alpha_i(0)=\alpha_i(\beta)} \prod_i D\mu_S(\alpha_i)\exp\left(S\int_0^\beta \frac{\dot{\bar{\alpha}}_i\alpha_i - \bar{\alpha}_i\dot{\alpha}_i}{1+|\alpha_i|^2}dt\right. \\ &\left. + S\int_0^\beta \psi_i^{(0)}\frac{1-|\alpha_i|^2}{1+|\alpha_i|^2}dt - S\int_0^\beta \frac{\psi_i\alpha_i}{1+|\alpha_i|^2}dt - S\int_0^\beta \frac{\bar{\psi}_i\bar{\alpha}_i}{1+|\alpha_i|^2}dt\right), \end{aligned} \qquad (2)$$

where we have put $\bar{\psi} = \psi_x + i\psi_y$, $\psi = \psi_x - i\psi_y$ and $\psi^{(0)} = \psi_z$. The functional measure in eq. (2) is formally defined as an infinite pointwise product of the $SU(2)$ invariant measures,

$$D\mu_S(\alpha) = \prod_t \frac{2S+1}{\pi}\frac{d^2\alpha}{(1+|\alpha|^2)^2}.$$

This integral can be evaluated by the change [2]

$$\alpha \to \alpha = \frac{u(t)\alpha + v(t)}{-\bar{v}(t)\alpha + \bar{u}(t)},$$

where

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in SU(2).$$

The result is

$$Z_0 = \exp L(\vec{\psi}), \quad L(\vec{\psi}) = \sum_i \log \frac{\sinh(S+\frac{1}{2})\int_0^\beta (\psi_i^{(0)} + \bar{\psi}_i z_i)dt}{\sinh\frac{1}{2}\int_0^\beta (\psi_i^{(0)} + \bar{\psi}_i z_i)dt}, \tag{3}$$

where the function $z = \bar{u}/v$ depends upon ψ via the Riccati-type equation

$$\dot{z} - \psi^{(0)} z - (\bar{\psi}/2)z^2 + \psi/2 = 0, \quad z(0) = z(\beta). \tag{4}$$

The partition function thus becomes

$$Z = \int D\vec{\psi} \exp(-S(\vec{\psi})) \tag{5}$$

$$S(\vec{\psi}) = -\frac{1}{2}\int \vec{\psi}_i(t) J_{ij}^{-1} \vec{\psi}_j(t) dt + L(\vec{\psi}) \tag{6}$$

Representation (5,6) is a starting point for the systematic mean field expansion.

2 Mean field theory

Let the stationary mean field $\vec{\Phi}$ be chosen in the z direction: $\vec{\Phi}_i = (0, 0, \Phi_i)$. The saddle-point equation

$$\frac{\delta S}{\delta \vec{\psi}_j(t)} |_0 = 0$$

(the subscript $|_0$ denotes a quantity at the stationary point) reads

$$\Phi = J_0 b(\beta\Phi), \tag{7}$$

where $b(x) = SB_S(Sx)$ and $B_S(x)$ is the Brillouin function. To derive (7), we have put $J_0 = \sum_j J_{ij}$ and $\Phi_i = \Phi$ assuming translation invariance for the system. Expanding $S(\vec{\psi})$ around Φ up to the second order, one arrives at ($\vec{\eta} = \vec{\psi} - \vec{\psi}\,|_0$)

$$\begin{aligned} Z = \quad & \exp\{-S(\Phi)\} \int D\vec{\eta} \exp\left\{ -\frac{1}{2}\int_0^\beta \eta_i^{(0)}(t) (J^{-1}_{eff;ln})_{ij}(t,s) \eta_j^{(0)}(s) dt ds \right. \\ & \left. -\frac{1}{2}\int_0^\beta \bar{\eta}_i(t) (J^{-1}_{eff;tr})_{ij}(t,s) \eta_j(s) dt ds \right\}, \end{aligned} \tag{8}$$

the effective inverse longitudinal and transverse interactions being given by

$$\begin{aligned} (J^{-1}_{eff;ln})_{ij}(t,s) &= J_{ij}^{-1}\delta(t-s) - \frac{\delta^2 L}{\delta\psi_i^{(0)}(t)\delta\psi_j^{(0)}(s)} |_0 \\ &= J_{ij}^{-1}\delta(t-s) - b'(\beta\Phi)\delta_{ij}, \end{aligned} \tag{9}$$

$$\begin{aligned} (J^{-1}_{eff;tr})_{ij}(t,s) &= J_{ij}^{-1}\delta(t-s) - 2\frac{\delta^2 L}{\delta\bar{\psi}_i(t)\delta\psi_j(s)} |_0 \\ &= J_{ij}^{-1}\delta(t-s) - 2b(\beta\Phi)\frac{\delta z_i(t)}{\delta\psi_j(s)} |_0 . \end{aligned} \tag{10}$$

