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QUANTUM PHASE OPERATOR AND STATISTICS OF THE OPTICAL FIELD

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The success in research of the photon number statistics of the optical fields supplied the interest to the photon phase operator Φ. This variable Φ was proposed to have connection to the polar decomposition [1]

$$a^+ \sim \sqrt{N}e^{-i\Phi}, \qquad a \sim e^{i\Phi}\sqrt{N}, \qquad N = a^+a, \qquad [a, a^+] = 1 \qquad (1)$$

for a, a^+ Bose-oscillator operators. There proved some mathematical difficulties in the construction (1), which were overcame in different ways in papers [2, 3, 4, 5]. The correct form of the formulae (1) was found to be [3] the following

$$a^+ = \sqrt{N}V^+, \qquad a = V\sqrt{N}, \qquad \Phi = \pi + i[\ln(1 - V^+) - \ln(1 - V)].$$

The matrix elements of Φ in the basis $N|n >= n|n >$ looks like [3, 6]

$$< m|\Phi|n >= \lim_{r\to\infty} < m|\Phi_r|n >= \begin{cases} \pi, & n = m, \\ \frac{i}{n-m} & n \neq m \end{cases} \qquad (2)$$

The finite dimensional operator Φ_r is defined in the Bose-oscillator subspace produced by the projection operator P_r:

$$P_r = \sum_{m-0}^{r-1} |m >< m|, \qquad |\varphi_r^{(k)} >= \frac{1}{\sqrt{r}} \sum_{m=0}^{r-1} \exp\left(i\varphi_r^{(k)}m\right)|m >$$

$$a_r = e^{i\Phi_r}\sqrt{N_r}, \qquad < m|\Phi_r|n >= \frac{2\pi}{r^2}\sum_{k=0}^{r-1} k \exp[i\varphi_r^{(k)}(m-n)], \qquad (3)$$

$$\varphi_r^{(k)} = \frac{2\pi k}{r}, \qquad \Phi_r|\varphi_r^{(k)} >= \varphi_r^k|\varphi_r^{(k)} >, \qquad \Phi_r = P_r\Phi P_r.$$

The $r \to \infty$ limit in $< m|\Phi_r|n >$ is taken with the fixed numbers m, n via the Chezaro procedure [7]

$$< m|\Phi_r|n >= \lim_{p\to\infty, r\to\infty} \frac{1}{r}\sum_{k=0}^{r-1} f_p\left(\frac{2\pi k}{r}\right) = \frac{i}{m-n}, \qquad (4)$$

$$f_p(x) = \sum_{\nu=-p}^{p} \left(1 - \frac{|\nu|}{p}\right)c_\nu e^{-i\nu x}, \qquad c_\nu = \frac{1}{2\pi}\int_0^{2\pi} e^{i\nu x} f(x)dx.$$

The canonical conjugation between Φ, N operators is fulfilled on the functions, equal to zero at $\varphi = 0$ and $\varphi = 2\pi$.

As far as the operator Φ is a hermitian one, the eigen function of it may be found

$$\Phi|\varphi>=\varphi|\varphi>, \qquad 0\le\varphi\le 2\pi. \tag{5}$$

Their scalar product with the eigen function of the operator N within the semiclassical approximation [3, 6] has the first term as follows

$$<n|\varphi>\sim\frac{1}{\sqrt{2\pi}}e^{in\varphi},\ n\gg 1 \tag{6}$$

The formulae (2,3) were introduced in [5] also and a certain "continuum limit" $r\to\infty$ for them was successfully used [5, 8] in some problems of coherent optics.

We are going to make here an additional calculation, that gives a new illustration of the application of the formulae (5,6). In order to step forward in this question we consider the correction term for the approximated formula (6) and try to appreciate the contribution of this term to the dispersion of the phase in various states of optical field.

Let us start with the formula of matrix element [3, 6, 10] with the unit decomposition over the complete set $|\varphi>$

$$<m|W(\Phi)|n>=\int_0^{2\pi}W(\varphi)<m|\varphi><\varphi|n>d\varphi$$

for the arbitrary function W of the phase operator Φ and introduced the auxiliary operator G

$$\begin{aligned}<m|W(\Phi)|n> &= \frac{1}{2i\pi}\oint_{|z|=1}W(z)G_{mn}(z)dz,\\ G_{mn}(z) &= <m|G|n>, \qquad (z-\Phi)G=1.\end{aligned}\tag{7}$$

Operator G is the resolvent for the phase operator Φ and its matrix elements are determined by the equation

$$zG_{mn}-\sum_{k=0}^{\infty}\Phi_{mk}G_{kn}(z) = \delta_{mn}, \qquad \Phi_{mn}=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,\varphi e^{i\varphi(m-n)}. \tag{8}$$

