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SU(1,1) COHERENCE-PRESERVING HAMILTONIANS: A PATH-INTEGRAL APPROACH

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1 Introduction

The aim of this paper is to develop a self-consistent path-integral approach for studying quantum systems driven by hamiltonians with $SU(1,1)$ dynamical symmetry. We provide here the correct expression for the $SU(1,1)$ path integral (PI) and the direct receipt for its calculation. For the conventional flat PI based on the coherent states (CS) associated with the oscillator group this receipt means nothing but the linear shift of integration variables. In the case of $SU(1,1)$ the integration space turns out to be a curved one so that the needed transformation looks a bit complicated.

Dynamical $SU(1,1)$ symmetry means that a given hamiltonian h can be cast into a linear combination of the $SU(1,1)$ generators. It then follows that the state space is embedded in the UIR' of $SU(1,1)$. The associated CS' provide a convenient basis for the $SU(1,1)$ UIR'. It can be shown that a given CS evolves under time evolution driven by h in another CS. Thus, a general $SU(1,1)$ coherence preserving hamiltonian looks as a linear time-dependent combination of the $SU(1,1)$ generators [1, 2].

A large variety of modelled systems with the $SU(1,1)$ coherence-preserving hamiltonians are relevant for quantum optics and nuclear physics. These are various parametric amplifiers associated with the production of nonclassical states of the quantized EM field [2, 3, 4], damped oscillator [5], superfluid Bose systems [6].

The most general $SU(1,1)$ coherence-preserving hamiltonian reads [2]

$$h = 2A(t)K_0 + f(t)K_+ + \bar{f}(t)K_-, \tag{1}$$

where operators $K_0, K_\pm$ span $SU(1,1)$ algebra

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0 \tag{2}$$

and $A(t)$ is a real, and $f(t)$ is an arbitrary complex, functions of time.

The probability amplitude is defined by

$$U_{fi} = \langle f| \, T\exp\left(-i\int_0^\tau h(s)ds\right)|i\rangle, \tag{3}$$

where the symbol T stands for the time-ordering operator. To proceed further with Eq.(3), one needs to get rid of time-ordering by employing a suitable disentangling procedure. This procedure has been proposed by Wie and Norman [7]. For time-independent hamiltonians the Wie-Norman method results in the well-known Baker-Hausdorff formula. A detailed analysis of (1) in the framework of the Wie-Norman method has been given by Dattoli et al [8, 9, 10].

The key point of the Wie-Norman approach is the following disentangling formula

$$T\exp\left(-i\int_0^\tau h(s)ds\right) = \exp(\psi_0(\tau)K_0)\exp(\psi_+(\tau)K_+)\exp(\psi_-(\tau)K_-),$$

where the set of auxiliary functions $\{\psi\}$ is to be determined through the set of coupled linear differential equations. The latter depend upon the $SU(1,1)$ structure constants.

The second approach to calculating (3) is just to rewrite it as an infinite product of short time-evolution operators

$$U_{fi} = \lim_{N\to\infty} \langle f| \prod_{j=1}^{N} U_\epsilon(t_j)|i\rangle ,$$

where

$$U_\epsilon(t_j) = \exp(-i\epsilon h(t_j)), \quad \epsilon = \tau/N.$$

By making multiple use of decomposition of unity in terms of $SU(1,1)$ CS' in the above equation and calculating appropriate matrix elements to order ϵ one arrives in the limit $N \to \infty$ at the Wie-Norman results. This has successfully been demonstrated by Gerry [11] for the $SU(1,1)$ case and later by Ellinas for the $SL(2;C)$ case [12].

In this paper we will show that PI over $SU(1,1)$ CS' provides a convenient approach to calculating (3). It turns out to be an alternative to the Wie-Norman procedure, but possesses the advantages common to all PI representations: compactness, convenience in handling and universality.

