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INSTABILITIES IN MHD TURBULENCE GENERATED BY WEAK ANISOTROPY

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1. Introduction

Unsolved problems in theory of turbulence permanently evoke new and new attempts to create effective theoretical approaches which could help to understand the nature of turbulent phenomena. Besides other approaches made over the last fifteen years, renormalization group technique (RNG) which turned out to be so fruitful in critical phenomena theory, has been widely applied in theory of developed turbulence. Apart from the critisizm addressed to the RNG by some theoreticians working in this area, it gave valuable contributions to elucidation of many phenomena in turbulent fluids ([1, 2, 3, 4], e.g.) based on the first principles (i.e. on the Navier-Stokes equation).

Two realisations of the RNG approach exist - the Wilson RNG [5] and the quantum-field RNG [6]. The Wilson RNG is physically more transparent but the quantum field RNG is technically more convenient and gives the possibility to extend the calculations to higher orders of perturbative theory. The physical meaning of the Wilson RNG approach consists in successive eliminations of short wave and high frequency degrees of freedom of a nonlinear field system to obtain effective equations describing its large scale behaviour. If Wilson RNG procedure of successive integrations over small scales with a consequent rescaling leads to a theory with parameters which do not change with recurent applying of this procedure, than a stable fixed point of RNG transformations exists. Unlike this procedure the quantum field RNG investigates the infrared asymptotic behaviour of effective variables such as turbulent viscosity, turbulent magnetic Prandtl number, etc.

The RNG approach has been used to investigate a fully developed turbulence governed by the stochastic Navier-Stokes equation with external random force **f** [1, 2, 7]. The existence of a stable fixed point has been demonstrated. The dissipative term in the effective equation of motion considerably differs from that of the Navier-Stokes equation. Molecular viscosity is replaced here by the wavenumber dependent effective turbulent viscosity. Due to the Galilean invariance of the Navier-Stokes equation its nonlinear term does not change essentially. The energy spectrum depends on the random force spectrum $\langle | \mathbf{f}^{\mathbf{v}}(\mathbf{k}) |^2 \rangle \sim k^{1-2\epsilon}$ ($k \equiv | \mathbf{k} |$), and for $\epsilon = 2$ it coincides with the Kolmogorov spectrum. This value corresponds to the case where the random force amplitude has the dimensionality of the energy injection rate.

The developed turbulence driven by a weak anisotropic random force has been investigated and the stable fixed point of RNG transformations has been found [8]. It corresponds to the Kolmogorov scaling. Additional terms describing effective anisotropic viscosity are generated by nonlinear interaction. These terms are small if the anisotropy is weak.

The isotropic MHD turbulence has been studied by both approaches, the Wilson RNG [9] and the quantum field RNG [10]. The case of a strong magnetic forcing has been considered in these papers. The existence of two stable fixed points has been determined by free parameters of the theory. The effective Lorentz force in the kinetic fixed point becomes unimportant for the large scale behaviour of correlation functions and gives only some non-analytical corrections to these functions. The energy spectrum remains of the Kolmogorov form. The effective magnetic Prandtl number is very large in the magnetic fixed point and the energy spectrum is not of the Kolmogorov form.

The influence of Gaussian random forces with anisotropic two-point correlator on the MHD turbulence is investigated in the present paper. It turns out that nonlinear interactions generate effective anisotropic viscosities and additional vertices which correspond to anisotropic Lorentz forces. An important discrepancy compared to the above mentioned cases consists in the fact that these forces can play significant role in effective equations of motion. The strength of this effect depends on the power of the singularity in the magnetic force spectrum $\langle | \mathbf{f}^{\mathbf{b}}(\mathbf{k}) |^2 \rangle \sim k^{1-2a\epsilon}$ and grows with increasing the value of a. This increase does not break the Kolmogorov spectrum if $a < 0.65$. For the range $a > 0.65$, the Kolmogorov spectrum is preserved only if the value of anisotropy is strongly limited. Otherwise the magnetic field does not play the role of a passive admixture and the Kolmogorov regime does not exist. It is necessary to examine the existence of a new stable fixed point and to study the corresponding critical scaling regime in this region.

2. The functional formulation of the problem

The MHD equations for incompressible conductive turbulent fluid governed by random forces have the following form (see e.g. [10]):

$$\partial_t \mathbf{v} = \nu_0 \nabla^2 \mathbf{v} - (\mathbf{v}\nabla)\mathbf{v} + (\mathbf{b}\nabla)\mathbf{b} - \nabla p + \mathbf{f}^{\mathbf{v}}, \quad \nabla.\mathbf{f}^{\mathbf{v}} = 0, \quad \nabla.\mathbf{v} = 0, \quad (1)$$

$$\partial_t \mathbf{b} = u_0 \nu_0 \nabla^2 \mathbf{b} - (\mathbf{v}\nabla)\mathbf{b} + (\mathbf{b}\nabla)\mathbf{v} + \mathbf{f}^{\mathbf{b}}, \quad \nabla.\mathbf{f}^{\mathbf{b}} = 0, \quad \nabla.\mathbf{b} = 0. \quad (2)$$

where ν_0 is the molecular viscosity, $u_0\nu_0$ is the molecular magnetic diffusivity, u_0 is the inverse magnetic Prandtl number, p is pressure. The first equation is the well-known Navier-Stokes equation for transverse velocity field $\mathbf{v} \equiv \mathbf{v}(\mathbf{x}, t)$ with additional nonlinear contribution of the Lorentz force. The second equation for magnetic field $\mathbf{b} \equiv \mathbf{b}(\mathbf{x}, t)$ follows from the Maxwell equations for a continuous medium. The magnetic field is measured in Alfven velocity units. The random forces are assumed to have Gaussian distribution with $\langle \mathbf{f} \rangle = 0$ and with given 2×2 diagonal matrix of two-point correlators $\langle \mathbf{f}\,\mathbf{f} \rangle$. The problem (1), (2) is equivalent to the quantum field theory (QFT) with double number of the fields $\Phi = (\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}')$ in accordance with the general theorem of stochastic quantization [7]. The corresponding action has the following form:

$$\begin{aligned} S(\Phi) = & \frac{1}{2}\left[\mathbf{v}'\langle \mathbf{f}^{\mathbf{v}}\, \mathbf{f}^{\mathbf{v}} \rangle \mathbf{v}' + \mathbf{b}'\langle \mathbf{f}^{\mathbf{b}}\, \mathbf{f}^{\mathbf{b}} \rangle \mathbf{b}'\right] + \\ & + \mathbf{v}'\left[-\partial_t \mathbf{v} + \nu_0 \nabla^2 \mathbf{v} - (\mathbf{v}\nabla)\mathbf{v} + (\mathbf{b}\nabla)\mathbf{b}\right] \\ & + \mathbf{b}'\left[-\partial_t \mathbf{b} + u_0\nu_0 \nabla^2 \mathbf{b} + (\mathbf{b}\nabla)\mathbf{v} - (\mathbf{v}\nabla)\mathbf{b}\right] \end{aligned} \quad (3)$$

The integration over $(\mathbf{x},t)$ and summation over the vector indices are implied in (3) and in the similar expressions. The explicit form of the matrix $\langle \mathbf{f}\,\mathbf{f} \rangle$ will be specified later. The auxiliary fields $\mathbf{v}', \mathbf{b}'$ are vectorial and transverse. As usual in QFT the action (3) is considered to be unrenormalized with the bare parameters marked by the subscript "0"; symbols without this subscript denote renormalized parameters. The main objects of the study are the Green functions G^{Φ} of the fields Φ or, equivalently, the correlation functions and the response functions in the terminology of the original problem (1),(2):

$$G^{\Phi} \equiv < \Phi\Phi ...\Phi >= \int \mathcal{D}[\Phi]\, \Phi\Phi...\Phi\, \exp[S(\Phi)]\,, \qquad \mathcal{D}[\Phi] \equiv \mathcal{D}\mathbf{v}\mathcal{D}\mathbf{v}'\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{b}'. \tag{4}$$

Here $\mathcal{D}[\Phi]$ denotes functional measure of the integration over the fields Φ with all normalization coefficients. The equivalence of (1),(2) and QFT (3) means that Green functions determined by (4) coincide with Green functions, which are obtained directly averaging the solution of the equations (1),(2) over forces $\mathbf{f}$ with a weight of $\exp\left[-\frac{1}{2}\mathbf{f}\,\langle \mathbf{f}\,\mathbf{f}\rangle^{-1}\,\mathbf{f}\right]$.

