объединенный
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исследований
дубна

E17-92-58

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STATISTICS OF THE PARAMETRIC OSCILLATOR IN THE THERMOSTAT

Submitted to "ТМФ"

1992

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Статистика параметрического осциллятора
в термостате

Определены области значений частоты, температуры, оптической накачки и времени, допускающих суб-пуассоновскую статистику чисел заполнения для параметрического осциллятора, имеющего начальное состояние в виде суперпозиции сигнала и шума. Показано, что задержка суб-пуассоновского эффекта вызывается тепловыми флуктуациями в длинноволновом диапазоне, подавление этого эффекта проявляется в присутствии порога по частоте при низких температурах, обусловленного квантовым шумом.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1992

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Statistics of the Parametric Oscillator
in the Thermostat

The range of values of the frequency, temperature, optical pumping and time admitting the sub-Poisson statistics of the occupation number for the parametric oscillator with the initial superposition of the signal and noise is defined. The delay of sub-Poisson effect is caused by the thermal fluctuations in the long-wave region and the suppression of this effect manifests itself in the presence of the frequency threshold at low temperatures caused by the quantum noise.

Tne investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1992

The coherent bose-oscillator states are usually used as a model of the coherent radiation in the laser theory, the Poisson distribution for the occupation number of which is interpreted as a light quantum number. The oscillator states with the sub-Poisson statistics of the occupation number are investigated in recent time in connection with the experimental observation of weak fluctuations This statistics corresponds to the variability of the occupation number which is smaller than that in the case of the Poisson statistics. This effect is relative to the squeezed light. The criterion of the sub-Poisson effect existence is that the average occupation number n exceeds its dispersion $D(n)$ [1]- [3]. A mathematical problem is to find the evolution of the initial statistical distribution of the system with Hamiltonian H. It is reached by the calculation of the average occupation number and its dispersion as the time function according to the formula for the average value A

$$< A >= Tr(U^+ A U R_0), \qquad U = T \exp(-i \int\limits_0^t d\tau H(\tau)). \tag{1}$$

Usually, in the theoretical works R_0 is taken as the Poisson distribution [2, 3] $\rho_0 = |z_0 >< z_0|$, where the dimensionless parameter z_0 defines coherent states of the oscillator. The parameter z_0 is associated with the laser field , the transition of which to the sub-Poisson statistics regime takes place at the same evolution law U. To estimate the limits of the criterion of the sub-Poisson statistics $D(t) < n(t)$ one should find the connection between the parameter z_0 and dimensional physical values. It is interesting to investigate a more general than ρ_0 initial distribution. In the present paper, the problem of the sub-Poisson statistics has been solved for the initial distribution R_0, which is a superposition of the signal and noise in the case of Hamiltonian H describing the parametric excitement of a bose-oscillator. The parameter z is expressed over observables according to $R_0 \to \rho_0$ at low temperatures.

1. The parametric amplification of the signal and noise superposition

The Hamiltonian of the bose-oscillator

$$H = \omega\psi^+\psi + \frac{g}{2}(\psi^2 e^{i2\omega t - i\varphi} + (\psi^+)^2 e^{-i2\omega t + i\varphi}) \qquad (2)$$

describes the parametric excitement on the frequency 2ω (pumping) of the field with the frequency ω of the fundamental wave, φ is a pumping phase, g is a pumping amplitude multiplied by the interaction constant. The initial distribution is assumed to be Gibbs distribution:

$$R_0 = \frac{1}{Q} e^{-\beta(\omega\psi^+\psi + j\psi^+ + j^*\psi)},$$

$$Q = \frac{\exp(|j|^2/(\omega\beta))}{1 - \exp(-\omega\beta)} = Tr R_0, \qquad \beta = T^{-1}, \qquad (3)$$

describing the radiation with the frequency ω in the state , which is a superposition of the coherent signal with the intensity j and of the thermal noise at the temperature T. The average photon number for the distribution (3) is the sum of its equilibrium number n_0 and the number caused by coherent signal

$$n = \frac{\partial \ln Q}{\partial(-\omega\beta)} = n_0 + <\psi><\psi^+> = <\psi^+\psi>,$$