2 $SU(1,1)$-coherent states and the path integral

Let us remind briefly the properties of the $SU(1,1)$ CS'. The $3D$ Lie algebra of $SU(1,1)$ consists in the Cartan basis of the operators (2). We are only interested in the positive discrete series representation $\mathcal{D}^+(k)$ with basis states denoted as $\{|m,k\rangle\}$, where

$$K_0|m,k\rangle = (m+k)|m,k\rangle \quad m = 0,1,2,\dots$$

and k is the Bargmann (representation) index [13]. Due to Ref. [15] the $SU(1,1)$ CS can be represented as

$$|\zeta;k\rangle = (1-|\zeta|^2)^k e^{\zeta K_+}|0;k\rangle , \tag{4}$$

where the complex parameter ζ belongs to the coset $SU(1,1)/S(U_1 \times U_1)$ which is isomorphic to the Lobachevskii plane L^2 in the Poincaré model. The latter can be realized in the form of the interior of the unit disk in C^1 with the $SU(1,1)$ invariant metric

$$ds^2 = \frac{4d\zeta d\bar\zeta}{(1-|\zeta|^2)^2}, \quad |\zeta| < 1 . \tag{5}$$

The manifold (5) appears as a classical phase space for system (1). Note that CS (4) depends upon the representation index k.

The overlap of two states is given by

$$\langle\zeta';k|\zeta;k\rangle = (1-|\zeta'|^2)^k(1-|\zeta|^2)^k(1-\bar\zeta'\zeta)^{-2k} . \tag{6}$$

An important property of these states is that they satisfy the completeness relation

$$\int |\zeta;k\rangle\langle\zeta;k|\, d\mu_k(\zeta) = I_k , \tag{7}$$

where the $SU(1,1)$ invariant measure looks as follows

$$d\mu_k(\zeta) = \frac{2k-1}{\pi}\frac{d^2\zeta}{(1-|\zeta|^2)^2} . \tag{8}$$

For the further use we need the $SU(1;1)$ averages

$$\langle\zeta;k|K_0|\zeta;k\rangle = k\frac{1+|\zeta|^2}{1-|\zeta|^2}, \quad \langle\zeta;k|K_+(K_-)|\zeta;k\rangle = 2k\frac{\bar\zeta(\zeta)}{1-|\zeta|^2} . \tag{9}$$

We proceed now to the construction of the PI over $SU(1,1)$ CS' for the probability amplitude (3), the final and the initial states being taken to be the coherent ones

$$\langle\zeta_1;k|T\exp(-i\int_0^\tau hdt)|\zeta_2;k\rangle \equiv G_k(\bar\zeta_1,\zeta_2;\tau) \tag{10}$$

Making multiple use of decomposition of unity (7) in the l.h.s. of Eq.(10) one obtains in the continuous limit

$$\begin{aligned} G_k(\bar\zeta_1,\zeta_2;\tau) &= \int_{\zeta(0)=\zeta_2}^{\bar\zeta(\tau)=\bar\zeta_1} D\mu_k(\zeta)\frac{(1-|\zeta_1|^2)^k(1-|\zeta_2|^2)^k}{(1-\bar\zeta_1\zeta(\tau))^k(1-\bar\zeta(0)\zeta_2)^k}\times \\ &\exp\left(k\int_0^\tau \frac{\dot{\bar\zeta}\zeta-\bar\zeta\dot\zeta}{1-|\zeta|^2}ds - i\int_0^\tau h_k(\bar\zeta,\zeta)ds\right), \end{aligned} \tag{11}$$

where

$$h_k(\bar\zeta,\zeta) = \langle\zeta;k|h|\zeta;k\rangle$$

and the following normalization holds

$$G_k|_{h=0} = \langle\zeta_1;k|\zeta_2;k\rangle .$$

The Euler-Lagrange equations yield

$$\begin{aligned} \dot\zeta &= \{\zeta,h_k\}, \quad \zeta(0)=\zeta_2 \\ \dot{\bar\zeta} &= \{\bar\zeta,h_k\}, \quad \bar\zeta(\tau)=\bar\zeta_1 , \end{aligned} \tag{12}$$

where $\{,\}$ is a Poisson bracket defined by

$$\{A,B\} = \frac{(1-|\zeta|^2)^2}{2ik}\left(\frac{\partial A}{\partial \zeta}\frac{\partial B}{\partial \bar\zeta} - \frac{\partial A}{\partial \bar\zeta}\frac{\partial B}{\partial \zeta}\right).$$

This indicates that the classical phase space spanned by ζ and $\bar\zeta$ is curved, in fact the Lobachevskii plane in the form of a unit disk (Poincaré model) [14].