The formula of the integral representation of Φ_{mn} in (8) corresponds the determination (2) of the operator Φ.[1] Our purpose is to calculate G from (8) in such a way that the correction terms for the approximation (6) will be found. It is useful to introduce an operator F, which satisfies the equation

$$zF_{mn}-\sum_{k=-\infty}^{\infty}\Phi^0_{mk}F_{kn}(z)=\delta_{mn}, \qquad F_{mn}(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi(m-n)}d\varphi}{z-\varphi}. \tag{9}$$

This operator F is the resolvent for the operator Φ^0

$$(z-\Phi^0)F=1$$

for the eigen functions of which the quality (6) is exact. It is seen that the equation (9) is close to the equation (8) for large m,n. Therefore the operator F may be used as the first approximation for G when $m,n\gg 1$. The indexes m,n of F,Φ^0 may be negative while the indexes of G,Φ are positive. We notice that the difference between the matrix elements Φ_{mn},Φ^0_{mn} consists in the values of indexes $0\le m,n<\infty$ and $-\infty<m,n<\infty$ correspondingly [2], so the same formula Φ_{mn} in (8) may be used both for Φ and Φ_0 as far as $0\le m,n<\infty$. Therefore we combine (8) and (9) to the new equation

$$z(G_{mn}-F_{mn})-\sum_{k=0}^{\infty}\Phi_{mk}(G_{kn}-F_{kn})=-\sum_{k=-\infty}^{-1}\Phi^0_{mk}F_{kn}.$$

After multiplying this equation by $G_{mk}(m,k=0,1,2,3,\ldots)$ and summing up the index k the last equation may be reduced to the form

$$\begin{aligned}\sum_{k=0}^{\infty}G_{mk}\left(zG_{kn}-\sum_{l=0}^{\infty}\Phi_{kl}G_{ln}\right) - \sum_{k=0}^{\infty}\left(F_{kn}zG_{mk}-\sum_{l=0}^{\infty}G_{mk}\Phi_{kl}F_{ln}\right) &=\\ = -\sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1}G_{mk}\Phi^0_{kl}F_{lk}.\end{aligned}$$

The δ_{kn} factor is present in the first brackets here because of the equation (8), and the same is true for the second brackets because of the equality

$$\sum_{k,l=0}^{\infty}(G_{mk}\Phi_{kl}F_{ln}-G_{ml}\Phi_{lk}F_{kn})=0.$$

[1] It was pointed out in [6], that the other (except $p=1$) powers of Φ^p, $p=2,3\ldots$ has no such representation.

[2] The integral representation (8) $(\Phi^0_{mn})^p=\frac{1}{2\pi}\int_0^{2\pi}\varphi^p d\varphi\exp(i\varphi(m-n))$ is valid for any $p=1,2,3,\ldots$ in the contrary to the case Φ^p_n, see note[1] above.

Now the equation for G_{mn} is found

$$G_{mn} - \sum_{k=0}^{\infty} F_{kn}\left(zG_{mk} - \sum_{l=0}^{\infty} G_{ml}\Phi_{kl}\right) = G_{mn} - F_{mn} = - \sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1} G_{mk}\Phi^0_{kl}F_{lk}.$$

This equation is exact and an iterative procedure for the last equation is possible so that

$$G^{(1)}_{mn} = F_{mn},$$
$$G^{(2)}_{mn} = F_{mn} - \sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1} F_{mk}\Phi^0_{kl}F_{ln},$$
$$\ldots\ldots\ldots\ldots\ldots$$

Thus we have the approximative formulae for operator G through the operator F, which has – in the contrary to G – a simple analytical representation $F_{mn}(z)$. Really G_{mn} function with the numbers $0 \le m, n < \infty$ corresponds to the Fock space of the Bose-oscillator with the spectrum on $[0, \infty)$, while F_{mn} with $-\infty < m, n < \infty$ corresponds to the quantum system with the spectrum on $(-\infty, +\infty)$ such as the "infinit spin". If we take $G = F$ as the zeroth approximation of the iterative procedure, it means that we ignore the vacuum state on the left side of $[0, \infty)$ similary to the quasiclassical approximation (6) and to the "continuum limit" construction (see the discussion later in this paper).

