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SU(2) PATH INTEGRAL AND LOCAL LINEAR FRACTIONAL TRANSFORMATIONS OF THE INTEGRATION VARIABLES

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1 Introduction

A starting point in the ordinary path-integral approaches to quantum spin systems is that to write down the spin operators in terms of boson (fermion) creation and annihilation operators. Once the spin hamiltonian is transformed into many- boson (fermion) problem, one can just use the well-known expression for the partition function as a path integral (PI) over the flat classical phase space associated with the oscillator boson (fermion) group (supergroup), whose Lie algebras consist of generators

$$\{a^{\dagger}a,\ a^{\dagger},\ a,\ I\} \text{ and } \{f^{\dagger}f,\ f^{\dagger},\ f,\ I\}, \tag{1}$$

respectively. The standard commutation (anticommutation) relations are as follows:

$$[a, a^{\dagger}] = \{f^{\dagger}, f\} = 1.$$

An important point is that the oscillator group (supergroup) appears as a group of motion of the phase plane of a harmonic Bose (Fermi) oscillator and acts in this phase space as a group of linear shifts, canonical transformations of a complex ordinary (Grassmann) plane. These canonical transformations provide a natural basis for calculating appropriate PI's. That is, let the oscillator group be a dynamical group for a hamiltonian H, e.g., H belongs to its Lie algebra (1). Then, PI for the partition function

$$Z = tr\, e^{-\beta H}$$

can be evaluated with the help of the linear shifts of integration variables.

It then follows that PI over the phase plane of harmonic oscillators turns out to be very convenient for quantum systems, provided their unperturbed hamiltonians look as the sums of uncoupled oscillators (Bose or/and Fermi). For perturbation theory based on the harmonic oscillator approximation to be effective, the originally flat phase space is to be only slightly deformed by an interaction. This is not the case for spin systems. As is known, the classical phase space for a single spin is a curved one, in fact a two-dimensional sphere S^2. Accordingly, the "flat" PI turns out to be rather "unnatural" for spin systems. This results in the fact that even for the simplest linear spin model the corresponding "flat" PI requires tiresome and lengthy calculations [1].

As $SU(2)$ group is the dynamical group for linear spin system, it seems to be quite natural to use the PI representation for spin systems starting from the $SU(2)$ PI. As is known, the $SU(2)$ PI can be constructed as a PI over $SU(2)$ coherent states (CS's). The CS's basis provides an "internal" PI representation for a large variety of Lie groups G whose coset space G/K with respect to the maximal compact subgroup K is hermitian symmetric. The relevant phase space appears as the orbit of the coadjoint representation. Note that the important cases of the $SU(1,1)$ group and the $OSP(1|2)$ supergroup have recently been discussed [2, 3].

The purpose of this paper is that to show that the $SU(2)$ PI representation when combined with $SU(2)$ canonical transformations of the integration space turns out to be very convenient for quantum spin systems. This should come as no surprise since the $SU(2)$ PI is the most appropriate one for quantum systems that could be expressed through the $SU(2)$ generators.

2 SU(2) coherent states and the path integral

Let us consider the quantum variables of a single spin $J_{\pm}, J_0$ which span $SU(2)$ algebra:

$$[J_+, J_-] = 2J_0\,, \qquad [J_0, J\pm] = \pm J_{\pm}. \tag{2}$$

$SU(2)$ CS can be represented then as follows [4, 5, 6]

$$|\alpha; j\rangle = (1+|\alpha|^2)^{-j} \exp(\alpha J_+)\; |j, -j\rangle, \tag{3}$$

where the lowest weight state $|j,-j\rangle$ is annihilated by the lowering operator J_-. The complex number α belongs to the coset space $SU(2)/U(1)$ which is isomorphic to the complex projective space $CP^1 \simeq S^2$. Note that CS (3) depends upon the representation index $j = n/2$ ($n = 0, 1, 2, ...$) - the eigenvalue of the quadratic Casimir operator $\vec{J}^2$. The overlap of two states $|\alpha'; j\rangle$ and $|\alpha; j\rangle$ is given as

$$\langle\alpha'; j|\alpha; j\rangle = (1+|\alpha'|^2)^{-j}\,(1+|\alpha|^2)^{-j}\,(1+\bar{\alpha}'\alpha)^{2j}. \tag{4}$$