By taking the functional derivative of eq. (4) one finds

$$(\frac{d}{dt} - \Phi)\frac{\delta z_i(t)}{\delta\psi_j(s)} |_0 = -\frac{1}{2}\delta_{ij}\delta(t-s), \quad \frac{\delta z_i(t=0)}{\delta\psi_j(s)} |_0 = \frac{\delta z_i(t=\beta)}{\delta\psi_j(s)} |_0,$$

which results in

$$\begin{aligned} 2b(\beta\Phi)\frac{\delta z_i(t)}{\delta\psi_j(s)} |_0 &= \frac{1}{2}\langle T S_i^-(t) S_j^+(s)\rangle_0 \\ &= \delta_{ij} b \exp(\Phi(t-s)) \{ n_\Phi \theta(t-s) + (1+n_\Phi)\theta(s-t) \} \equiv G_{ij}^{(0)}(t-s|\Phi), \end{aligned} \tag{11}$$

the temperature Green function for noninteracting spins with the Hamiltonian $H = H_0 = \Phi\sum S_i^{(z)}$; $n_\Phi = (e^{\beta\Phi}-1)^{-1}$.

Turning back to eq. (10), one finds

$$(J^{-1}_{eff;tr})_{ij}(t-s) = J_{ij}^{-1}\delta(t-s) - G_{ij}^{(0)}(t-s|\Phi). \tag{12}$$

In the energy-momentum representation this reads

$$J_{eff}(\omega_n, \vec{q}) = \frac{J(\vec{q})}{1 - J(\vec{q})G^{(0)}(\omega_n)}, \tag{13}$$

In view of eq. (12), the Dyson equation for the whole propagator

$$G = G_0 + G_0 J_{eff} G_0 \tag{14}$$

gives ($G_0(\omega_n) = b/(i\omega_n + \Phi)$; $\omega_n = 2\pi n/\beta$)

$$G(\omega_n, \vec{q}) = \frac{b(\beta\Phi)}{\epsilon_{\vec{q}} + i\omega_n}, \tag{15}$$

where

$$\epsilon_{\vec{q}} = b(\beta\Phi)(J(0) - J(\vec{q}))$$

is the temperature-dependent energy of the spin wave excitation. Path integral (8) is easily calculated to yield a contribution to the partition function coming from the Gaussion fluctuations around the mean field. The result coincides exactly with that of ref. [1]. What is important is that our approach does not invoke any Fermi (Bose) oscillator-like representations for the spin operators which would perturb the original problem and complicate calculations [3]. A further point is that this formalism provides convenient expansions for the Green functions around the mean field.

3 Green functions

For definiteness, we shall consider the transverse Green function

$$\begin{aligned} G_{ij}(\tau,\sigma) &= \frac{1}{2}\langle T\, S_i^-(\tau)S_j^+(\sigma)\rangle \\ &= \frac{\int D\vec{\psi} e^{-S(\vec{\psi})} G_{ij}^{(0)}(\tau,\sigma|\vec{\psi}(t))}{\int D\vec{\psi} e^{-S(\vec{\psi})}} \equiv \langle\langle G_{ij}^{(0)}(\tau,\sigma|\vec{\psi}(t))\rangle\rangle, \end{aligned} \quad (16)$$

where $G_{ij}^{(0)}(\tau,\sigma|\vec{\psi}(t))$ is a Green function of noniteracting spins in an external fluctuating field $\vec{\psi}(t)$. It can be evaluated as an appropriate functional derivative of (3), which finally results in