In this way we receive the first correction term D_{mn} for the matrix element $< m|W|n >$

$$< m|W(\Phi)|n >= \frac{1}{2\pi}\int_0^{2\pi} W(\varphi)e^{i(m-n)\varphi}d\varphi + D^{(2)}_{mn}, \tag{10}$$

$$D^{(2)}_{mn} = \frac{1}{(2\pi)^2}\sum_{k=0}^{\infty}\sum_{l=-\infty}^{-1}\Phi^0_{kl}\int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \frac{W(\varphi) - W(\varphi')}{\varphi - \varphi'}\, e^{i[(m-k)\varphi + (l-n)\varphi']}.$$

and it will be seen later that the term D_{mn} is really a small corrective term corresponding to the main value F_{mn}. It can be shown, that D_{mn} is the contribution of the integral around the cut [0,1] in the $z-$ plane

in the integral over the unit circle. The first term in (10) corresponds to the approximation (6) and as it was explained in [3, 6] represents the contribution of the pole of the function G. Some simplification of $D_{m,n}$ can be reached by the use of the operations

$$\Phi^0_{kl} = \frac{i}{l-k}, \qquad l \to -l, \qquad \sum_{l=-\infty}^{-1} \to \sum_{l=1}^{\infty}, \qquad \frac{1}{k+l} = \int_0^{\infty} ds\; e^{-s(k+l)}$$

so that the final formula is

$$D^{(2)}_{mn} = \frac{i}{(2\pi)^2}\int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \frac{W(\varphi) - W(\varphi')}{\varphi - \varphi'} \times \frac{\ln\frac{1-\exp(-i\varphi)}{1-\exp(-i\varphi')}}{\exp(i\varphi) - \exp(i\varphi')} \cdot e^{i[(m+1)\varphi - n\varphi']}.$$

The most important variables for optical experiments are sin – and cos – functions of the phase operator Φ. That is why we choose $W(\Phi) = \exp i\Phi$ and in the case of m, $n >> 1$ the approximated formula for the matrix element is received in the form

$$< m|e^{i\Phi}|n > \simeq \delta_{m+1,n} + d_{mn}, \tag{11}$$

$$d_{mn} = \frac{i}{(2\pi)^2}\int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi' \frac{\ln\frac{1-\exp(-i\varphi)}{1-\exp(-i\varphi')}}{\varphi - \varphi'} \cdot e^{i[(m+1)\varphi - n\varphi']} \simeq \simeq \frac{\ln^2(\frac{m+1}{n})}{4\pi^2(n-m-1)}, \qquad m, n \gg 1.$$

To define an importance of the supplementary term as compared with (6) we consider the variance of the phase operator cosine for the occupation number, Gauss and coherent states. First of all for arbitrary state $|\psi >$ which can be expended to the number states

$$|\psi >= \sum_{n=0}^{\infty} b_n|n >,$$

when b_n is real, we shall write the general phormula for cosine of the phase operator using the formula (11)

$$D(\cos\Phi) = <\cos^2\Phi> - <\cos\Phi>^2 = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} b_n b_{n+2} +$$
$$+ \frac{1}{8\pi^2}\sum_{mk}^{\infty}\left\{\frac{1}{4\pi^2}\sum_{l}^{\infty}\frac{\ln^2\frac{l}{m+1}\ln^2\frac{k}{l+1}}{(k-l-1)(l-m-1)} +\right.$$
$$+ \left.\frac{\ln^2\frac{k}{m+2}}{k-m-2} + \frac{\ln^2\frac{k-1}{m+1}}{k-m-2}\right\} b_n b_m - \left[\sum_{n=0}^{\infty} b_n b_{n+1} +\right.$$
$$+ \left.\frac{1}{4\pi^2}\sum_{m,k=0}^{\infty} b_k b_m \cdot \frac{\ln^2\frac{k}{m+1}}{k-m-1}\right]^2, \qquad k,m,l \gg 1. \quad (12)$$

At first we shall find the variance of the phase operator cosine for the occupation number (Fock) states $N|n>= n|n>$, which is the function of the occupation number. Taking into account $b_n = 1$ is only nonzero for these states, we have

$$D(\cos\Phi)_n = \frac{1}{2} - \frac{1}{16\pi^4}\left\{\frac{1}{2}\sum_{k}^{\infty}\frac{\ln^2\frac{k}{n+1}\cdot\ln^2\frac{n}{k+1}}{(n-k)^2-1} +\right.$$
$$+ \left.\pi^2\left(\ln^2\frac{n}{n+2} + \ln^2\frac{n-1}{n+1}\right) + \ln^4\frac{n}{n+1}\right\}, \; k \gg 1. (13)$$

In the Fig.1 one can see the variance depending on the occupation number. For the number states (also the vacuum) phase operator defined by (6) has regular distribution, which is represented by solid line in Fig.1. In other words, the variance of the phase operator cosine is equal $\frac{1}{2}$ for approximation (6) and is not constant for (11) approximation described by the dotted line. It is important to note that the photon number n is the smaller the larger the difference between (6) and (11) definitions. The same difference was found in [9] between the results [2] (which are similar to those presented here) and [5].