An important property of these states is that they satisfy the completeness relation

$$\int |\alpha;j\rangle\langle\alpha;j|\, d\mu_j(\alpha) = I_j = \sum_{m=-j}^{j} |j,m\rangle\langle m,j|, \tag{5}$$

where the $SU(2)$-invariant integration measure looks as follows:

$$d\mu_j(\alpha) = \frac{2j+1}{\pi}\frac{d^2\alpha}{(1+|\alpha|^2)^2}. \tag{6}$$

It is easily seen that

$$Tr\, F = \sum_m \langle m,j|F|j,m\rangle = \int d\mu_j(\alpha)\, \langle\alpha;j|F|\alpha;j\rangle. \tag{7}$$

The averages over $SU(2)$ CS's look as follows

$$\begin{aligned}
\langle\alpha;j|(J_+)^p|\alpha;j\rangle &= \frac{(2j)!}{(2j-p)!}\frac{\bar{\alpha}^p}{(1+|\alpha|^2)^p}, \\
\langle\alpha;j|(J_-)^p|\alpha;j\rangle &= \frac{(2j)!}{(2j-p)!}\frac{\alpha^p}{(1+|\alpha|^2)^p},\ p=0,1,...2j, \\
\langle\alpha;j|J_0|\alpha;j\rangle &= -j\frac{1-|\alpha|^2}{1+|\alpha|^2}.
\end{aligned} \tag{8}$$

We proceed now to the path integral over $SU(2)$ CS's for the partition function

$$Z = Sp\, e^{-\beta H},$$

where the Hamiltonian H belongs to the $SU(2)$ enveloping algebra. Due to Eq.(7) one has

$$Z = \int d\mu_j(\alpha)\langle\alpha;j|e^{-\beta H}|\alpha;j\rangle.$$

Defining ϵ as β/N and using Eq.(5) we write in the usual manner

$$\begin{aligned}
Z &= \int d\mu_j(\alpha)\langle\alpha;j|\alpha_N;j\rangle\langle\alpha_N;j|e^{-\epsilon H}|\alpha_{N-1};j\rangle\langle\alpha_{N-1};j|e^{-\epsilon H} \\
&\quad \ldots e^{-\epsilon H}|\alpha_0;j\rangle\langle\alpha_0;j|\alpha;j\rangle\, d\mu_j(\alpha_N)...d\mu_j(\alpha_0).
\end{aligned}$$

Up to the second order in ϵ one has

$$\langle\alpha_k;j|e^{-\epsilon H}|\alpha_i;j\rangle = \langle\alpha_k;j|\alpha_i;j\rangle \exp\left(-\epsilon \mathcal{H}_j(\bar{\alpha}_k,\alpha_i)\right),$$

where

$$\mathcal{H}_j(\bar{\alpha}_k,\alpha_i) = \frac{\langle\alpha_k;j|H|\alpha_i;j\rangle}{\langle\alpha_k;j|\alpha_i;j\rangle}.$$

The integration over $d\mu_j(\alpha)$ in accordance with Eq.(7) yields

$$\int d\mu_j(\alpha)\langle\alpha; j|\alpha_N; j\rangle\langle\alpha_0; j|\alpha; j\rangle = Tr\,|\alpha_N; j\rangle\langle\alpha_0; j|$$
$$= \langle\alpha_0; j|\alpha_N; j\rangle,$$

so that

$$Z = \int \prod_{i=0}^{N} d\mu_i \prod_{i=1}^{N} \langle\alpha_i|\alpha_{i-1}\rangle\langle\alpha_0|\alpha_N\rangle \exp\left(-\epsilon\sum_{i=1}^{N}\mathcal{H}_j(\bar{\alpha}_i, \alpha_{i-1})\right), \tag{9}$$