$$n_0 = (e^{\omega\beta} - 1)^{-1}, \qquad <\psi> = \frac{j}{\omega}.$$

In the coherent basis the matrix elements of the operator R_0 are shown to have the form:

$$\frac{<z|R_0|z'>}{<z|z'>} = (1 - e^{-\omega\beta}) \exp\left((e^{-\omega\beta} - 1)(z^* + \frac{j^*}{\omega})(z' + \frac{j}{\omega})\right),$$

$$\frac{<z|R_0|z'>}{<z|z'>} \to \begin{cases} (1 - e^{-\omega\beta}) \exp[z^* z'(e^{-\omega\beta} - 1)], & j \to 0 \\ \frac{<z|-\frac{j}{\omega}><-\frac{j}{\omega}|z'>}{<z|z'>}, & T \to 0. \end{cases} \qquad (4)$$

It follows that the Poisson distribution $\rho_0 = |z_0><z_0|$ is the limit of the initial distribution (3) at the zero- temperature and the parameter z_0 is connected with the distribution parameter (3) by the formula $z_0 = \frac{j}{\omega}$. At high temperatures distribution (3) is the Plank distribution. The competition between these distributions leads to the formation of a new distribution. Our aim is to define this distribution which corresponds to the evolution with the Hamiltonian H.

2. The matrix transition elements and generating function

It is convenient to define the average photon number n and its dispersion $D(n)$ for the problem of the sub-Poisson statistics of the model (2) with the initial condition (3) by the generating function

$$P = Tr(U^+ e^{-\varepsilon\psi^+\psi} U R_0),$$

$$n = -\frac{\partial P}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \qquad D(n) = \frac{\partial^2 P}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} - n^2. \qquad (5)$$

One should calculate P in the coherent state representation

$$P = \int d\mu <z|U^+|z'><z'|e^{-\varepsilon\psi^+\psi}|z''><z''|U|z'''><z'''|R_0|z>,$$

$$d\mu = \pi^{-4} d^2z d^2z' d^2z'' d^2z'''.$$

In formula (4) the dotproduct of the coherent states for the matrix elements of the operators R_0 and $\exp(-\varepsilon\psi^+\psi)$ is equal to

$$< z|z' >= \exp(-\frac{|z|^2}{2} - \frac{|z'|^2}{2} + z^* z').$$

Let us describe the matrix element of the evolution operator with the functional integral method

$$< z|U|z' >= J \exp(-\frac{|z|^2}{2} - \frac{|z'|^2}{2}),$$

$$J = \int D\psi^* D\psi e^{iS + z^* \psi(t)} = \frac{e^{iS_0 + z^* \psi_0(t)}}{\sqrt{Det \delta^2 S_0}},$$

$$z^* = \psi^*(t), \qquad z' = \psi(0), \qquad iS_0 + z^* \psi_0 = \Phi,$$

where the subscript '0' means that the action S and the determinant of its second variation are calculated on the extreme path ψ_0, ψ_0^*. The equation for these paths:

$$i\frac{d}{d\tau}\psi_0 - \omega\psi_0 - \psi_0^* g e^{-i\omega\tau + i\varphi_0} = 0,$$

$$-i\frac{d}{d\tau}\psi_0^* - \omega\psi_0^* - \psi_0 g e^{i\omega\tau - i\varphi_0} = 0,$$

$$\varphi_0 = \varphi - \omega t$$

with the boundary conditions for ψ and ψ^* have the solutions:

$$\tilde{\psi}(\tau) = z'\big(ch(g\tau) - sh(g\tau)th(gt)\big) - iz^* e^{i\varphi_0}\frac{sh(g\tau)}{ch(gt)},$$

$$\tilde{\psi}^*(\tau) = z^* \frac{ch(g\tau)}{ch(gt)} + iz' e^{-i\varphi_0}\big(sh(g\tau) - ch(g\tau)th(gt)\big), \tag{6}$$

$$\psi_0 = \tilde{\psi} e^{-i\omega\tau}, \qquad \psi_0^* = \tilde{\psi}^* e^{i\omega(\tau - t)}.$$

The phase of the integral J on the extreme paths (6) is equal to

$$\Phi_0 = \frac{z' z^* e^{-i\omega t}}{ch(gt)} - \frac{i}{2} th(gt)\big((z')^2 e^{-i\varphi} + (z^*)^2 e^{i(\varphi_0 - \omega t)}\big).$$