For the Gauss states the contribution $\triangle$ of the term d_{mn}(11) in the variance

$$D(\cos\Phi)_G = \frac{1}{2} - \triangle, \qquad (14)$$

$$\triangle = \frac{1}{16\pi^4}\frac{1}{1+n_0}\sum_{n}^{\infty}\left(\frac{n_0}{1+n_0}\right)^n\left\{\frac{1}{2}\sum_{k}^{\infty}\frac{\ln^2\frac{k}{n+1}\cdot\ln^2\frac{n}{k+1}}{(n-k)^2-1} +\right.$$
$$\left. +\pi^2\left(\ln^2\frac{n}{n+2} + \ln^2\frac{n-1}{n+1}\right)\right\} - \frac{1}{16\pi^4}\frac{1}{(1+n_0)^2}\times$$
$$\times\left\{\sum_{n}^{\infty}\left(\frac{n_0}{1+n_0}\right)^n\ln^2\frac{n}{n+1}\right\}^2, \quad n,k \gg 1, \quad n_0 = Sp(e^{-\beta\omega a^+ a})$$

is smaller than one for the occupation number states (Fig.2). It means that, the thermal fluctuations suppress the influence of this term in this case.

One can show by the similar calculation that the contribution of the d_{mn} term to the variance of $\cos\Phi$ can not be distinguished for the coherent states. Thus, the cases above considered show that the defect of the approximation (6) can be observed only for the occupation number states as the occupation number is small.

So we have the following situation with Φ variable. The formulae (5-14) are based on the calculations [3, 6, 10] of the eigen functions (5) for the operator $< m|\Phi|n >= \lim < m|\Phi_r|n >, \quad r\to\infty$. The "continuum limit" treatment $\varphi_r^{(k)} \to \varphi$ is based on the formula for average value of any function F of Φ in quantum "physical" state ρ [5, 8]

$$< W(\Phi) >= \lim_{r\to\infty}\sum_{m=0}^{r-1} < \varphi_r^{(m)}|\rho|\varphi_r^{(m)} > W(\varphi_r^{(m)}) = \int_0^{2\pi} W(\varphi)\Pi(\varphi)d\varphi, (15)$$
$$\Pi(\varphi) = \frac{1}{2\pi}\sum_{m,n=0}^{\infty} < m|\rho|n > \exp[i(n-m)\varphi].$$

The $r\to\infty$ limit in this formulae may be understood in the way similar to (4) because Toeplitz matrix

$$M_r = \int_0^{2\pi} e^{i(n-m)\varphi}W(\varphi)d\varphi, \qquad n,m = 1,\dots r$$

used in (15), is approximately (with $r\to\infty$) close [12] to the matrix

$$L_r = \frac{1}{r}\sum_{k=1}^{r} e^{2i\pi\frac{k}{r}(n-m)}W_p(2\pi\frac{k}{r})$$

defined for Chezaro functions W_p. If we omit index p in the last symbol and substitute $W_p \to W$ we receive (15). It was shown [6, 8, 9, 11] that the approaches [2, 3, 6, 10] and [5, 8, 11] for the phase variable Φ give the same result for quantum state of the Bose-oscillator with the large occupation number $n \gg 1$ and different result for small n. The same conclusion follows from the formulae (12-14) of this paper. The origin of this discrepancy seems to be as it was also assumed in [9, 11], in the different ways of taking a limit $r \to \infty$ in the formulae(3).

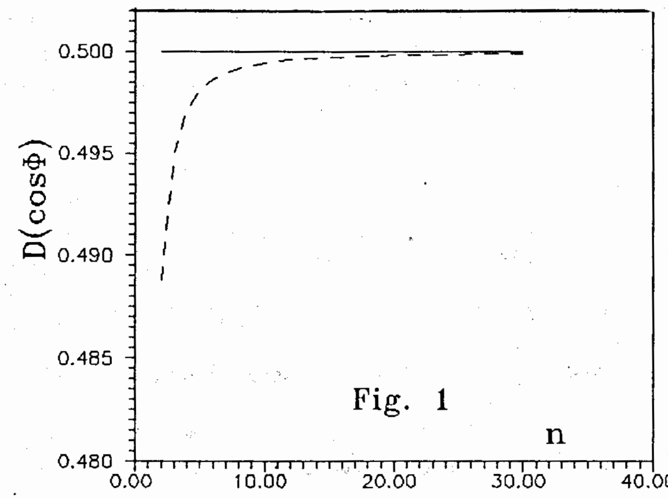
It was noticed also that for the slightly excited oscillator the phase distribution [2, 3, 6, 10] has an anisotropy and it is phase fluctuations are smaller then those in [5, 8, 11]. Namely the advantage of formula (15) is that it gives the isotropy distribution of φ values in $[0, 2\pi]$ for any $n-$ photon state. This advantage follows from the definition (15) as the "continuum limit" procedure for average value $< W(\Phi) >$ and this fact was underlined earlier [9, 11].