Now, we choose the explicit form of the matrix $\langle \mathbf{f}\,\mathbf{f}\rangle$ and we assume for simplicity that the non diagonal elements $\langle \mathbf{f}^{\mathbf{v}}\mathbf{f}^{\mathbf{b}}\rangle$ vanish. Then the anisotropic energy forcing is introduced by the tensors $\langle f_j^{\mathbf{v}} f_s^{\mathbf{v}}\rangle$ and $\langle f_j^{\mathbf{b}} f_s^{\mathbf{b}}\rangle$ which can be expressed in terms of transverse projectors $P_{js} = \delta_{js} - k_j k_s k^{-2}$ and $P_{jl} n_l n_m P_{ms}$, where $\mathbf{n}$ is the unit vector which represents an unaxial anisotropy. The weak anisotropy, which we are interested in, is controlled by the small free dimensionless parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ ($|\,\alpha_i\,| \ll 1$) which are included in the two-point correlators as it has been made in papers devoted to RNG application to anisotropically driven turbulent transport [11], [12]:

$$\begin{aligned}\left\langle f_j^{\mathbf{v}}(\hat{\mathbf{k}},\mathbf{n}) f_s^{\mathbf{v}}(\hat{\mathbf{k}}',\mathbf{n})\right\rangle &= g_{\mathbf{v}0}\nu_0^3 u_0 \delta(\hat{\mathbf{k}}+\hat{\mathbf{k}}') D_{js}^{\mathbf{vv}}(\mathbf{k},\mathbf{n}),\\ \left\langle f_j^{\mathbf{b}}(\hat{\mathbf{k}},\mathbf{n}) f_s^{\mathbf{b}}(\hat{\mathbf{k}}',\mathbf{n})\right\rangle &= g_{\mathbf{b}0}\nu_0^3 {u_0}^2 \delta(\hat{\mathbf{k}}+\hat{\mathbf{k}}') D_{js}^{\mathbf{bb}}(\mathbf{k},\mathbf{n}),\end{aligned} \tag{5}$$

$$\begin{aligned} D_{js}^{\mathbf{vv}} &= c\,(P_{js} + n_l n_m[\alpha_1(\delta_{lm} - P_{lm})P_{js} + \alpha_2 P_{jl} P_{ms}])\, k^{4-2\epsilon-d},\\ D_{js}^{\mathbf{bb}} &= c\,(P_{js} + n_l n_m[\alpha_3(\delta_{lm} - P_{lm})P_{js} + \alpha_4 P_{jl} P_{ms}])\, k^{4-2a\epsilon-d}, \end{aligned} \tag{6}$$

where $\hat{\mathbf{k}} \equiv \mathbf{k}, \omega$, the constant $c = d(d+2)(4\pi)^{d/2}\Gamma(d/2)$ (Γ is gamma function) and factors $\nu_0^3 u_0$, $\nu_0^3 u_0^2$ have been extracted for convenience of calculations. Here $\mathbf{D}^{\mathbf{vv}}$ and $\mathbf{D}^{\mathbf{bb}}$ are the most general parametrizations of the nonhelical tensors which satisfy the symmetry relations:

$$D_{js} = D_{sj}, \quad \mathbf{D}(\hat{\mathbf{k}}) = \mathbf{D}(-\hat{\mathbf{k}}), \quad \mathbf{D}(\mathbf{n}) = \mathbf{D}(-\mathbf{n}).$$

The a and ϵ are free parameters of the theory and d is the dimension of the $\mathbf{k}$ space. The constants $g_{\mathbf{v}0}$, $g_{\mathbf{b}0}$ play the role of coupling constants. For $\epsilon = 2$ their dimensionality is the same as that of energy injection rate. Note that parameter ϵ is independent of the space dimension. In the used dimensional regularisation it plays the same role as the analogical parameter in the known $(4-\epsilon)$ Wilson scheme [5].

One can consider more general case when the cross-correlator $< f_j^{\mathbf{v}} f_s^{\mathbf{b}} >$ is non zero. Than, in the case of non helical anisotropic MHD, the correlator $< f_j^{\mathbf{v}} f_s^{\mathbf{b}} >$ is parametrized as

$$< f_j^{\mathbf{v}} f_s^{\mathbf{b}} >= g_0 \nu_0^3 u_0^{3/2}\, c\, i\, [\,\epsilon_{jsm}\, k_m + \alpha_5\, \epsilon_{jml} k_m n_s n_l]\, k^{3-\epsilon-a\epsilon-d}, \tag{7}$$

where $g_0 = \sqrt{g_{\mathbf{v}0} g_{\mathbf{b}0}}$ [10] (ρ and α_5 are new free parameters of the theory). In ref. [10] has been shown that in the isotropic MHD turbulence ($\alpha_5 = 0$) the ultraviolet divergences of all Green functions, which are obtained in one loop approximation, does not depend on parameter ρ, because there are not any divergences proportional to the antisymmetric tensor ϵ_{jml} in (7). Consequently, stability of fixed points of RNG is independent on ρ in this approximation. The same fact is valid also for the anisotropic part of the cross-correlator with small parameter α_5. Therefore, the case of $< f_j^{\mathbf{v}} f_s^{\mathbf{b}} >= 0$ is considered in the next.

Green functions (4) are calculated by means of Feynman diagramatic perturbative technique. The matrix of propagators $\Delta = \mathbf{K}^{-1}$ (the lines in Feynman diagrams) can be obtained from the squared term $(1/2)\Phi\mathbf{K}\Phi$ of the action (3). All non-vanishing propagators with anisotropic terms are presented in Appendix I.

3. Renormalization analysis of anisotropic MHD

Calculations show that perturbation series for correlation functions possess terms like $(gk^{-2\epsilon})^n$, where n is the order of expansion. These terms are too large in the infrared range $k \to 0$ for positive ϵ (i.e. also for the physical value of $\epsilon = 2$) and therefore summation of infinite series is necessary. It is the same situation as in the critical phenomena theory. This nontrivial "infrared" problem can be overcome using RNG technique. In order to use this approach, the values of ϵ close to zero must be considered. For ϵ exactly equal to zero the infrared singularities disappear but ultraviolet divergences ($k \to \infty$) occur. As consequence the problem of ultraviolet divergences elimination has to be overcame. In limiting case $\epsilon = 0$ the power of ultraviolet divergences does not depend on the order of the diagramatic expansion and they can be eliminated by the procedure of the ultraviolet renormalization [13]. Subsequently the RNG technique can be applied and the former physical value of the parameter ϵ can be assigned again.

In dimensional regularization the ultraviolet divergences manifest themselves like poles of ϵ. They can be eliminated adding the appropriate counterterms to the "basic" action, which can be obtained from (3) replacing the bare parameters e_0 by the renormalized ones e: $e_0 \to e\mu^{d^p_{e_0}}$. Here μ is a scale setting parameter. Momentum dimensions $d^p_{e_0}$ will be defined later. The counterterms are formed by the superficial ultraviolet divergences which are present in one-particle irreducible Green functions (see also [13]). If these counterterms have the same form as the terms of action (3), the ultraviolet divergences can be eliminated redefining the

parameters of the original QFT and the theory becomes to be multiplicatively renormalizable. To make the considered (3) theory by multiplicative renormalizable it is neccessary to add formally new terms to the basic action:

$$\begin{array}{c}\nu\chi_1\mathbf{v}'(\mathbf{n}\nabla)^2\mathbf{v}, \quad \nu\chi_2(\mathbf{n}\mathbf{v}')\nabla^2\mathbf{n}\mathbf{v}, \quad u\nu\chi_3\mathbf{b}'(\mathbf{n}\nabla)\mathbf{b}, \quad u\nu\chi_4(\mathbf{n}\mathbf{b}')\nabla^2\mathbf{n}\mathbf{b}, \\ \lambda_1(\mathbf{v}'\mathbf{b})(\mathbf{n}\nabla)\mathbf{n}\mathbf{b}, \quad \lambda_2(\mathbf{n}\mathbf{v}')(\mathbf{n}\nabla)\mathbf{b}, \quad \lambda_3(\mathbf{n}\mathbf{v}')(\mathbf{b}\nabla)\mathbf{n}\mathbf{b},\end{array} \tag{8}$$

where χ_j and λ_j are some additional constants. In the following text these and any other dimensionless constants (renormalized coupling constants $g_\mathbf{v}, g_\mathbf{b}$, Prandtl number u) will commonly be referred as charges. The corresponding unrenormalized action S^A is of the following form

$$\begin{aligned} S^A(\Phi) &= S(\Phi) + \mathbf{v}'\left[\nu_0\chi_{10}(\mathbf{n}\nabla)^2\mathbf{v} + \nu_0\chi_{20}\mathbf{n}\nabla^2\mathbf{n}\mathbf{v}\right. \\ &+ \left.\lambda_{10}\mathbf{b}(\mathbf{n}\nabla)\mathbf{n}\mathbf{b} + \lambda_{20}\mathbf{n}(\mathbf{n}\nabla)\mathbf{b}^2 + \lambda_{30}\mathbf{n}(\mathbf{b}\nabla)\mathbf{n}\mathbf{b}\right] \\ &+ \mathbf{b}'\left[u_0\nu_0\chi_{30}(\mathbf{n}\nabla)^2\mathbf{b} + u_0\nu_0\chi_{40}\mathbf{n}\nabla^2(\mathbf{n}\mathbf{b})\right]. \end{aligned} \tag{9}$$

Here $S(\Phi)$ is the action (3).