where the use has been made of the notation

$$d\mu_i \equiv d\mu_j(\alpha_i).$$

Note that the reproducing kernel $\langle\alpha; j|\beta; j\rangle$ acts as a delta function with respect to the measure $d\mu_j(\alpha)$. Due to this the integration over $d\mu_0$ in (9) can be carried out explicitly to yield

$$Z = \int d\mu_1 \ldots d\mu_N \prod_{i=1}^{N} \langle\alpha_i|\alpha_{i-1}\rangle \exp\left(-\epsilon\sum_{i=1}^{N}\mathcal{H}_j(\bar{\alpha}_i, \alpha_{i-1})\right)_{\alpha_0=\alpha_N}. \tag{10}$$

With $\alpha_{i-1} = \alpha_i - \delta_i$ it then follows

$$\ln\langle\alpha_i|\alpha_{i-1}\rangle = j\frac{\alpha_i\bar{\delta}_i - \bar{\alpha}_i\delta_i}{1+|\alpha_i|^2} + O(\delta_i^2). \tag{11}$$

In the continuous limit this yields

$$Z = \int_{\alpha(0)=\alpha(\beta)} D\mu_j(\alpha) \exp\left(-j\int_0^\beta \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}ds - \int_0^\beta \mathcal{H}_j(\bar{\alpha}, \alpha)ds\right), \tag{12}$$

where the following normalization holds:

$$\int_{\alpha(0)=\alpha(\beta)} D\mu_j(\alpha) \exp\left(-j\int_0^\beta \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}ds\right) = Tr\, I_j = 2j+1.$$

3 Evaluation of the path integral

In order to illustrate how the general formula (12) works, let us take H to be the linear function in the spin variables

$$H = 2\omega J_0 + \bar{\lambda}J_+ + \lambda J_-, \tag{13}$$

where ω and λ are real and complex c-numbers, respectively. The hamiltonian (13) describes a single paramagnetic spin in a constant field. For the partition function, in accordance with Eqs.(12), one obtains

$$Z_j = \int_{\alpha(0)=\alpha(\beta)} D\mu_j(\alpha) \exp\left(-\int_0^\beta \mathcal{L}_j(\bar{\alpha},\alpha)ds\right), \tag{14}$$

where the quantity

$$\mathcal{L}_j = j\frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2} - 2j\omega\frac{1-|\alpha|^2}{1+|\alpha|^2} + 2j\bar{\lambda}\frac{\bar{\alpha}}{1+|\alpha|^2} + 2j\lambda\frac{\alpha}{1+|\alpha|^2} \tag{15}$$

can be defined as the Lagrangian. The Euler-Lagrange equations lead to the equations of motion

$$\dot{\alpha} = \{\mathcal{H}_j, \alpha\},$$

where $\{,\}$ is a Poisson bracket defined by

$$\{A, B\} = \frac{(1+|\alpha|^2)^2}{2j}\left[\frac{\partial A}{\partial \alpha}\frac{\partial B}{\partial \bar{\alpha}} - \frac{\partial A}{\partial \bar{\alpha}}\frac{\partial B}{\partial \alpha}\right].$$

This indicates that the classical phase space spanned by α and $\bar{\alpha}$ is curved — in fact the complex projective line $CP^1 \simeq S^2$ [7].

$SU(2)$ being the dynamical group for the hamiltonian (13), PI (14) can be evaluated by means of $SU(2)$ transformation of integration variables. The $SU(2)$ group element can be taken to be

$$SU(2) = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix},$$

where $|u|^2 + |v|^2 = 1$. $SU(2)$ acts in the integration space $SU(2)/U(1)$ through the following canonical (projective) transformations:

$$\alpha \to \alpha = \frac{u\alpha + v}{-\bar{v}\alpha + \bar{u}}, \tag{16}$$

where the group parameters u and v are kept constant. The integration measure $d\mu_j(\alpha)$ is invariant under transformations (16). The same is true for the kinetic term, for example