The calculation of the determinant of the second variation operator

$$Det\delta^2 S_0 = Det K, \qquad S = \frac{1}{2}\begin{pmatrix}\psi \\ \psi^*\end{pmatrix} K \begin{pmatrix}\psi \\ \psi^*\end{pmatrix}$$

may be reduced to defining the inverse operator K^{-1} with the zero boundary conditions

$$Tr(\ln K) = \ln(Det K), \qquad \frac{\partial}{\partial g}\ln(Det K) = Tr(K^{-1}\frac{\partial K}{\partial g}),$$

$$Det K = \exp\left(\int dg Tr(K^{-1}\frac{\partial K}{\partial g})\right),$$

$$\begin{pmatrix}\psi \\ \psi^*\end{pmatrix} = K^{-1}\begin{pmatrix}A \\ A^*\end{pmatrix}, \qquad \psi(0) = 0, \qquad \psi^*(t) = 0.$$

By means of the substitution

$$\begin{pmatrix}\psi \\ \psi^*\end{pmatrix} = U\begin{pmatrix}\eta \\ \eta^*\end{pmatrix}, \qquad U = \begin{pmatrix} e^{-i\omega\tau} & 0 \\ 0 & e^{i\omega\tau}\end{pmatrix}$$

the new unknown operator is introduced K_0^{-1} :

$$K_0\begin{pmatrix}\eta \\ \eta^*\end{pmatrix} = \begin{pmatrix}A \\ A^*\end{pmatrix}, \qquad K_0 = \begin{pmatrix} g & -i\frac{d}{d\tau} \\ i\frac{d}{d\tau} & g\end{pmatrix}, \tag{7}$$

$$K^{-1}\frac{\partial K}{\partial g} = UK_0^{-1}\frac{\partial}{\partial g}K_0 U^+, \qquad \eta(0) = 0, \qquad \eta^*(t) = 0.$$

The solution of the differential equations system (7) with the zero boundary conditions is equal to

$$\begin{pmatrix}\eta\\ \eta^*\end{pmatrix} = K_0^{-1}\begin{pmatrix}A\\ A^*\end{pmatrix} = -i\int_0^t d\tau'\Big\{\Theta(\tau-\tau')e^{i\sigma g(\tau-\tau')}+$$

$$+e^{i\sigma g\tau}\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix}\Big\}\sigma\begin{pmatrix}A\\ A^*\end{pmatrix},$$

$$\sigma = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad a = -(\mathrm{ch(gt)})^{-1}.$$

The calculation of the diagonal part of the operator

$$Tr(K^{-1}\frac{\partial K}{\partial g}) = \int_0^t d\tau\ \left[(K_0)^{-1}_{11} + (K_0)^{-1}_{22}\right]\Big|_{\tau=\tau'} = t\ th(gt)$$

results in:

$$Det K = \exp\left(t\int dg\ th(gt)\right) = ch(gt),$$

$$< z|U|z' >= \frac{1}{\sqrt{ch(gt)}}\exp\left(-\frac{|z|^2}{2} - \frac{|z'|^2}{2} + \Phi_0\right). \tag{8}$$

It is necessary to note that the functional method of the integral calculation involved here is distinguished by the compactness and an universality compared with the finite - multiplicity approximation method used earlier [5] for deriving formula (8).

Now, formulae (4) and (3) are substituted to the function P and the four-fold integral is calculated in the complex plane of the coherent states. Using consecutively the formula

$$\int\frac{d^2z}{\pi}\exp\left(-|z|^2 + az + bz^* + cz^2 + d(z^*)^2\right) =$$

$$= \frac{1}{\sqrt{1-4cd}}\exp\left(\frac{ab + cb^2 + da^2}{1-4cd}\right), \tag{9}$$

one can obtain

$$P = \left(1 - e^{-\omega\beta}\right)\exp\left((e^{-\omega\beta} - 1)\frac{|j|^2}{\omega^2}\right)\sqrt{E}\times$$

$$\times\exp\left(-\frac{i}{2}th(gt)e^{-i\varphi}(1 - e^{-\omega\beta})^2\frac{j^2}{\omega^2}x\right)F, \tag{10}$$