The classification of the ultraviolet divergences is possible in terms of power counting. The stochastic MHD considered above is invariant under two independent scale transformations - in time and length, with all the physical quantities, on which the action depends, transforming in accordance to their dimension. Formally, it can be described by momentum d^p, frequency d^ω and overall $\overline{d} = d^p + 2d^\omega$ scaling dimensions for all parameters and fields:

$$\begin{aligned} d^\omega_\mathbf{v} &= d^\omega_\mathbf{b} = 1, \quad d^p_\mathbf{v} = d^p_\mathbf{b} = -1, \quad d^\omega_{\mathbf{v}'} = d^\omega_{\mathbf{b}'} = -1, \\ d^p_{\mathbf{v}'} &= d^p_{\mathbf{b}'} = d+1, \quad d^\omega_{\nu_0} = d^\omega_\nu = 1, \quad d^p_{\nu_0} = d^p_\nu = -2, \\ d^\omega_{g_{\mathbf{b}0}} &= ad^\omega_{g_{\mathbf{v}0}} = 0, \quad d^p_{g_{\mathbf{b}0}} = ad^p_{g_{\mathbf{v}0}} = 2a\epsilon, \quad d^\omega_\mu = 0, \\ d^{\omega,p}_{g_\mathbf{v}} &= d^{\omega,p}_{g_\mathbf{b}} = d^{\omega,p}_{u_0} = d^{\omega,p}_{\chi_{j0}} = d^{\omega,p}_{\lambda_{m0}} = d^{\omega,p}_u = d^{\omega,p}_{\chi_j} = d^{\omega,p}_{\lambda_m} = 0, \\ d^p_\mu &= 1, \quad j = 1,2,3,4, \qquad m = 1,2,3. \end{aligned}$$

The superficial ultraviolet divergences are simple polynomials of the momentum and the frequency. The power of these polynomials is determined by the formal ultraviolet divergence index δ. The total scaling dimension of one-particle irreducible Green function Γ^{Φ} with N_Φ external legs is [10]

$$\overline{d_\Gamma} = d + 2 - N_\Phi d_\Phi, \quad N_\Phi d_\Phi = N_{\rm v} d_{\rm v} + N_{\rm b} d_{\rm b} + N_{\rm v'} d_{\rm v'} + N_{\rm b'} d_{\rm b'}. \quad (10)$$

In the logarithmic theory ($\epsilon = 0$) δ is $\delta = \overline{d_\Gamma}$. If $\delta \geq 0$ the diagram possesses ultraviolet divergences. It has been demonstrated [10] that in isotropic MHD only one-particle irreducible Green functions $\Gamma^{\rm v'v}$, $\Gamma^{\rm b'b}$ ($\delta_{\Gamma^{\rm v'v}} = \delta_{\Gamma^{\rm b'b}} = 2$) and vertex $\Gamma^{\rm v'bb}$ ($\delta_{\Gamma^{\rm b'bb}} = 1$) possess the superficial ultraviolet divergences. Other Green functions $\Gamma^{\rm v'b}$, $\Gamma^{\rm b'v}$, $\Gamma^{\rm v'vv}$, $\Gamma^{\rm v'vb}$, $\Gamma^{\rm b'vb}$, $\Gamma^{\rm b'vv}$, $\Gamma^{\rm b'bb}$ with $\delta \geq 0$ do not show ultraviolet divergences due to the Galilean invariance and reflection symmetry. In anisotropic MHD the same divergences are present but now the divergent parts of $\Gamma^{\rm v'v}$, $\Gamma^{\rm b'b}$ $\Gamma^{\rm v'bb}$ are proportional to all admissible tensor structures. The anisotropic structures proportional to $n_i n_j$ give new additional counterterms (8). All ultraviolet divergences can be eliminated using independent renormalization constants $Z_j, j = 1, 2, .., 10$ and the renormalized action is in form

$$\begin{aligned} S_R^A(\Phi) &= \frac{1}{2}[g_{\rm v}\mu^{2\epsilon}\nu^3 u {\rm v}' D^{\rm vv}{\rm v}' + g_{\rm b}\mu^{2a\epsilon}\nu^3 u^2 {\rm b}' D^{\rm bb}{\rm b}'] \quad (11)\\ &+ {\rm v}'\,[-\partial_t {\bf v} + Z_1\nu\nabla^2{\bf v} + Z_4\nu\chi_1({\bf n}\nabla)^2{\bf v} + Z_5\nu\chi_2{\bf n}\nabla^2{\bf n}{\bf v} \\ &- ({\bf v}\nabla){\bf v} + Z_3({\bf b}\nabla){\bf b} + Z_8\lambda_1{\bf b}({\bf n}\nabla){\bf n}{\bf b} \\ &+ Z_9\lambda_2{\bf n}({\bf n}\nabla){\bf b}^2 + Z_{10}\lambda_3{\bf n}({\bf b}\nabla){\bf n}{\bf b}] + {\rm b}'\,[-\partial_t{\bf b} + Z_2 u\nu\nabla^2{\bf b} \\ &+ Z_6 u\nu\chi_3({\bf n}\nabla)^2{\bf b} + Z_7 u\nu\chi_4{\bf n}\nabla^2{\bf n}{\bf b} + ({\bf b}\nabla){\bf v} - ({\bf v}\nabla){\bf b}]\,. \end{aligned}$$

The action (11) is connected with unrenormalized one (9) by the standart formula of the multiplicative renormalization : $S_R^A(\Phi, e) = S^A(Z_\Phi\Phi, e_0)$, where $Z_\Phi\Phi \equiv (Z_{\rm v'}{\rm v}', Z_{\rm b'}{\rm b}', Z_{\rm v}{\rm v}, Z_{\rm b}{\rm b})$ are renormalization constants of the fields Φ. The renormalized parameters are related to the bare parameters by relations

$$\begin{aligned} g_{\rm v0} &= g_{\rm v}\mu^{2\epsilon}Z_{g_{\rm v}}, \quad g_{\rm b0} = g_{\rm b}\mu^{2a\epsilon}Z_{g_{\rm b}}, \quad \nu_0 = \nu Z_\nu \quad , \quad (12)\\ u_0 &= u Z_u, \quad \chi_{j_0} = \chi_j Z_{\chi_j}, \quad \lambda_{j_0} = \lambda_j Z_{\lambda_j}, \end{aligned}$$

and all renormalization constants depend on Z_j as follows

$$\begin{aligned} Z_{g_{\rm v}} &= Z_1^{-2}Z_2^{-1}, \quad Z_{g_{\rm b}} = Z_1^{-1}Z_2^{-2}Z_3, \quad Z_\nu = Z_1, \quad Z_u = Z_2 Z_1^{-1}, \\ Z_{\chi_1} &= Z_4 Z_1^{-1}, \quad Z_{\chi_2} = Z_5 Z_1^{-1}, \quad Z_{\chi_3} = Z_6 Z_2^{-1}, \quad Z_{\chi_4} = Z_7 Z_2^{-1}, \\ Z_{\lambda_m} &= Z_{m+7} Z_3^{-1}, \quad m = 1, 2, 3, \\ Z_{\rm v} &= Z_{\rm v'} = 1, \quad Z_{\rm b} = Z_{\rm b'}^{-1} = Z_3^{1/2}. \quad (13) \end{aligned}$$

The RNG-functions (β–functions, the anomalous dimensions of the fields γ_Φ and of the parameters γ_e) can be expressed in terms of renormalization constants Z:

$$\begin{aligned} \gamma_\Phi &\equiv \overline{D}_\mu \ln Z_\Phi, \quad \gamma_e \equiv \overline{D}_\mu \ln Z_e, \quad (14)\\ \beta_g &= \overline{D}_\mu g, \quad g \equiv (g_{\rm v}, g_{\rm b}, u, \chi_1, \chi_2, \chi_3, \chi_4, \lambda_1, \lambda_2, \lambda_3). \end{aligned}$$