$$\begin{aligned} G_{ij}(\tau,\sigma) = 2\langle\langle \frac{\delta z_j(\sigma)}{\delta\psi_i(\tau)} b(\vec{\psi}_j) + \sum_l \int_0^\beta dt \bar{\psi}_l(t) \frac{\delta^2 z_l(t)}{\delta\psi_i(\tau)\delta\psi_j(\sigma)} b(\vec{\psi}_l) \\ + \sum_m \int_0^\beta dt \bar{\psi}_m(t) \frac{\delta z_m(t)}{\delta\psi_i(\tau)} \left\{ \delta_{jm} z_m(\sigma) b'(\vec{\psi}_m) + z_j(\sigma) b(\vec{\psi}_m) b(\vec{\psi}_j) \right\} \\ + \sum_{lm} \int_0^\beta dt_1 dt_2 \bar{\psi}_l(t_1) \bar{\psi}_m(t_2) \frac{\delta z_l(t_1)}{\delta\psi_j(\sigma)} \frac{\delta z_m(t_2)}{\delta\psi_i(\tau)} \left\{ b(\vec{\psi}_l) b(\vec{\psi}_m) + \delta_{lm} b'(\vec{\psi}_l) \right\} \rangle\rangle, \end{aligned} \quad (17)$$

where we have introduced the abbreviation

$$b(\vec{\psi}_i) \equiv b\left(\int_0^\beta (\psi_i^{(0)} + z_i \bar{\psi}_i) dt \right).$$

If one is interested in calculating corrections to the Green function coming from the Gaussion fluctuations over the mean field, one should expand $S(\vec{\psi})$ in (16) up to the second order in $\vec{\eta}$ as is done in eq. (8). Expanding then the functional in $\langle\langle\ \rangle\rangle$ in eq. (17) in powers of $\vec{\eta}$ and calculating integrals, one will reach the goal. As is seen from (8), corrections would appear in powers of the effective interactions $J_{eff;ln}$ and $J_{eff;tr}$. The needed functional derivatives of z_i with respect to $\vec{\psi}_j$ at the stationary point are to be calculated with eq. (4).

The zeroth approximation gives

$$G_{ij}^{(0)} = 2\frac{\delta z_j(\sigma)}{\delta\psi_i(\tau)} |_0\, b(\beta|\Phi) = G_{ij}^{(0)}(\tau,\sigma|\Phi)$$

as it should be. Let us single out the contribution to the first-order term coming from the forth term in eq. (17):

$$\begin{aligned} &\langle\langle 4b^2(\beta\Phi) \sum_{km} \int_0^\beta du_1 du_2 \frac{\delta z_m(u_1)}{\delta\psi_i(\tau)} |_0 \frac{\delta z_j(\sigma)}{\delta\psi_k(u_2)} |_0 (J_{eff;tr})_{mk}(u_1-u_2) \rangle\rangle \\ &= \sum_{km} \int_0^\beta du_1 du_2 G_{ik}^{(0)}(\tau-u_1)(J_{eff;tr})_{km}(u_1-u_2) G_{mj}^{(0)}(u_2-\sigma) \\ &= G^{(0)} J_{eff} G^{(0)}. \end{aligned} \quad (18)$$

These two terms are summed up to yield eq. (14) and hence the energy of spin wave excitations.

The reason for treating the term (18) separately lies in that it is of the zeroth order in the inverse effective interaction volume v, whereas the subsequent terms in (17) starting from a one-loop expansion are of higher orders (a one-loop contribution $\sim v$) and contribute to the irreducible part of G. As v is usually regarded as a small parameter in a mean-field theory, Green function (14) is to be considered as the zeroth order mean-field Green function. It can be easily checked that (17) comprises all the diagrams that appear in the conventional temperature-dependent spin diagrammatic technique, with the interaction lines being replaced by the effective ones $(J_{ln;eff})_{ij}$ and $(J_{tr;eff})_{ij}$.

4 Conclusion

We have presented a new method for studying the thermodynamics of the quantum Heisenberg model based on the $SU(2)$ path integral technique. It is a natural path integral representation for spin systems that apparently simplifies calculations. The very same technique for the path integral over coherent states associated with more complicated groups could be easily developed. The $U(2|1)$ supergroup relevant for the Hubbard $(t-J)$ model provides quite a nontrivial example of this kind [4]. Another interesting point is that this scheme naturally allows for the existence of the time-dependent mean fields relevant to the excited eigenstates of the Hamiltonian [5]. These and related problems will be discussed elsewhere.

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