$$\int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}ds \to \int_0^\beta \frac{\dot{\alpha}}{1+|\alpha|^2}\frac{\bar{u}\bar{\alpha}+\bar{v}}{-\bar{v}\alpha+\bar{u}}ds =$$

$$\int_0^\beta \frac{\bar\alpha\dot\alpha}{1+|\alpha|^2}ds+\int_0^\beta \frac{ds}{1+|\alpha|^2}\left(\dot\alpha\frac{\bar u\bar\alpha+\bar v}{-\bar v\alpha+\bar u}-\bar\alpha\dot\alpha\right)=$$
$$\int_0^\beta \frac{\bar\alpha\dot\alpha}{1+|\alpha|^2}ds-\int_0^\beta d\ln(\bar v\alpha-\bar u)=\int_0^\beta \frac{\bar\alpha\dot\alpha}{1+|\alpha|^2}ds\,,$$

where the total derivative can be dropped since $\alpha(0)=\alpha(\beta)$.

Upon taking

$$\begin{aligned}
&u=\frac{\omega}{|\omega|}\sqrt{\frac{\bar\lambda}{|\lambda|}}\cos\theta\,,\quad v=-\sqrt{\frac{\bar\lambda}{|\lambda|}}\,sin\theta,\\
&\cos\theta=\frac{1}{\sqrt2}(1+|\omega|/\Omega)^{1/2}\,,\ \sin\theta=\frac{1}{\sqrt2}(1-|\omega|/\Omega)^{1/2}\,,\\
&\Omega=\sqrt{\omega^2+|\lambda|^2}\,,
\end{aligned}\tag{17}$$

one eliminates the terms linear in λ and $\bar\lambda$ in the the exponent of Eq. (14) which results in

$$Z_j=\int\limits_{\alpha(0)=\alpha(\beta)} D\mu_j(\alpha)\exp\left(-j\int_0^\beta\frac{\bar\alpha\dot\alpha-\dot{\bar\alpha}\alpha}{1+|\alpha|^2}+2j\Omega\int_0^\beta\frac{1-|\alpha|^2}{1+|\alpha|^2}ds\right). \tag{18}$$

The latter integral can be evaluated directly through the definition (10):

$$Z_j=\lim_{N\to\infty}\int\limits_{\alpha_0=\alpha_N} d\mu_1\dots d\mu_N\prod_{i=1}^N\langle\alpha_i|e^{-2\epsilon\Omega J_0}|\alpha_{i-1}\rangle.$$

Namely, taking into account that

$$e^{-2\epsilon\Omega J_0}|\alpha_{i-1}\rangle=e^{2j\epsilon\Omega}|\alpha_{i-1}e^{-2\epsilon\Omega}\rangle\ .$$

one gets

$$Z_j=e^{2j\beta\Omega}\lim_{N\to\infty}\int\limits_{\alpha_0=\alpha_N} d\mu_1\dots d\mu_N\prod_{i=1}^N\langle\alpha_i|\alpha_{i-1}e^{-2\epsilon\Omega}\rangle. \tag{19}$$

By using decomposition of unity (5) the integration over $d\mu_1\dots d\mu_{N-1}$ can be carried out explicitly. This yields

$$\begin{aligned}
Z_j\,e^{-\beta\Omega 2j}&=\int\limits_{\alpha_N=\alpha_0} d\mu_N\langle\alpha_N|\alpha_0e^{-2N\epsilon\Omega}\rangle=\int d\mu_j(\alpha)\langle\alpha|\alpha e^{-2\beta\Omega}\rangle\\
&=(2j+1)e^{-2\beta\Omega(2j+1)}\int_0^\infty dx\frac{(1+x)^{2j}}{(e^{-2\beta\Omega}+x)^{2j+2}}=\frac{1-e^{-2\beta\Omega(2j+1)}}{1-e^{-2\beta\Omega}},
\end{aligned}$$

which gives

$$Z_j = \frac{\sinh \beta\Omega(2j+1)}{\sinh \beta\Omega},$$

which is the correct expression.