Here $\overline{D}_\mu \equiv \mu\partial/\partial\mu|_{e_0}$ denotes the derivative with respect to the parameter μ at fixed values of bare parameters e_0. Later similar operation $D_\mu \equiv \mu\partial/\partial\mu|_e$ will be used at fixed values of the renormalized parameters e. Using (12), (13), (14) one obtains:

$$\begin{aligned} \gamma_{\rm v} &= \gamma_{\rm v'} = 0, \quad \gamma_{\rm b} = -\gamma_{\rm b'} = \frac{1}{2}\gamma_3, \quad \gamma_\nu = \gamma_1, \quad \beta_u = u(\gamma_1 - \gamma_2), \\ \beta_{g_{\rm v}} &= g_{\rm v}(-2\epsilon + 2\gamma_1 + \gamma_2), \quad \beta_{g_{\rm b}} = g_{\rm b}(-2\epsilon a + \gamma_1 + 2\gamma_2 - \gamma_3), \quad (15)\\ \beta_{\chi_1} &= \chi_1(\gamma_1 - \gamma_4), \quad \beta_{\chi_2} = \chi_2(\gamma_1 - \gamma_5), \\ \beta_{\chi_3} &= \chi_3(\gamma_2 - \gamma_6), \quad \beta_{\chi_4} = \chi_4(\gamma_2 - \gamma_7), \\ \beta_{\lambda_m} &= \lambda_m(\gamma_3 - \gamma_{m+7}), \quad m = 1, 2, 3. \end{aligned}$$

4. RNG calculations. Fixed points

In this section the constants Z and RNG-functions are calculated. We use the minimum subtraction scheme [14], where the constants Z possess only ϵ poles. The one-particle irreducible Green functions $\Gamma^{\rm v'v}$, $\Gamma^{\rm b'b}$ and $\Gamma^{\rm v'bb}$ have been calculated by the standard Feynman diagrammatic technique. The calculations of the singular parts of the graphs give the following expressions for constants Z_j:

$$Z_j = 1 - \frac{1}{\epsilon}(g_v C_j^v + g_b C_j^b \frac{1}{a}), \quad j = 1, 2..10\,. \tag{16}$$

The coefficients C_j^v, C_j^b which depend on $d, u, \alpha_j, \chi_j, \lambda_j$ are given in Appendix II. RNG-functions $\gamma_j = D_\mu \ln z_j$ can be determined from the relation $\gamma_j = (\sum_g \beta_g \partial_g) \ln Z_j$ and then

$$\gamma_j = 2(g_v C_j^v + g_b C_j^b). \tag{17}$$

The final goal of RNG analysis is to establish the infrared asymptotic behaviour of the considered model. The Green functions G^Φ of renormalized theory (11) satisfy the basic RNG equations which express the invariance of the unrenormalized theory with respect to the scale setting parameter μ:

$$\left[D_\mu + \sum_g \beta_g \partial_g - \gamma_\nu D_\nu + \gamma_b N_b + \gamma_{b'} N_{b'} \right] G^\Phi = 0; \tag{18}$$

where N_b, $N_{b'}$ denote the numbers of the external lines of the fields **b**, **b**$'$ (let us remind that the anomalous dimensions γ_v, γ_v' of fields **v**, **v**$'$ vanish). Formally the asymptotic behaviour of the Green functions may be inferred finding the fixed points g^*, which are the solutions of the equations $\beta_g(g^*) = 0$. A set of ten algebraic equations with free parameters a, α_j, $j = 1, 2, 3, 4$ is obtained from (15). In the case of $\alpha_j = 0$ it yields to the fixed points of the isotropic MHD [10]. The fixed points g^* are infrared stable, if the matrix $\Omega \equiv \frac{\partial \beta_g}{\partial g}|_{g^*}$ is positively definite. Two infrared stable fixed points are known - the kinetic and magnetic. The first of them represents the Kolmogorov critical regime. It is clear that the fixed points of anisotropic MHD for small anisotropy are close to isotropic MHD fixed points.

The anisotropic *kinetic fixed point* corresponds to two zero charges $g_b^* = \chi_4{}^* = 0$. The other charges are given by equations

$$\gamma_1{}^* = \gamma_2{}^* = \gamma_\nu{}^* = \frac{2\epsilon}{3}, \quad . \tag{19}$$

$$\gamma_{j+3}{}^* = \gamma_1{}^*, \quad \gamma_{j+7}{}^* = \gamma_3{}^*, \quad j = 1, 2, 3.$$

where $\gamma^* \equiv \gamma(g^*)$.

In this fixed point it results in

$$\begin{pmatrix} u \\ \tilde{g}_v \\ g_b \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}^* = \begin{pmatrix} 1.3930 \\ 0.1595 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} -0.0019 \\ -0.0556 \\ 0 \\ 0.1429 \\ -0.1714 \\ 0.0121 \\ 0 \\ -1.4667 \\ 2.3087 \\ -1.4666 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0.1109 \\ 0.0503 \\ 0 \\ 0.3571 \\ 0.3714 \\ 0.4896 \\ 0 \\ 0.0573 \\ 0.5484 \\ 0.0573 \end{pmatrix}, \tag{20}$$

where $\tilde{g}_v \equiv g_v / \epsilon$.

The corresponding $\Omega-$ matrix on the diagonalized form is

$$\Omega = \epsilon\, \mathrm{diag}\, (1.0547,\ 2,\ 2(1.1595 - a),\ 0.7778 \mp i0.3685,\ 0.5890, \tag{21}$$
$$0.8218,\ 0.1013,\ -0.1857,\ -1.0141)\,.$$

In anisotropic *magnetic fixed point* another two charges are equal to zero, $g_v{}^* = u^* = 0$. The remaining charges could be found from equations

$$\gamma_1{}^* + 2\gamma_2{}^* - \gamma_3{}^* = 2a\epsilon \tag{22}$$

$$\gamma_4^* = \gamma_5^* = \gamma_1^*, \quad \gamma_6^* = \gamma_7^* = \gamma_2^*, \quad \gamma_{m+7}{}^* = \gamma_3{}^*, \quad m = 1, 2, 3.$$

In this fixed point for $u^* = 0$ the equality $\beta_{\lambda 1} = \beta_{\lambda 3}$ holds, therefore one of the charges becomes free (we choose λ_3) and for the remaining ones it yields

$$\begin{pmatrix} u \\ g_{\rm v} \\ \tilde{g}_{\rm b} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 0.229 \\ 0.121 \\ -0.146 \\ 0.826 \\ -0.189 \\ -0.146 \\ -0.095 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 0.081 \\ 0.368 \\ -0.285 \\ 0.117 \\ 0.611 \\ 0.285 \\ -0.249 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ -0.304 \\ -0.571 \\ 0.787 \\ -0.475 \\ 0.516 \\ 0.212 \\ -0.297 \end{pmatrix} \quad (23)$$

where $\tilde{g}_{\rm b} \equiv g_{\rm b} / a\epsilon$.

The diagonalized matrix in the magnetic regime is given by the relation

$$\begin{aligned} \Omega &= a\epsilon \,\mathrm{diag}\,(4,\, 2(4-1/a),\, 2,\, 12.3749, \\ &\quad -4.0189 \mp i3.7460,\, 2.1701 \mp i2.4401,\, 4.5369,\, 0)\,. \end{aligned} \quad (24)$$

5. Solutions of RNG equations

As it follows from relations (21),(24), $\Omega-$ matrix eigenvalues with negative real parts are in both kinetic and magnetic regimes. This usually means that these fixed points are unstable. Nevertheless if in theory with many charges some of them are equal to zero then the Green functions may depend on such products of charges which eliminate the instability. In other words, a transformed set of charges, in which the fixed point is stable, can be found.