The eigen vectors $|\varphi >$ in (5) shows the anisotropy in φ in the scalar products $< n|\varphi >$ for the slightly excited states of the field oscillator [6, 9, 11]. This anisotropy may be considered as a certain "repulsion" between the phase values in $[0, 2\pi]$. This case is close to that one in the theory of complex molecules and heavy nuclei [13], where the energy levels also demonstrate the extraordinary "repulsion". Such a phenomena shows, that the canonical quantum theory may give an unusual behavior of the system when the semiclassical approximation is invalid. This situation is known over the title "quantum chaos" due to the analogy with chaotization for the irregular motion in classical mechanic. Bose-oscillator is a simple system in p, q variables, but it is not so in N, Φ variables. The reason is that the combination of requirements to have the polar decomposition and the canonical conjugation for N, Φ complicate the situation for these variables near the ground state. All these problems disappears for excited oscillator, where the formulae (5-14) and (15) give the same result.

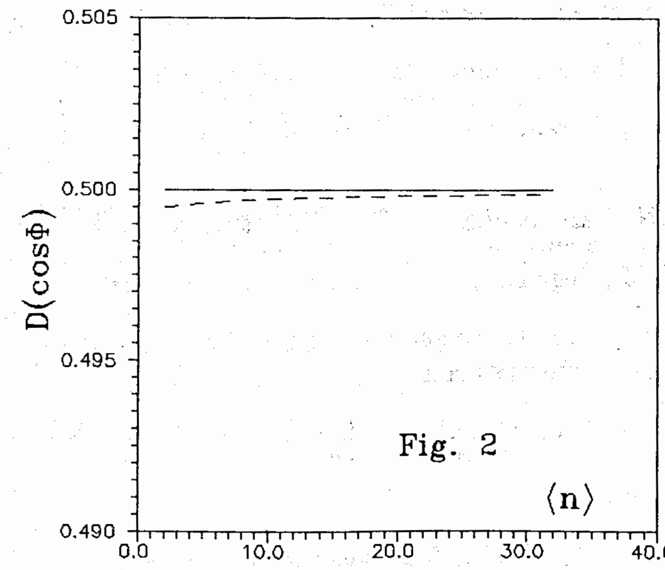
To see the role of such "repulsion" it is necessary to examine the states with the small occupation number. At the same time, it is known [14] that the normalized term g of the factorial decomposition for the one-mode optical field

$$g = \frac{< n^2 > - < n >}{< n >^2}$$

is equal to $1 - (n)^{-1}$ for the occupation number state $|n >$. The sub-



1. Dependence of the variance of the phase operator cosine upon the occupation number for the number occupation states. Solid line corresponds to the approximation (2), dashed line - to the (7) approximation.



2. Dependence of the variance of the phase operator cosine upon the average occupation number for Gauss states. Solid line corresponds to the approximation (2), dashed line - to the (7) approximation.

Poisson effect for photons is associated with the values $g < 1$. Therefore the term d_{mn} in (11), which represents the correction of the approximation (6), may be valid for the case of the strong sub-Poisson effect in a weak optical field.

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Квантовый оператор фазы
и статистика оптического поля

Введен поправочный член к полуклассическому приближению матричных элементов оператора фазы. Рассчитан вклад этого дополнительного члена в дисперсию косинуса оператора фазы в представлении чисел заполнения, а также для гауссовских и когерентных состояний. Обсуждается причина различия между результатами данной работы и теми, которые следуют из непрерывного предела. Ожидается, что поправочный член существен в случае сильного субпуассоновского эффекта в слабых полях.

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Quantum Phase Operator
and Statistics of the Optical Field

The corrective term for the semiclassical approximation of the phase operator matrix elements is defined. An influence of this supplementary term on the variance of the phase operator cosine is determined for the occupation number, Gauss, coherent states. The origin of the difference between these results and «continuum limit» treatment is discussed. The corrective term is expected to be valid for the case of the strong sub-Poisson effects in the weak field.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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