It is of particular importance to note that the additional functional in Eq.(11)

$$\frac{(1-|\zeta_1|^2)^k(1-|\zeta_2|^2)^k}{(1-\bar\zeta_1\zeta(\tau))^k(1-\bar\zeta(0)\zeta_2)^k} \tag{13}$$

reflects the difference in the boundary conditions: at $t=0$ $\zeta(t)$ is fixed while at $t=\tau$ $\bar\zeta(t)$ is fixed. It should be stressed that $\bar\zeta(0)$ and $\zeta(0)=\zeta_2$ as well as $\zeta(\tau)$ and $\bar\zeta(\tau)=\bar\zeta_1$ are to be considered to be independent. For example, we integrate over $\zeta(\tau)$ while $\bar\zeta(\tau)$ is fixed. It is the additional term (13) that provides the correct equation of motion (12). Sometimes there appear rather formal $SU(1,1)$ path-integral representations for the transition amplitude (10) with the boundary fixing term (13) being omitted. For the ordinary flat PI over Glauber CS' the boundary fixing terms have been discussed in Ref. [16].

3 Evaluation of the path integral

PI (11) can be evaluated directly, provided h belongs to the $SU(1,1)$ algebra. This is in a complete analogy with the flat case, where the gaussian PI stems from the particular form of the hamiltonian

$$h = \omega a^\dagger a + \lambda a + \bar\lambda a^\dagger$$

and can be calculated with the help of linear shifts of the integration variables. From the group-theoretical point of view these shifts can be thought of as the oscillator group action in the classical phase space.

Let us take h in the form (1), where we allow for the explicit time-dependence of entering parameters. Then in view of Eq. (9) one has

$$h_k(\bar\zeta,\zeta) = 2kA(t)\frac{1+|\zeta|^2}{1-|\zeta|^2} + 2kf(t)\frac{\bar\zeta}{1-|\zeta|^2} + 2k\bar f(t)\frac{\zeta}{1-|\zeta|^2}. \tag{14}$$

Let us take the $SU(1,1)$ action in C^2 in the form

$$g \in SU(1,1) = \begin{pmatrix} u & v \\ \bar v & \bar u \end{pmatrix},$$

where $|u|^2-|v|^2=1$. $SU(1,1)$ acts in the coset space $SU(1,1)/S(U_1\times U_1)$ through the following canonical (projective) transformations

$$\zeta \to \zeta = \frac{u\eta+v}{\bar v\eta+\bar u}. \tag{15}$$

Under the action of (15) an open disk in C^1 transforms into itself. Equation (15) leaves the metric (5) invariant. In other words, $SU(1,1)$ acts as a group of motion of the Lobachevskii plane.

Let us change the integration variables in (11) by means of substitution (15), parameters u and v being time-dependent. The integration measure in (11) remains invariant. Let us then fix the $SU(1,1)$ parameters by demanding that the terms linear in f and $\bar f$ in PI (11) with hamiltonian (14) vanish after substitution (15). After some tedious but straightforward algebra one gets

$$\begin{aligned} G_k(\bar\zeta_1,\zeta_2;\tau) = {} & \left(\frac{v(\tau)\bar\eta_1+u(\tau)}{\bar v(\tau)\eta_1+\bar u(\tau)}\right)^k\left(\frac{\bar v(0)\eta_2+\bar u(0)}{v(0)\bar\eta_2+u(0)}\right)^k\int_{\eta(0)=\eta_2}^{\bar\eta(\tau)=\bar\eta_1} D\mu_k(\eta)\times \\ & \frac{(1-|\eta_1|^2)^k(1-|\eta_2|^2)^k}{(1-\bar\eta_1\eta(\tau))^k(1-\bar\eta(0)\eta_2)^k}\times \\ & \exp\left(k\int_0^\tau\frac{\dot{\bar\eta}\eta-\bar\eta\dot\eta}{1-|\eta|^2}ds - 2ik\int_0^\tau\Omega(s)\frac{1+|\eta|^2}{1-|\eta|^2}ds\right) \end{aligned} \tag{16}$$