Let us consider the *kinetic fixed point.* We will examine as an example the pair equal-time correlation functions $G^{\rm v}(\mathbf{k}) = \langle \mid \mathbf{v}(\mathbf{k}) \mid^2 \rangle$, $G^{\rm b}(\mathbf{k}) = \langle \mid \mathbf{b}(\mathbf{k}) \mid^2 \rangle$ with traced indices, which define the spectrum of kinetic and magnetic energy $E^{\rm v,b}(\mathbf{k}) = 2\pi k^2 G^{\rm v,b}(\mathbf{k})$. Equation (18) for these quantities taking into account (15) is in the form

$$(D_\mu + \sum_g \beta_g \partial_g - \gamma_1 D_\nu) G^{\rm v} = 0\,, \quad (25)$$

$$(D_\mu + \sum_g \beta_g \partial_g - \gamma_1 D_\nu + \gamma_3) G^{\rm b} = 0\,. \quad (26)$$

Correlation functions $G^{\rm v,b}$ can be expressed in renormalized variables and expanded into a series of $g_{\rm v}, g_{\rm b}$. The lowest order terms coincide with the corresponding propagator (44) changing

$\nu_0 \to \nu$, $u_0 \to u$, $g_{\rm v0} \to g_{\rm v}\mu^{2\epsilon}$, $g_{\rm b0} \to g_{\rm b}\mu^{2a\epsilon}$, and tracing indices. Hence

$$\begin{aligned} G^{\rm v} &= \frac{1}{2}\mu^{2\epsilon} k^{2-2\epsilon-d} \nu^2 g_{\rm v} u \left\{ \left[1 + (\alpha_1 - \chi_1) \frac{(\mathbf{kn})^2}{k^2} \right] (d-1) \right. \\ &\quad \left. + (\alpha_2 - \chi_2)(1 - \frac{(\mathbf{kn})^2}{k^2}) \right\} + \cdots = \nu^2 \mu^{2\epsilon} k^{2-2\epsilon-d} R^{\rm v}, \end{aligned} \quad (27)$$

$$\begin{aligned} G^{\rm b} &= \frac{1}{2}\mu^{2a\epsilon} k^{2-2a\epsilon-d} \nu^2 g_{\rm b} u \left\{ \left[1 + (\alpha_3 - \chi_3) \frac{(\mathbf{kn})^2}{k^2} \right] (d-1) \right. \\ &\quad \left. + (\alpha_4 - \chi_4)(1 - \frac{(\mathbf{kn})^2}{k^2}) \right\} + \cdots = \nu^2 \mu^{2a\epsilon} k^{2-2a\epsilon-d} R^{\rm b}, \end{aligned} \quad (28)$$

where the dots indicate higher order contributions of $g_{\rm v}, g_{\rm b}$ and $R^{\rm v,b}(s, g)$ are functions of dimensionless arguments g, $s = k/\mu$. Substituting (27),(28) into equations (25),(26) one obtains

$$\left(2\epsilon - D_s + \sum_g \beta_g \partial_g - 2\gamma_1 \right) R^{\rm v} = 0, \quad (29)$$

$$\left(2a\epsilon - D_s + \sum_g \beta_g \partial_g - 2\gamma_1 + \gamma_3 \right) R^{\rm b} = 0. \quad (30)$$

The invariant charges $\overline{g}_i(s, g)$ are introduced as solutions of equations

$$D_s \overline{g}_i = \beta_i(\overline{g}), \quad \overline{g}_i(1, g) = g_i\,. \quad (31)$$

The solution of equations (29),(30) for known RNG functions $\overline{g}_i(s,g)$ can be written in the form

$$R^{\mathrm{v}}(s,g) = s^{2\epsilon}\Psi_1^{-2}R^{\mathrm{v}}(1,\overline{g}(s)), \tag{32}$$

$$R^{\mathrm{b}}(s,g) = s^{2a\epsilon}\Psi_1^{-2}\Psi_3 R^{\mathrm{b}}(1,\overline{g}(s)), \tag{33}$$

where

$$\Psi_j(s,g) = \exp\left[\int_1^s \frac{ds'}{s'}\gamma_j\left(\overline{g}(s')\right)\right]. \tag{34}$$

The relations (32),(33),(34) are the main results of RNG approach, which enable to find the asymptotic behaviour of the Green functions for $s \to 0$. The calculation of the functions $R^{\mathrm{v,b}}$ using the perturbation theory gives an expansion over the dimensionless parameters $s^{-2\epsilon}g_{\mathrm{v}}$, $s^{-2a\epsilon}g_{\mathrm{b}}$, which tend to infinity if $s \to 0$. The RNG equation shows that the powers $s^{-2\epsilon}$ form combinations of $\overline{g}(s)$ with simple asymptotic behaviour $\overline{g}(s) \to g^*$ for $s \to 0$.

Notice that the approach of $s \to 0$ cannot be taken literally. Indeed, in the developed turbulence the inertial interval is limited by an infrared cutoff $k_{min} = L^{-1}$, where L is the external scale of turbulence. So, the minimum value of s is equal to $k_{min}/\mu \sim (Re)^{-3/4}$ (Re being the Reynolds number) because the μ^{-1} is of the same order as the dissipation length l, which is connected with L by relation $l = Re^{-3/4}L$. Therefore the maximum values of expansion parameters $s^{-2\epsilon}g_{\mathrm{v}}$ and $s^{-2a\epsilon}g_{\mathrm{b}}$ are approximately equal to $(Re)^{3\epsilon/2}$ and $(Re)^{3a\epsilon/2}$, respectively.

Near the fixed point g^* the invariant charges behave like $\overline{g}(s) = g^* + \delta\overline{g}(s)$. The linearized equations for $\delta\overline{g}$ in form of $D_s\delta\overline{g}_i = \Omega_{ij}\delta\overline{g}_j$ are obtained from (31). They have the solution

$$\delta\overline{g}_i = \sum_j c_{ij}s^{\Omega_j}, \tag{35}$$

where c_{ij} denotes matrix which diagonalizes Ω-matrix , $\sum_j$ is the sum over the all eigenvalues Ω_j of $\Omega-$ matrix. If all of them are positive then $\delta\overline{g}_i(s) \to 0$ and $\overline{g}_i(s) \to g_i^*$ if $s \to 0$ and the fixed point is stable.

Firstly, *the isotropic case* with all positive Ω_j is considered. The asymptotic form of the expression (34) for $s \to 0$ is determined by small s':

$$\Psi_j(s) = \exp\left(\int_1^s \frac{ds'}{s'}\left[\gamma_j^* + \left(\gamma_j(\overline{g}(s')) - \gamma_j^*\right)\right]\right) = A_j s^{\gamma_j^*},$$

where

$$A_j = \exp\left\{-\int_0^1 \frac{ds'}{s'}\left[\gamma_j(\overline{g}(s')) - \gamma_j^*\right]\right\}.$$

The substitution of (36) into (32),(33) gives

$$R^{\mathrm{v}}(s,g) \simeq s^{2\epsilon-2\gamma_1^*}A_1^{-2}R^{\mathrm{v}}(1,g^*), \tag{36}$$

$$R^{\mathrm{b}}(s,g) \simeq s^{2a\epsilon-2\gamma_1^*+\gamma_3^*}A_1^{-2}A_3R^{\mathrm{b}}(1,g^*). \tag{37}$$

It follows from (27),(28) that $G^{\mathrm{v}}, G^{\mathrm{b}}$ are of the form

$$G^{\mathrm{v}} = \nu^2\mu^{4\epsilon/3}k^{2-d-4\epsilon/3}A_1{}^{-2}R^{\mathrm{v}}(1,g^*) \tag{38}$$

$$G^{\mathrm{b}} = \nu^2\mu^{4\epsilon/3-\gamma_3^*}k^{2-d-4\epsilon/3+\gamma_3^*}A_1{}^{-2}A_3R^{\mathrm{b}}(1,g^*). \tag{39}$$

For $\epsilon = 2$ and $d = 3$ the Kolmogorov spectrum of the kinetic energy $E^{\mathrm{v}} \sim k^{-5/3}$ holds. One can see from (28) for the magnetic energy spectrum that in the lowest order $R^{\mathrm{b}}(1,g^*) \simeq g_{\mathrm{b}}^* = 0$. The structure of the vertices in action (11) maintain this property in all orders of perturbation theory. To obtain a nontrivial result $R^{\mathrm{b}}(1,\overline{g}(s))$ is to be considered instead of $R^{\mathrm{b}}(1,g^*)$ and the asymptotic formula (35) should be used. It gives $R^{\mathrm{b}}(1,\overline{g}(s)) \sim \overline{g}_{\mathrm{b}}(s) \approx \delta\overline{g}_{\mathrm{b}}(s) = s^{\Omega_{g_b}}$, where

$$\Omega_{g_b} = -2a\epsilon + \gamma_1^* + 2\gamma_2^* - \gamma_3^* = 2\epsilon(1-a) - \gamma_3^*. \tag{40}$$

Thus the spectrum of magnetic energy is in the form $E^{\mathrm{b}} \sim k^{4-d+2\epsilon(1-a)-4\epsilon/3}$ as it has been obtained earlier [9]. For $\epsilon = 2$, $a = 1$ it coincides with the Kolmogorov spectrum. *These results for exponents of kinetic and magnetic energy spectrum are exact and do not include the usual corrections* $\sim \epsilon^2,\ \epsilon^3 \cdots$.