$$E = \frac{1}{ch^2(gt) - e^{-2\varepsilon}sh^2(gt)}, \qquad x = (1 - e^{-2\varepsilon}E) \to_{\varepsilon\to 0} 0,$$

$$F = \int\frac{d^2z}{\pi}\exp\Big\{-|z|^2(1 - e^{-\varepsilon-\omega\beta}E) + z^*e^{-\varepsilon}(e^{-\omega\beta} - 1)\frac{j}{\omega}E+$$

$$+z\left((e^{-\omega\beta} - 1)\frac{j^*}{\omega} + i\ th(gt)e^{-i\varphi-\omega\beta}(1 - e^{-\omega\beta})\frac{j}{\omega}x\right)+$$

$$+(z^*)^2\frac{i}{2}th(gt)e^{i\varphi}x - z^2\frac{i}{2}th(gt)e^{-i\varphi-2\omega\beta}x\Big\}.$$

As all preceding integrals the integral (10) is also Gauss which is calculated by the formula (9). To simplify the differentiation P with respect to ε, it is convenient to decompose the phase and normalization quantity of the functional P in the series on the function $x(\varepsilon)$. The nonvanishing contribution as $\varepsilon \to 0$ to n and D is determined by the expression

$$P_{n,D} = (1 - e^{-\omega\beta})\,\frac{\sqrt{E}}{\gamma}\left(1 + \frac{x^2}{2\gamma^2}th^2(gt)e^{-2\omega\beta}\right)\times$$

$$\times\exp\Big\{\frac{(1 - e^{\omega\beta})^2}{\gamma}\left(\frac{|j|^2}{\omega^2}\left(\frac{1 - e^{-\varepsilon}E}{e^{-\omega\beta} - 1}+\right.\right.$$

$$\left.\left.+x^2\frac{e^{-\omega\beta}}{\gamma^2}th^2(gt)\right) + \frac{i}{2}\left(\frac{(j^*)^2}{\omega^2}e^{i\varphi} - \frac{j^2}{\omega^2}e^{-i\varphi}\right)\frac{x}{\gamma}th(gt)\right)\Big\},$$

$$\gamma = 1 - \exp^{-\varepsilon-\omega\beta}E.$$

Using the definition (5) for the function obtained we derive the following result:

$$< n >= \left[sh^2(gt) + \frac{|j|^2}{\omega^2}(ch(gt) + sh(2gt)sin(\varphi - 2\theta)) \right] + < n >_\beta, \quad (11)$$

$$< n >_\beta = n_0 ch(2gt),$$

$$D- < n >= sh(gt) \left[\frac{sh(3gt) - sh(gt)}{2} + 2\frac{|j|^2}{\omega^2} sh(3gt) + \right.$$

$$\left. +2\frac{|j|^2}{\omega^2} ch(3gt) sin(\varphi - 2\theta) \right] + (D- < n >)_\beta, \quad (12)$$

$$(D- < n >)_\beta = n_0^2 ch(4gt) + 2n_0 sh(gt) sh(3gt) +$$

$$+2n_0 \frac{|j|^2}{\omega^2}(ch(4gt) + sh(4gt) sin(\varphi - 2\theta)),$$

$$\theta = arg(\alpha),$$

which is the exact solution of the problem with the quadratic Hamiltonian and initial distribution.

3. Dispersion and average occupation number

The sub-Poisson photon statistics of the field with the frequency ω is said to take place if the inequality $D - n$ is fulfilled. At zero temperatures in formulae (11,12) the terms $< n >_\beta$ and $(D- < n >)_\beta$ are equal to zero and these formulae coincide with the well known one [2] which is obtained for the initial distribution ρ_0. The inequality $D < n$ is fulfilled when the relation between the phases is equal to $\varphi - 2\theta = -\frac{\pi}{2}$. With increasing temperature the positive addend $(D- < n >)_\beta$ describes the influence

of the thermal noise and prevents the establishment of the sub-Poisson photon statistics. Also, this is prevented by the first terms in the square brackets (12) which are independent of the initial field j and describe the quantum noise. It can be noted that the thermal fluctuations are important at a very small evolution time. On the contrary, the quantum fluctuations are important at large times.