$$\begin{aligned} = {} & \left(\frac{v(\tau)\bar\eta_1+u(\tau)}{\bar v(\tau)\eta_1+\bar u(\tau)}\right)^k\left(\frac{\bar v(0)\eta_2+\bar u(0)}{v(0)\bar\eta_2+u(0)}\right)^k\times \\ & \frac{(1-|\eta_1|^2)^k(1-|\eta_2|^2)^k}{(1-\bar\eta_1\eta_2\exp(-2i\int_0^\tau\Omega ds))^{2k}}\exp(-2ik\int_0^\tau\Omega ds), \end{aligned} \tag{17}$$

where

$$\eta_2 = \frac{\bar u(0)\zeta_2 - v(0)}{-\bar v(0)\zeta_2+u(0)} \quad \bar\eta_1 = \frac{u(\tau)\bar\zeta_1-\bar v(\tau)}{-v(\tau)\bar\zeta_1+\bar u(\tau)} \tag{18}$$

and

$$\Omega(s) = \frac{i}{2}(\dot{\bar v}v - \dot u\bar u - \dot v\bar v + \dot{\bar u}u) + f(s)\bar u\bar v + \bar f(s)uv + A(s)(|u|^2+|v|^2). \tag{19}$$

In passing from (16) to (17) we use that the PI in Eq.(16) due to its pointwise definition is equal to

$$\langle\eta_1;k|\exp(-2iK_0\int_0^\tau\Omega ds)|\eta_2;k\rangle = \langle\eta_1;k|\eta_2e^{-2i\int_0^\tau\Omega ds};k\rangle\exp(-2ik\int_0^\tau\Omega ds).$$

Inserting back Eq.(18) into Eq.(17) one finally arrives at

$$G_k(\bar{\zeta}_1,\zeta_2;\tau)=\frac{(1-|\zeta_1|^2)^k(1-|\zeta_2|^2)^k}{[\Lambda(\bar{\zeta}_1,\zeta_2;k)]^{2k}}\exp(-2ik\int_0^\tau\Omega(s)ds)\,, \qquad (20)$$

where

$$\begin{aligned}&\Lambda(\bar{\zeta}_1,\zeta_2;k)=\\&\bar{u}(\tau)u(0)-\bar{v}(\tau)v(0)e^{-2i\int_0^\tau\Omega ds}+\zeta_2\left(\bar{v}(\tau)\bar{u}(0)e^{-2i\int_0^\tau\Omega ds}-\bar{v}(0)\bar{u}(\tau)\right)+\\&\bar{\zeta}_1\left(u(\tau)v(0)e^{-2i\int_0^\tau\Omega ds}-v(\tau)u(0)\right)+\bar{\zeta}_1\zeta_2\left(v(\tau)\bar{v}(0)-u(\tau)\bar{u}(0)e^{-2i\int_0^\tau\Omega ds}\right)\end{aligned} \qquad (21)$$

Functions u,v are to be taken to satisfy

$$\begin{aligned}i(\dot{\bar{v}}u-\dot{u}\bar{v})+2A\bar{v}u+f\bar{v}^2+\bar{f}u^2=0\\-i(\dot{v}\bar{u}-\dot{\bar{u}}v)+2a\bar{u}v+\bar{f}v^2+f\bar{u}^2=0.\end{aligned} \qquad (22)$$

The sufficient conditions for Eqs.(22) to be fulfilled are as follows

$$\begin{aligned}\dot{\bar{v}}&=iA\bar{v}+i\bar{f}u\\\dot{u}&=-iAu-if\bar{v}\end{aligned} \qquad (23)$$

It is easy to convince oneself that

$$\Omega(s)=0$$

on the solutions of system (23). Equation (20) then reads

$$G_k(\bar{\zeta}_1,\zeta_2;\tau)=\frac{(1-|\zeta_1|^2)^k(1-|\zeta_2|^2)^k}{[\bar{a}(\tau)+\bar{b}(\tau)\zeta_2-b(\tau)\bar{\zeta}_1-a(\tau)\bar{\zeta}_1\zeta_2]^{2k}}, \qquad (24)$$

where

$$a(t)=u(t)\bar{u}(0)-v(t)\bar{v}(0)\,,\quad b(t)=v(t)u(0)-u(t)v(0)\,. \qquad (25)$$

Due to Eqs.(23) $a(t)$ and $b(t)$ satisfy

$$\begin{aligned}\dot{a}&=-iAa-if\bar{b}\\\dot{b}&=-iAb-if\bar{a}\end{aligned} \qquad (26)$$

with the boundary conditions

$$a(0)=1\,,\quad b(0)=0\,.$$

Formulae (24,26) coincide exactly with those obtained by Gerry [11] and Ellinas [12].