Our attempt shows that it is more convenient to use the rescaled magnetic field $\tilde{\mathbf{b}} = \mathbf{b}/\sqrt{g_{\mathrm{b}}}$ than $\mathbf{b}$. Then the quantity $\tilde{G}^{\mathrm{b}} = G^{\mathrm{b}}/g_{\mathrm{b}}$ is finite for $s \to 0$. Using (29) for $\tilde{R}^{\mathrm{b}} = R^{\mathrm{b}}/g_{\mathrm{b}}$ one obtains:

$$\left(2a\epsilon - D_s + \frac{\beta_{g_b}}{g_{\mathrm{b}}} + \sum_g \beta_g \partial_g - 2\gamma_1 + \gamma_3\right)\tilde{R}^{\mathrm{b}} = 0\,.$$

Substitution of $\beta_{g_b}/g_{\mathrm{b}}$ from (15) results in

$$\left(-D_s + \sum_g \beta_g \partial_g + 2\gamma_2 - \gamma_1\right)\tilde{R}^{\mathrm{b}} = 0\,.$$

The solution of this equation is of the form

$$\tilde{R}^{\mathrm{b}}(s) = \Psi_2^2 \Psi_1{}^{-1} \tilde{R}^{\mathrm{b}}(1, \overline{g}(s)). \qquad (41)$$

To calculate $\tilde{R}^{\mathrm{b}}(1, \overline{g}(s))$ it is necessary to rewrite the action (11) in terms of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{b}}' \equiv \mathbf{b}'\sqrt{g_{\mathrm{b}}}$, $\tilde{\Phi} \equiv (\mathbf{v}, \tilde{\mathbf{b}}, \mathbf{v}', \tilde{\mathbf{b}}')$. It yields

$$\begin{aligned} S_R^A(\tilde{\Phi}) &= \frac{1}{2}\left[g_{\mathrm{v}}\mu^{2\epsilon}\nu^3 u\, \mathbf{v}'\mathbf{D}^{\mathrm{vv}}\mathbf{v}' + \mu^{2a\epsilon}\nu^3 u^2\, \tilde{\mathbf{b}}'\mathbf{D}^{\mathrm{bb}}\tilde{\mathbf{b}}'\right] + \mathbf{v}'\left[-\partial_t \mathbf{v}\right. \\ &+ Z_1\nu\nabla^2\mathbf{v} + Z_4\nu\chi_1(\mathbf{n}\nabla)^2\mathbf{v} + Z_5\nu\chi_2\mathbf{n}\nabla^2\mathbf{n}\mathbf{v} - (\mathbf{v}\nabla)\mathbf{v} \\ &+ Z_3 g_{\mathrm{b}}(\tilde{\mathbf{b}}\nabla)\tilde{\mathbf{b}} + Z_8 g_{\mathrm{b}}\lambda_1\tilde{\mathbf{b}}(\mathbf{n}\nabla)\mathbf{n}\tilde{\mathbf{b}} + Z_9 g_{\mathrm{b}}\lambda_2\mathbf{n}(\mathbf{n}\nabla)\tilde{\mathbf{b}}^2 \\ &+ \left. Z_{10} g_{\mathrm{b}}\lambda_3\mathbf{n}(\tilde{\mathbf{b}}\nabla)\mathbf{n}\tilde{\mathbf{b}}\right] + \tilde{\mathbf{b}}'\left[-\partial_t\tilde{\mathbf{b}} + Z_2 u\nu\nabla^2\tilde{\mathbf{b}}\right. \\ &+ \left. Z_6 u\nu\chi_3(\mathbf{n}\nabla)^2\tilde{\mathbf{b}} + Z_7 u\nu\chi_4\mathbf{n}\nabla^2\mathbf{n}\tilde{\mathbf{b}} + (\tilde{\mathbf{b}}\nabla)\mathbf{v} - (\mathbf{v}\nabla)\tilde{\mathbf{b}}\right]. \end{aligned} \qquad (42)$$

Now, the nonzero quantity $\tilde{R}^{\mathrm{b}}(1, g^*)$ can be calculated in perturbation theory with action (42) (without anisotropic terms). The vertex $\mathbf{v}'(\tilde{\mathbf{b}}\nabla)\tilde{\mathbf{b}}$, related to Lorentz force $(\tilde{\mathbf{b}}\nabla)\tilde{\mathbf{b}}$ asymptotically for $s \to 0$ does not contribute and may be neglected. Thus the magnetic field does not influence the velocity field dynamics and behaves like a passive admixture. From (41) it follows that the magnetic spectrum E^{b} preserves the previous form.

In *the anisotropic case* among the eigenvalues (21) of the $\Omega-$ matrix there are two negative values. From the Ω - matrix structure one can prove that the negative eigenvalues contribute only to $\delta\overline{g}_i \equiv \delta\overline{\lambda}_i$ (i=1,2,3) in (35). The main singular part of these charges increases as $s^{-1.0141\epsilon}$ for $s \to 0$. However they are included in action (42) with the multiplier g_{b}. Hence the asymptotic form of functions $R^{\mathrm{v,b}}(1, \overline{g}(s))$ contains the product of invariant charges $\overline{g}_{\mathrm{b}}(s)\overline{\lambda}_i(s)$, which is proportional to $s^{(\Omega_{g_b} - 1.0141\epsilon)} = s^{(1.30-2a)\epsilon}$. For $a < 0.65$ the exponent is positive and therefore all $\lambda-$ vertexes vanish in infrared range. The same result is obtained if rescaled charges $\tilde{\lambda}_i = \lambda_i g_{\mathrm{b}}$ with redefined β-functions

$$\beta_{\tilde{\lambda}_i} = \tilde{\lambda}_i\left(\frac{\beta_{\lambda i}}{\lambda_i} + \frac{\beta_{g_b}}{g_{\mathrm{b}}}\right)$$

and with corresponding $\tilde{\Omega}-$ matrix are introduced. The minimum eigenvalue of $\tilde{\Omega}$ which defines the stability of the fixed point equals to $\tilde{\Omega}_{inf} = (1.30 - 2a)\epsilon$. Thus the Lorentz forces become infrared important for $a > 0.65$ in anisotropic case, whereas in isotropic case they become important for $a > 1.16$. *It makes a significant difference between isotropic and anisotropic MHD because "physical" value $a = 1$ belongs to the instability range of a in anisotropic MHD.*

The *magnetic fixed* point will briefly be investigated. The invariant charges $\overline{g}(s)$ and $\overline{u}(s)$ tend now to zero as $\overline{g}_{\mathrm{v}} \sim s^{\Omega_{g_v}}$, $\overline{u} \sim s^{\Omega_u}$ ($\Omega_{g_v} = 2\epsilon(4a - 1)$, $\Omega_u = 4\epsilon a$, see (24)). This fact complicates the analysis of functions $R^{\mathrm{v,b}}(1, \overline{g}(s))$, because u occurs in denominators of perturbation series. In this case it is convenient to introduce rescaled fields $\hat{\mathbf{b}} = \mathbf{b}/\sqrt{u}$, $\hat{\mathbf{v}} = \mathbf{v}/u$, $\hat{\mathbf{b}}' = \mathbf{b}'\sqrt{u}$, $\hat{\mathbf{v}}' = \mathbf{v}'$, $\hat{\Phi} = (\hat{\mathbf{v}}, \hat{\mathbf{b}}, \hat{\mathbf{v}}', \hat{\mathbf{b}}')$ and time $\hat{t} = ut$. Taking into account the integration over the time then in new variables action (11) looks like