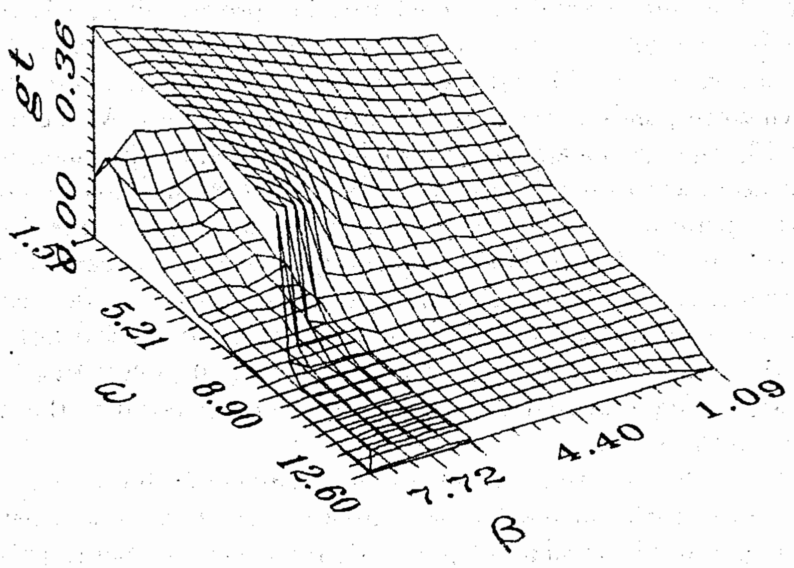
For a more detailed research of the Sub-Poisson field statistics the numerical calculation was undertaken. The points of sign-change of the function $D- < n >$ were determined by the use of the net of three variables gt, j, ω, β values. A special indication of the change of the $D- < n >$ sign from "plus" to "minus" and vice versa showed the existence of two surfaces with such a change. The scale of the step in the net of values was determined by the requirement of "smoothness" for the function $D- < n >$. The calculation shows that the region of the parameters gt, ω, and β admitting a sub-Poisson process when j is constant, is embedded between two surfaces drawn in the figure. This figure shows the case $< n > << 1$ and the interval $1 \div 0,1$ for the parameter $\alpha\omega^{-1}$ which corresponds to the optical region of frequencies. At the frequency $\omega \geq 10\alpha$ the disappearance of this effect is caused by the quantum noise. It is necessary to emphasize that the frequency threshold is eroded as the temperature rises. When a frequency is small $\omega \sim 0$, the disappearance of this effect is explained by joining two surfaces because of thermal fluctuations. For this reason the time delay exists. In decreasing the pumping α the shape of the surfaces is not changed but they are brought together, thus reducing the volume admitting the sub-Poisson photon statistics. One may find a short statement of this result in the work [6] of the authors.

Thus, the quantum as well as thermodynamic fluctuations destroy light states with the sub-Poisson statistics. The latter may exist only in the region of finite values of pumping, frequency, temperature and evolution time. Suppression of this effect is caused by thermal fluctuations in the long-wave region and manifests itself in the delay of the effect. In the short-wave region the suppression is caused by the quantum noise and manifests itself in the presence of the frequency threshold at low temperatures.

The increasing of the sub-Poisson effect was found [7] in the model (2) with the growth of the temperature. The photon statistics and waiting-time distribution were investigated in [8]. The cross- correlation of two modes associated with the two-photon process of more general type was treated in [9].

In conclusion let us note, that in the mathematical aspect the Hamiltonian (2) corresponds to the Hamiltonian of the bosons excited above the condensate in Bogolubov's superfluidity theory [10]. We suppose that the investigation of the occupation number statistics in this case can give new information about the condensate. So, the "squeezing" theory in the sphere of light may be extended to the condensed matter physics.

The authors are thankful to N.N.Bogolubov for the discussion of this aspect of their work and for the support.



1. The range of parameter values admitting the sub-Poisson distribution is embedded between two surfaces. The interval of the parameter $\omega(sec)^{-1}$ $1 \div 10$ corresponds to the frequency interval $10^{13} \div 10^{14}$ when the pumping α is equal to 10^{12}, the interval of values β $8 \div 1$ corresponds to the temperature interval $120 \div 1200^0$K.

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Received by Publishing Department
on February 17, 1992.