PI (18) is straightforwardly generalized for time-dependent Ω's. The same calculations as that above immediately yield

$$\int\limits_{\alpha(0)=\alpha(\beta)} D\mu_j(\alpha) \exp \left(-j \int_0^\beta \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2} + 2j \int_0^\beta \Omega(s) \frac{1-|\alpha|^2}{1+|\alpha|^2} ds \right) = \frac{\sinh(2j+1) \int_0^\beta \Omega(s) ds}{\sinh \int_0^\beta \Omega(s) ds}. \tag{20}$$

Formula (20) can directly be applied in the parasupersymmetric quantum mechanics. Let us consider the $1D$ parasupersymmetric hamiltonian [8]

$$H = \frac{1}{2}p^2 + V(x) + 2B(x)J_0,$$

where the order p of paraquantization is fixed by the spin j: $p = 2j$. The hamiltonian H describes $1D$ motion of a particle with spin $p/2$ in a particular potential $V(x)$ and magnetic field $B(x)$ related to this potential (see for details [8]). With the help of Eqs. (8) and (20) one gets

$$\begin{aligned} Z &= Tr\, e^{-\beta H} \\ &\sim \int Dx(t) D\mu_{p/2}(\alpha) \exp \left(-\frac{1}{2} \int_0^\beta \dot{x}^2 ds - \int_0^\beta V(x) ds \right. \\ &\left. -j \int_0^\beta \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2} + 2j \int_0^\beta B(x) \frac{1-|\alpha|^2}{1+|\alpha|^2} ds \right) \\ &= \int Dx(t) \exp \left(-\frac{1}{2} \int_0^\beta \dot{x}^2 ds - \int_0^\beta V(x) ds \right) \frac{\sinh(2j+1) \int_0^\beta B(x) ds}{\sinh \int_0^\beta B(x) ds}. \end{aligned} \tag{21}$$

In the supersymmetric case ($p = 1$) this reads as

$$Z \sim \int Dx(t)D \exp\left(-\frac{1}{2}\int_0^\beta \dot{x}^2 ds - \int_0^\beta V(x)ds\right) \cosh \int_0^\beta B(x)ds.$$

This formula is usually derived in the framework of the fermionic PI technique [9].

4 $SU(2)$ group contraction to the oscillator group

As is known, the $U(2)$ algebra with generators

$$J_+ \, , \, J_- \, , \, J_0 \, , \, I \, ,$$

where I is identity operator, can be contracted to the harmonic-oscillator algebra

$$a^\dagger \, , \, a \, , \, a^\dagger a \, , \, I \, ,$$

where

$$[a, a^\dagger] = 1.$$

This is done under the identification [5]

$$\lim_{j\to\infty} \frac{J_+}{\sqrt{2j}} = a^\dagger, \ \lim_{j\to\infty} \frac{J_-}{\sqrt{2j}} = a \, , \\ \lim_{j\to\infty} (J_0 + j) = a^\dagger a. \tag{22}$$

In this limit, the Bloch sphere surface contracts to the phase space of the harmonic oscillator, the ordinary complex plane. The spin CS goes over into the ordinary (Glauber) CS [5, 6]. The physical meaning of the U_2 algebra contraction can be easily read off the Holstein-Primakoff representation for spin operators [10]:

$$J_+ = \sqrt{2j} a^\dagger \left(1 - \frac{a^\dagger a}{2j}\right)^{\frac{1}{2}} \, , \ J_- = \sqrt{2j} \left(1 - \frac{a^\dagger a}{2j}\right)^{\frac{1}{2}} a, \\ J_0 = -j + a^\dagger a \tag{23}$$

The transition from Eq. (23) to Eq. (22) implies in particular

$$\frac{\langle a^\dagger a \rangle}{j} \ll 1, \tag{24}$$

where $\langle ... \rangle$ means the averaging over an unperturbed wave function.