$$\begin{aligned} S_R^A(\hat{\Phi}) &= \frac{1}{2}\left[g_{\mathrm{v}}\mu^{2\epsilon}\nu^3\hat{\mathbf{v}}'\mathbf{D}^{\mathrm{vv}}\hat{\mathbf{v}}' + g_{\mathrm{b}}\mu^{2a\epsilon}\nu^3\hat{\mathbf{b}}'\mathbf{D}^{\mathrm{bb}}\hat{\mathbf{b}}'\right] + \hat{\mathbf{v}}'\left[-u\partial_{\hat{t}}\hat{\mathbf{v}}\right. \\ &+ Z_1\nu\nabla^2\hat{\mathbf{v}} + Z_4\nu\chi_1(\mathbf{n}\nabla)^2\hat{\mathbf{v}} + Z_5\nu\chi_2\mathbf{n}\nabla^2\mathbf{n}\mathbf{v} \\ &- u(\hat{\mathbf{v}}\nabla)\hat{\mathbf{v}} + Z_3(\hat{\mathbf{b}}\nabla)\hat{\mathbf{b}} + Z_8\lambda_1\hat{\mathbf{b}}(\mathbf{n}\nabla)\mathbf{n}\hat{\mathbf{b}} + +Z_9\lambda_2\mathbf{n}(\mathbf{n}\nabla)\hat{\mathbf{b}}^2 \\ &+ \left. Z_{10}\lambda_3\mathbf{n}(\hat{\mathbf{b}}\nabla)\mathbf{n}\hat{\mathbf{b}}\right] + \hat{\mathbf{b}}'\left[-\partial_{\hat{t}}\hat{\mathbf{b}} + Z_2\nu\nabla^2\hat{\mathbf{b}}\right. \end{aligned} \qquad (43)$$

$$+ \; Z_6\nu\chi_3(\mathbf{n}\nabla)^2\hat{\mathbf{b}} + Z_7\nu\chi_4\mathbf{n}\nabla^2\mathbf{n}\hat{\mathbf{b}} + (\hat{\mathbf{b}}\nabla)\hat{\mathbf{v}} - (\hat{\mathbf{v}}\nabla)\hat{\mathbf{b}}\Big] .$$

It follows from (30),(15) that the function $\hat{R}^{\mathbf{b}} = R^{\mathbf{b}}/u$ satisfies the equation

$$\left(2a\epsilon - D_s + \sum_g \beta_g\partial_g - \gamma_1 - \gamma_2 + \gamma_3\right)\hat{R}^{\mathbf{b}} = 0 ,$$

which has the solution

$$\hat{R}^{\mathbf{v},\mathbf{b}}(s,g) = s^{2a\epsilon}\Psi_1{}^{-1}\Psi_2{}^{-1}\Psi_3\hat{R}^{\mathbf{v},\mathbf{b}}(1,\overline{g}(s)) \simeq A_1{}^{-1}A_2{}^{-1}A_3 s^{\gamma_2^*}\hat{R}^{\mathbf{v},\mathbf{b}}(1,g^*),$$

where the relation (22) has been used. The quantities $\hat{R}^{\mathbf{v},\mathbf{b}}(1,g^*)$ may be calculated in perturbation theory with action (43). Having in mind that $\gamma_2{}^* = 0$ for $d = 3$, the spectrum of magnetic energy is in the form $E^{\mathbf{b}} \sim k^{1-2a\epsilon}$. This result has been obtained for *isotropic case* in [9].

In *the anisotropic case* $\Omega-$ matrix has some eigenvalues with negative real part (24). It leads to an increasing anisotropic part of invariant charges. Action (43) implies that this increase is not compensated with decreasing charges $\overline{g}_{\mathbf{v}}(s)$, $\overline{u}(s)$ for $s \to 0$.

6. Conclusions

Using the RNG approach it has been shown that small anisotropic terms ($\sim \alpha_i$) in the random forces two-point correlators play the role of corrections in the kinetic regime only for $a < 0.65$. In the interval $0.65 < a < 1.16$ these anisotropic forces give corrections only at strongly limited values of their amplitude: $\alpha_j \ll s_{min}^{(2a-1,30)\epsilon}$, where $s_{min} \approx Re^{-3/4}$. For the magnetic regime the same situation occurs when $a > 1/4$ and the restriction is $\alpha_j \ll s_{min}^{4.01a\epsilon}$. In case of the fully developed turbulence the relation $Re \gg 1$ implies $s_{min} \ll 1$. A quantitative analysis is required to find more precise restrictions on α. If the restrictions are not satisfied the linearization in the anisotropic terms is not valid even for $\alpha_j \ll 1$ and they must be considered as relevant terms. In the kinetic regime this relevancy is probably connected only with Lorentz force terms since namely their increase for $a \to 0.65$ that leads to the kinetic fixed point destabilization.

Therefore in MHD turbulence the critical regimes stability is sensitive to small departure from isotropy what is quite a different case from usual turbulence [11, 12]

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Appendix I: Evaluated propagators containing anisotropy parameters

The anisotropy is considered as a higher-order effect in the propagators, which has in momentum - time (k, t) representation the following form :

$$\begin{aligned}
\Delta_{js}^{\mathbf{vv}'}(\mathbf{k},t) &= \exp(-\nu_0 k^2 t)\theta(t)\left([1-\chi_{10}\nu_0(\mathbf{kn})^2 t]P_{js}\right.\\
&\quad \left. - \chi_{20}\nu_0 k^2 t P_{jl}n_l n_m P_{ms}\right), \qquad (44)\\
\Delta_{js}^{\mathbf{bb}'}(\mathbf{k},t) &= \exp(-\nu_0 u_0 k^2 t)\theta(t)\left([1-\chi_{30}\nu_0 u_0(\mathbf{kn})^2 t]P_{js}\right.\\
&\quad \left. - \chi_{40}\nu_0 u_0 k^2 t P_{jl}n_l n_m P_{ms}\right),\\
\Delta_{js}^{\mathbf{v}'\mathbf{v}}(\mathbf{k},t) &= \Delta_{js}^{\mathbf{vv}'}(-\mathbf{k},-t),\ \Delta_{js}^{\mathbf{b}'\mathbf{b}}(\mathbf{k},t) = \Delta_{js}^{\mathbf{bb}'}(-\mathbf{k},-t),\\
\Delta_{js}^{\mathbf{vv}}(\mathbf{k},t) &= \frac{1}{2}c g_{\mathbf{v}0}\nu_0{}^2 u_0 \exp(-\nu_0 k^2|t|)k^{2-2\epsilon-d}\\
&\times \left(\left[1+\alpha_1\frac{(\mathbf{kn})^2}{k^2} - \chi_{10}(1+\nu_0 k^2\mid t\mid)\frac{(\mathbf{kn})^2}{k^2}\right]P_{js}\right.\\
&+ \left.[\alpha_2 - \chi_{20}(1+\nu_0 k^2\mid t\mid)]\,P_{jl}n_l n_m P_{ms}\right),
\end{aligned}$$

$$
\begin{aligned}
\Delta^{\mathbf{bb}}_{js}(\mathbf{k},t) &= \frac{1}{2} c g_{\mathrm{b}0} {\nu_0}^2 {u_0}^2 \exp(-\nu_0 u_0 k^2 |t|) k^{2-2a\epsilon-d} \\
&\times \left(\left[1 + \alpha_3 \frac{(\mathbf{kn})^2}{k^2} - \chi_{30}(1 + \nu_0 u_0 k^2 \mid t \mid) \frac{(\mathbf{kn})^2}{k^2} \right] P_{js} \right. \\
&+ \left. \left[\alpha_4 - \chi_{40}(1 + \nu_0 u_0 k^2 \mid t \mid) \right] P_{jl} n_l n_m P_{ms} \right),
\end{aligned}
$$

where $\theta(t)$ denotes usual step function. The propagators $\Delta^{\mathbf{vv}'}_{js}(\mathbf{k},t)$, $\Delta^{\mathbf{bb}'}_{js}(\mathbf{k},t)$ are retarded, and propagators $\Delta^{\mathbf{v}'\mathbf{v}}_{js}(\mathbf{k},t)$, $\Delta^{\mathbf{b}'\mathbf{b}}_{js}(\mathbf{k},t)$ advanced.

Appendix II: Calculated coefficients C^v_j, C^b_j incoming to Z_j, γ_j and β_j- functions

This section reviews constants C^v_j, C^b_j involved in expressions (16), (17). They were derived from divergent parts of the one particle irreducible functions $\Gamma^{\mathbf{v}'\mathbf{v}}, \Gamma^{\mathbf{b}'\mathbf{b}}$ and $\Gamma^{\mathbf{v}'\mathbf{bb}}$.