In conclusion let us briefly comment on the case of the time-independent $SU(1,1)$ hamiltonian. Consider

$$G_k^{(0)}(\bar{\zeta}_1,\zeta_2;\tau)=\langle\zeta_1;k|\exp-i\tau\left(2AK_0+fK_++\bar{f}K_-\right)|\zeta_2;\tau\rangle\,, \qquad (27)$$

where A and f are regarded to be constants. $SU(1,1)$ acts in the Hilbert space of analytic functions defined on $|z|<1$ as follows [14]

$$\left(U_gf\right)(z)=(\bar{v}z+\bar{u})^{-2k}f\left(\frac{uz+v}{\bar{v}z+\bar{u}}\right),$$

where $g\in SU(1,1)$. This implies that

$$U_g|\zeta;k\rangle=\left(\frac{v\bar{\zeta}+u}{\bar{v}\zeta+\bar{u}}\right)^k\left|\frac{u\zeta+v}{\bar{v}\zeta+\bar{u}}\right\rangle.$$

Let U_g be taken to diagonalize h:

$$U_ghU_g^\dagger=2\Omega_0K_0\,,\ \Omega_0=\sqrt{A^2-|f|^2},\ |A|>|f|.$$

The operator we need is just

$$U_g=\exp\theta(\bar{f}K_--fK_+)\,,$$

with

$$\sinh 2|f|\theta=|f|/\Omega_0\,,\quad\cosh 2|f|\theta=A/\Omega_0\,.$$

Then

$$\begin{aligned}G^{(0)}&=\langle\zeta_1|U_g\exp(-2i\tau\Omega_0K_0)U_g^\dagger|\zeta_2\rangle\\&=\left\langle\frac{u\zeta_1-v}{-\bar{v}\zeta_1+\bar{u}}\middle|\frac{u\zeta_2-v}{-\bar{v}\zeta_2+\bar{u}}e^{-2i\tau\Omega_0}\right\rangle=\frac{(1-|\zeta_1|^2)^k(1-|\zeta_2|^2)^ke^{-2i\tau k\Omega_0}}{[\Lambda^{(0)}(\bar{\zeta}_1,\zeta_2;k)]^{2k}},\end{aligned} \qquad (28)$$

where

$$\begin{aligned}\Lambda^{(0)}(\bar{\zeta}_1,\zeta_2;k)&=|u|^2-e^{-2i\tau\Omega_0}|v|^2+\bar{v}u\zeta_2(e^{-2i\tau\Omega_0}-1)\\&+\ \bar{u}v\bar{\zeta}_1(e^{-2i\tau\Omega_0}-1)+\bar{\zeta}_1\zeta_2(|v|^2-e^{-2i\tau\Omega_0}|u|^2)\,,\end{aligned} \qquad (29)$$

$$u=\cosh|f|\theta\,,\quad v=-\frac{f}{|f|}\sinh|f|\theta\,.$$

On the other hand, the solution of Eq.(26) at A and f fixed looks as follows

$$a(t)=\cos\Omega_0t-i\cosh 2|f|\theta\sin\Omega_0t\,,\ \bar{b}(t)=i\frac{\bar{f}}{|f|}\sinh 2|f|\theta\sin\Omega_0t\,. \qquad (30)$$

Being inserted into Eq.(24) it results in Eq.(28).

The last remark is that the PI over $SU(1,1)$ CS' might be used for constructing the probability amplitude for a system with a hamiltonian that belongs to the $SU(1,1)$ enveloping algebra. All the needed averages are given in [19]. A lot of systems are relevant for the case [17, 18, 19], the interesting point being that they exhibit the nontrivial behavior and rich vacuum structure.

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