Under the contraction procedure the PI (12) turns into the "flat" one with the measure $D\bar{\alpha}D\alpha$. This is easily seen with the help of the substitution

$$\alpha \to \alpha/\sqrt{2j}\,, \quad j \to \infty.$$

For example, PI (14) is readily seen to take the form

$$\begin{aligned} Z_j \;&=\; e^{j\int\limits_0^\beta \Omega(s)ds} \int\limits_{\alpha(0)=\alpha(\beta)} D\alpha D\bar{\alpha} \exp\left(\frac{1}{2}\int\limits_0^\beta (\dot{\bar{\alpha}}\alpha - \bar{\alpha}\dot{\alpha})ds \right. \\ &\left. - \int\limits_0^\beta \Omega(s)|\alpha|^2 ds - \sqrt{2j}\int\limits_0^\beta \bar{\lambda}\bar{\alpha}ds - \sqrt{2j}\int\limits_0^\beta \lambda\alpha ds \right), \end{aligned} \tag{25}$$

where the parameters λ and Ω can be viewed now to be time-dependent. The limiting flat forms of the $SU(2)$ PI's might be of some use in studying the spin-field interaction models with high spins. There should be mentioned, for instance, quantum-optical models based on the Jaynes-Cummings and Rabi hamiltonians [11].

5 Quantum Heisenberg model and $SU(2)$ local transformations

In this section, we will show that local linear fractional transformations (16), with u and v being time dependent, turn out to be very helpful for the PI representations of spin systems effectively interacting with the external fields. The tiresome procedure of disentangling spin operators out of the symbol of the T-ordering operator looks in the $SU(2)$ PI technique as a simple algebraic operation. As an instructive example we will consider here the quantum Heisenberg model.

A partition function for quantum Heisenberg ferromagnet

$$Z = Tr\,\exp(-\beta H)\,, \quad H = -\frac{1}{2}\sum_{i,j} K_{ij}\vec{J}_i\vec{J}_j\,,$$

where $\vec{J}_i$ are $SU(2)$ generators of dimensionality $2J+1$ and K_{ij} is the positively defined symmetric matrix of the exchange interaction can be written as [12]

$$Z = \int \prod_i D\vec{\phi}_i(t) \exp\left(-\frac{1}{2}\int_0^\beta \vec{\phi}_i(t) K_{ij}^{-1} \vec{\phi}_j(t) dt\right)$$
$$\times Tr\left[T \exp\left(-\int_0^\beta \vec{\phi}_i(t)\vec{J}_i(t)dt\right)\right], \qquad (26)$$

where the symbol T stands for a chronological product and summation over repeated indices is assumed.

In order to proceed any further with Eq.(26), one has to disentangle the operators $\vec{J}$ out of the T-ordering symbol. This has been done by Kolokolov [12] by the direct identification

$$T \exp\left(-\int_0^t \vec{\phi}(s)\vec{J}(s)ds\right) = \exp\left(J_+\psi_-(t)\right)\exp\left(J_0\int_0^t \psi_0(s)ds\right)$$
$$\times \exp\left[J_-\int_0^t \psi_+(s)\exp\left(\int_0^s \psi_0(s')ds'\right)ds\right]\exp\left(-J_+\psi_-(0)\right). \qquad (27)$$

Eq. (27) implies

$$Tr\, T \exp\left(-\int_0^\beta \vec{\phi}(s)\vec{J}(s)ds\right) = \frac{\sinh(J+1/2)\int_0^\beta \psi_0(s)ds}{\sinh\frac{1}{2}\int_0^\beta \psi_0(s)ds}. \qquad (28)$$

The set of auxiliary functions $\psi_\alpha, \alpha = 0, +, -$ are to be determined by comparing the time-derivatives of the left- and right-hand sides of Eq. (27). Proceeding in this way one finally arrives at some nonlinear differential equations on functions ψ_α, which play a key role in the disentangling procedure [12].