Using definitions $d_j = d + j, u_j = u + j$ it is straightforward to verify that:

$$
\begin{aligned}
C^v_1 &= u\left[\alpha_1(dd_3+2) + \alpha_2 d_{-2} - 2\chi_1(d_{-1}d_4+7)\right. \\
&+ \left.\chi_2(2-3d) + d_4 d_{-1} d\right] / \, 4d_4, \\
C^b_1 &= [\alpha_3(d_2d_4 - d_6) + \alpha_4 d_6 + 2\chi_3(d_7 \\
&- d_2 d_4) - \chi_4 d_{10} + d_4 d_2 d_{-1} - 2d_4] \, / \, 4d_4, \\
C^v_2 &= \left[\alpha_1 u_1 d_1 + \alpha_2 u_1 - \chi_1 d_1 u_2 - \chi_2 u_2 - \chi_3 u d_1 - \chi_4 u + d_2 d_{-1} u_1\right] / \, 2u_1^2, \\
C^b_2 &= \left[\alpha_3 u_1 d_{-1} + \alpha_4 u_1 - \chi_1 d_{-1} - \chi_2 - \chi_3(1+2u)d_{-1} - \chi_4(1+2u)\right. \\
&- \left.\lambda_1 u_{-1} + \lambda_2(d_{-1}u_1 - 2) - \lambda_3 u_1 + d_2 u_1 d_{-3}\right] / \, 2u_1^2, \\
C^v_3 &= \left[2u_1(\alpha_2 - \alpha_1 + 2\chi_1 - 2\chi_2 + \chi_3 - \chi_4)\right. \\
&- \left.(\lambda_1+\lambda_2)d_2 u + 2\lambda_3 u - 2d_4 u_1\right] / \, 2d_4 u_1, \\
C^b_3 &= \left[2u_1(\alpha_3 - \alpha_4 - \chi_1 + \chi_2 - 2\chi_3 + 2\chi_4)\right. \\
&+ \left.(\lambda_1+\lambda_2)d_2(2u+1) - 2\lambda_3(2u+1) + 2d_4 u_1\right] / \, 2d_4 u_1, \\
C^v_4 &= u\left[-4\alpha_1 d_{-2} + 2\alpha_2(d_2 d_4 d_{-4} + 10 d_4 - 12) + 4\chi_1(3d-2)\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \left.\chi_2(3dd_2d_4 - 13d_2d_4 + 38d + 96)\right] / \, 8\chi_1 d_4, \\
C^b_4 &= \left[-4\alpha_3 d_6 + 2\alpha_4(d_4 d_2 d_{-2} + 4) + 4\chi_3 d_{10} - \chi_4(d_2 d_4(3d-5) - 6d)\right. \\
&+ \left.2\lambda_1 d_4(d_2 d_{-1} - 2)\right] / \, 8\chi_1 d_4, \\
C^v_5 &= u\left[-8\alpha_1 d_1 + 8\alpha_2 d_1 + 8\chi_1(2d+1)\right. \\
&- \left.\chi_2(d_2 d_4 d_{-5} + 26d + 48)\right] / \, 8\chi_2 d_4, \\
C^b_5 &= \left[-8\alpha_3 - 4\alpha_4 d_2 + 24\chi_3 - \chi_4(d_2 d_4 d_{-1} - 10d - 16)\right. \\
&+ \left.2\lambda_3 d_4(d_2 d_{-1} - 2)\right] / \, 8\chi_2 d_4, \\
C^v_6 &= \left[-2\alpha_1 u_1 + \alpha_2 u_1(d_2 d_{-2} + 2) + 2\chi_1 u_2\right. \\
&- \left.\chi_2 u_2(d_2 d_{-2} + 2) + 2\chi_3 u - \chi_4 u d\right] / \, 2\chi_3 u_1^2, \\
C^b_6 &= \left[-2\alpha_3 u_1 + \alpha_4 u_1(d_2 d_{-4} + 4) + 2\chi_1 + \chi_2 d_2 + 2\chi_3(2u+1)\right. \\
&- \chi_4(2u+1)(d_2 d_{-4} + 4) + \lambda_1(dd_2 u_1 - 2d_2 u_2 + 2u_4) \\
&+ \left.\lambda_2(3d_2 u_1 - dd_2 u_1 + 2d - 4u - 6) - \lambda_3 d u_1\right] / \, 2\chi_3 u_1^2, \\
C^v_7 &= -u\,(4 + d_2 d_{-3}) \, / \, 2u_1^2, \\
C^b_7 &= \left[-4\alpha_3 u_1 - 2\alpha_4 u_1 + 4\chi_1 - \chi_2(d_2 d_{-3} + 2)\right. \\
&+ 4\chi_3(2u+1) + 2\chi_4(2u+1) - 2\lambda_1(2u+3) \\
&- \left.2\lambda_2 u + \lambda_3 u_1(d_2 d_{-1} - 4)\right] / \, 2\chi_4 u_1^2, \\
C^v_8 &= \left[4u_1(-2\alpha_1 - \alpha_2 d_2 + 4\chi_1 + 2\chi_2 d_2 + 2\chi_3 + \chi_4 d_2) +\right. \\
&+ \lambda_1(d_2 d_4 u - 2d_4(3u+2) + 8u) + \lambda_2 u(d_{-2} d_4 + 8) \\
&- \left.4\lambda_3 u d_2\right] / \, 4\lambda_1 d_4 u_1, \\
C^b_8 &= \left[4u_1(2\alpha_3 + \alpha_4 d_2 - 2\chi_1 - \chi_2 d_2 - 4\chi_3 - 2\chi_4 d_2)\right. \\
&- \lambda_1((d_2 d_4 + 8)(2u+1) - 2d_4(6u+5)) \\
&- \left.\lambda_2(2u+1)(d_{-2} d_4 + 8) + 4\lambda_3 d_2(2u+1)\right] / \, 4\lambda_1 d_4 u_1, \\
C^v_9 &= ([2u_1[-2\alpha_1 + \alpha_2(dd_4+2) + 4\chi_1 - 2\chi_2(dd_4+2) + 2\chi_3 \\
&- \chi_4(dd_4+2) + 2\lambda_1(d_2 d_4 - d_4 u_1 + 2u) + \lambda_2(d_{-1} d_2 d_4 \\
&- 2d_4 u + 4u) + 2\lambda_3(dd_4 u_1 + d_4 + 2u)) \, / \, 2\lambda_2 d_4 u_1, \\
C^b_9 &= (2u_1[2\alpha_3 - \alpha_4(dd_4+2) - 2\chi_1 + \chi_2(dd_4+2) - 4\chi_3 \\
&+ 2\chi_4(dd_4+2)] + 2\lambda_1(-d_2 d_4 + 2d_4 u_1 - 4u - 2) + \lambda_2(-d_2 d_4 d_{-1} \\
&+ 2d_2(2u+1)) + 2\lambda_3(-2d_2 d_4 u_1 + d_3(4u+3) + 1)) \, / \, 2\lambda_2 d_4 u_1,
\end{aligned}
$$

$$C_{10}^{v} = \left[4u_1(-2\alpha_1 - \alpha_2 d_2 + 4\chi_1 + 2\chi_2 d_2 + 2\chi_3 + \chi_4 d_2) + \lambda_1 u(d_{-4}d_4 + 8) + \lambda_2 u d_2 d - 4\lambda_3(u_1 d_2 + 2)\right] / 4\lambda_3 d_4 u_1,$$

$$C_{10}^{b} = [4u_1(2\alpha_3 + \alpha_4 d_2 - 2\chi_1 - \chi_2 d_2 - 4\chi_3 - 2\chi_4 d_2) - \lambda_1(2u+1) \times (d_{-4}d_4 + 8) - \lambda_2(2u+1)dd_2 + 8\lambda_3(d_2 u_1 + 1)] / 4\lambda_3 d_4 u_1 .$$

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Неустойчивости в МГД-турбулентности, генерируемые слабой анизотропией

Методом ренормализационной группы исследуется модель анизотропной магнитогидродинамической турбулентности. Показано, что малая анизотропия увеличивает влияние силы Лоренца. Это полностью отличается от изотропного случая, где сила Лоренца не влияет на крупномасштабные свойства МГД-турбулентности и магнитное поле ведет себя как пассивная примесь в случайном поле скоростей. В анизотропной МГД нелинейности генерируют модифицированные «анизотропные силы Лоренца», которые при определенной форме внешнего магнитного шума ведут к неустойчивости обычного колмогоровского режима, и магнитное поле больше не является пассивной примесью.

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Adzhemyan L. Ts. et al. E17-93-429

Instabilities in MHD Turbulence Generated by Weak Anisotropy

The model of anisotropic magnetohydrodynamic turbulence is investigated by a renormalization group approach. It is demonstrated that including of small anisotropy into the model leads to increasing of the Lorentz force influence. It is quite different from the isotropic case where the Lorentz force has no influence on large scale properties of magnetohydrodynamic turbulence and the magnetic field behaves like a passive admixture in random velocity field. In the anisotropic MHD, nonlinear interactions generate modified «anisotropic Lorentz forces» which lead to instability of known Kolmogorov regime for given external magnetic noise and the magnetic field does not behave as a passive admixture.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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