Let us see now how the $SU(2)$ PI works in the quantum Heisenberg model. Using averages (8) one easily finds

$$Z = \int \prod_i D\phi_i^{(0)} D\phi_i D\bar{\phi}_i \exp\left(-2\int_0^\beta \phi_i^{(0)}(t) K_{ij}^{-1}\phi_j^{(0)}(t)dt\right.$$
$$\left. -2\int_0^\beta \bar{\phi}_i(t) K_{ij}^{-1}\phi_j(t)dt\right) Z_J(\vec{\phi}), \qquad (29)$$

where

$$Z_J(\vec{\phi}) = \int\limits_{\alpha_i(0)=\alpha_i(\beta)} \prod_i D\mu_J(\alpha_i) \exp\left(J\int_0^\beta \frac{\dot{\bar{\alpha}}_i\alpha_i - \bar{\alpha}_i\dot{\alpha}_i}{1+|\alpha_i|^2}dt \right.$$
$$\left. +2J\int_0^\beta \phi_i^{(0)}\frac{1-|\alpha_i|^2}{1+|\alpha_i|^2}dt - 2J\int_0^\beta \frac{\phi_i\alpha_i}{1+|\alpha_i|^2}dt - 2J\int_0^\beta \frac{\bar{\phi}_i\bar{\alpha}_i}{1+|\alpha_i|^2}dt \right). \tag{30}$$

In the ordinary PI formulation of the quantum Heisenberg model one should take care of fixing the value of spin J at every lattice cite. This is usually done "by hand" inserting in a PI the product of appropriate δ-functions. This trick leads in turn to the additional integration over a set of auxiliary fields [13]. In the $SU(2)$ PI formulation (29) the J-dependence of the partition function appears in a natural way due to that of the $SU(2)$ CS's.

In order to calculate PI (30) we use linear fractional substitution (16) with the $SU(2)$ group parameters u and v being now considered time-dependent:

$$\alpha \to \alpha = \frac{u(t)\alpha + v(t)}{-\bar{v}(t)\alpha + \bar{u}(t)}, \tag{31}$$

where we have dropped for the moment the index dependence of the functions entering into Eq. (31). We proceed further by choosing u and v to cancel the terms linear in ϕ and $\bar{\phi}$ in the exponent in Eq.(30). After some algebra, one finally gets

$$Z_J(\vec{\phi}) = \int\limits_{\alpha_i(0)=\alpha_i(\beta)} \prod_i D\mu_J(\alpha_i) \exp\left(J\int_0^\beta \frac{\dot{\bar{\alpha}}_i\alpha_i - \bar{\alpha}_i\dot{\alpha}_i}{1+|\alpha_i|^2}dt \right.$$
$$\left. +2J\int_0^\beta \Omega_i(t)\frac{1-|\alpha_i|^2}{1+|\alpha_i|^2}dt \right) = \prod_i \frac{\sinh(2J+1)\int_0^\beta \Omega_i(t)dt}{\sinh \int_0^\beta \Omega_i(t)dt}, \tag{32}$$

where

$$\Omega = \phi^{(0)}(|u|^2 - |v|^2) - \bar{\phi}\bar{u}\bar{v} - \phi uv - \bar{v}\dot{v} - u\dot{\bar{u}}.$$

The functions $u(t)$ and $v(t)$ must be taken to satisfy the following nonlinear differential equation:

$$\bar{u}\dot{v} - v\dot{\bar{u}} + 2\phi^{(0)}\bar{u}v + \bar{\phi}\bar{u}^2 - \phi v^2 = 0\,, \tag{33}$$

and the conjugate one

$$\bar{v}\dot{u} - u\dot{\bar{v}} + 2\phi^{(0)}\bar{v}u - \bar{\phi}\bar{v}^2 + \phi u^2 = 0\,, \tag{34}$$

with the boundary conditions

$$u(0) = u(\beta) \text{ and } v(0) = v(\beta).$$

In getting Eq. (34) there should be kept in mind that the conjugation operator changes the sign of the time derivatives. By setting $z = \bar{u}/v$ in Eq. (33) one is led to the equation

$$\dot{z} - 2\phi^{(0)}z - \bar{\phi}z^2 + \phi = 0\,, \quad z(0) = z(\beta)\,, \tag{35}$$

which is of the Riccati type. Eq. (34) then reads

$$\dot{\bar{z}} + 2\phi^{(0)}\bar{z} + \phi\bar{z}^2 - \bar{\phi} = 0\,, \quad \bar{z}(0) = \bar{z}(\beta)\,. \tag{36}$$

Consequently, parameter Ω entering Eq. (32) becomes

$$\Omega = -\phi^{(0)} - \frac{1}{2}(\bar{\phi}z + \phi\bar{z}).$$

Eq. (35) coincides exactly with the Kolokolov's disentangling condition. This is easily seen under the identification

$$\phi^{(-)} = -2\phi\,, \quad \phi^{(+)} = \bar{\phi}\,, \quad \phi^{(z)} = 2\phi_0\,,$$
$$\psi_- = 2z\,, \quad \psi_+ = \bar{\phi}\,, \quad \psi_0 = 2\phi_0 + 2\bar{\phi}z\,,$$

wherein the left-hand sides of the equalities are referred to the Kolokolov's notation; and the right-hand sides, to ours. It could also be checked that the product of traces (28) over the lattice sites coincides with expression (32). All this proves that the $SU(2)$ local linear fractional transformations (31) lead to the very same result as that of the direct $SU(2)$ disentangling procedure.

Inserting then Eq. (32) into Eq. (29), with the supplementary conditions (33,34) being in mind, one arrives at the PI representation for the Heisenberg partition function. To proceed further one could try to express the integration variables $\vec{\phi}$ in (29) through the complex functions u and v. To evaluate the jacobian $J(u, v)$ of this transformation one should take into account Eqs. (33,34) and the $SU(2)$ relation $|u|^2 + |v|^2 = 1$. The resulting expression is a PI over two interacting scalar Bose fields [12]. This and related problems will be discussed elsewhere.

There should be pointed out that Eqs. (33,34) have obvious time -independent solutions in the form of Eqs.(17), provided the functions $\vec{\phi}$ do not depend upon time. This means nothing but the mean-field approximation for the PI (29). Indeed, assuming the Hubbard-Stratonovich functions $\vec{\phi}_i$ to be constant, we have from (17,29,32)

$$Z = \int \prod_i d\vec{\phi}_i \exp\left(-2\beta \sum_{ij} \vec{\phi}_i K_{ij}^{-1} \vec{\phi}_j + \sum_{i=1}^{N} \ln \frac{\sinh(2J+1)\beta|\vec{\phi}_i|}{\sinh \beta|\vec{\phi}_i|}\right). \tag{37}$$

The saddle-point equation reads

$$\vec{\phi}_i = \frac{J}{2} \sum_{j=1}^{N} K_{ij} \frac{\vec{\phi}_j}{|\vec{\phi}_j|} B_J(2J\beta|\vec{\phi}_j|), \tag{38}$$

where

$$B_J(x) = \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} x\right) - \frac{1}{2J} \coth\left(\frac{x}{2J}\right)$$

is the Brillouin function. If we assume translational invariance for the system, it becomes

$$\phi = \frac{Jk}{2} B_J(2J\beta\phi), \tag{39}$$

where ϕ denotes the modulus of $\vec{\phi}_i$, and

$$k = \sum_j K_{ij}.$$

Eqs. (38,39) at $J = 1/2$ coincide exactly with those obtained by Liebler and Orland in the framework of conventional path- integral technique [14].

Note in conclusion that the $SU(2)$ PI bears all the "internal" $SU(2)$ group structure. This makes it to be very convenient in applications to interacting spin-spin or spin-field systems. In particular, the $SU(2)$ linear fractional transformations put into consideration the set of new functions $u_i(t)$ and $v_i(t)$ subjected to the constraints $|u_i|^2 + |v_i|^2 = 1$. The latter seem to be important in attempts to reveal the connection between the interacting spin systems and nonlinear σ-models.

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