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PATH INTEGRAL FOR COHERENT STATES OF THE DYNAMICAL U_2 GROUP AND $U_{2|1}$ SUPERGROUP

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1 Introduction

The standard path integral over fermionic and bosonic variables in the holomorphic representation is widely used in various quantum-mechanical problems. Such an integral can be thought of as an integral over the classical phase space associated with ordinary Fermi and Bose coherent states (CS). These states provide a convenient basis for unitary irreducible representations (UIR's) of the Bose oscillator group and Fermi oscillator supergroup, whose Lie algebras consist of generators

$$\{b^\dagger b,\, b^\dagger,\, b,\, I\} \quad \text{and} \quad \{f^\dagger f,\, f^\dagger,\, f,\, I\}, \tag{1}$$

respectively. The standard commutation (anticommutation) relations are as follows

$$[b, b^\dagger] = \{f^\dagger, f\} = 1 \ .$$

Symbolically CS's can be represented as

$$|CS\rangle_B = \exp(\alpha b^\dagger)\,|0\rangle_B \qquad |CS\rangle_F = \exp(\theta f^\dagger)\,|0\rangle_F, \tag{2}$$

where α is a complex number and θ is a Grassmann parameter. By using decomposition of unity in terms of states (2), one can obtain path integral with respect to the measure

$$D\bar{\alpha} D\alpha D\bar{\theta} D\theta, \tag{3}$$

which is invariant under linear shift transformations

$$\alpha \to \alpha + \alpha_0, \qquad \theta \to \theta + \theta_0. \tag{4}$$

The corresponding classical phase space can be thought of as a direct product of a complex plane and a complex flat Grassmann manifold. The Bose (Fermi) oscillator group (supergroup) acts in this space through linear shifts (4). To be more specific, unitary transformations

$$U(g) = \exp\left(\alpha_0 b^\dagger - \bar{\alpha}_0 b + \theta_0 f^\dagger - f\bar{\theta}_0\right)$$

induce in the classical phase space canonical transformations (4).

Flat path integral over measure (3) turns out to be very useful in the framework of perturbation theory, as unperturbed Hamiltonian H_0 looks as a linear combination of generators (1). But it is practically useless in attempts to go beyond perturbation expansion. This is merely due to the fact that the exact diagonalization of the whole Hamiltonian $H = H_0 + H_{int}$ requires more general transformations than that of the harmonic oscillator-type.

Let G be a group of transformations that result in diagonalization of H. Then its Lie algebra L is known to contain H as an element. L is then called the spectrum generating algebra (SGA) and the corresponding group G is known as a dynamical group. Note that a direct product of oscillator groups generated by (1) plays a role of the dynamical group for H_0.

A new phase space (an orbit of the coadjoint representation of L) turns out to be a curved one. Path integral over CS's associated with L is to be regarded as an integral in a curved space with measure more complicated than that of Eq.(3). An important point is that G acts in this space via linear fractional transformations, which induce in the corresponding path integral, an appropriate change of integration variables.

The concept of CS's associated with unitary irreducible representations (UIR') of Lie group G was firstly introduce by Perelomov (1972) and generalized to the case of supergroups by Bars and Günaydin (1983). Let us outline below the main features of this approach. Let L be a Lie algebra that has the so-called 3-grading decomposition with respect to Lie algebra L_0 of its maximal compact subgroup:

$$L = L^{-1} \oplus L^0 \oplus L^{+1}, \tag{5}$$

L_0 contains the generator Q of an Abelian U_1 factor, that gives the grading, i.e.,

$$L^0 = H \oplus Q$$

and

$$[Q, H] = 0, \ \ [Q, L^{+1}] = L^{+1}, \ \ [Q, L^{-1}] = -L^{-1}.$$

The elements $l^m \in L^m$ satisfy the formal commutation relations

$$[l^m, l^n] \in L^{n+n}, \quad n, m = -1, 0, +1, \tag{6}$$

where $L^{n+m} = 0$ for $|n+m| > 1$. For Lie superalgebras the same definitions (5, 6) hold, with bilinear product (6) is now to be understood as an anticommutator between any two odd elements of superalgebra L and as a commutator otherwise.The important point concerning the decomposition (5) is that if there exists a set of "lowest weight" states $|lw\rangle$, that are transformed irreducibly under the maximal compact subgroup action and are annihilated by all the annihilation operators L^{-1}, then the set of states

$$(L^{+1})^p |lw\rangle, \quad p = 0, 1, 2, \ldots \tag{7}$$

form the basis for the irreducible representation of the whole group G. It then follows that the generalized CS's associated with algebra L can be symbolically defined as

$$|CS\rangle = \exp\left(\sum_i l_i^{+1} \alpha_i\right) |lw\rangle, \tag{8}$$

where α_i are even or odd Grassmann parameters (depending on bosonic or fermionic nature of the raising operators l_i^{+1}). For ordinary Lie groups α_i are complex numbers.The CS vectors (8) provide a convenient basis for constructing path-integral representation for the systems with dynamical group G, the crucial point being that the irreducibility of the states (7) ensures decomposition of unity in terms of CS (8).

For the physical applications, especially in quantum-optical models, it is convenient to deal with oscillator-like representations of the L-algebra generators. Then, all the L-generators are expressed as bilinears of Bose(Fermi) creation and annihilation operators. As is well known, n^2 blinears

$$b_i b_j^\dagger, \ [b_i, b_j^\dagger] = \delta_{ij} \ i, j = 1, 2, \ldots n$$

generate the Lie algebra of U_n group. To extend U_n to the unitary $U_{n|m}$ supergroup, one has to add to n bosonic operators m fermionic ones f_μ, $\mu = 1, 2, \ldots m$. Bilinears $b_i b_j^\dagger$ and $f_\mu f_\nu^\dagger$ form the Lie algebras of U_n and U_m under commutation, respectively, whereas Bose-Fermi bilinears $b_i f_\mu^\dagger$ and $b_i^\dagger f_\mu$ close into the set $b_i b_j^\dagger$, $f_\mu f_\nu^\dagger$ under anticommutation

$$\{b_i f_\mu^\dagger, b_j^\dagger f_\nu\} = \delta_{ij} f_\mu^\dagger f_\nu + \delta_{\mu\nu} b_j^\dagger b_i. \tag{9}$$

In the subsequent sections we will be concerned with the simplest cases $n = 2$ and $n = 2, m = 1$.

2 U_2 Lie Algebra in the oscillator-like representation and U_2 CS's

In the Bose oscillator-like representation the generators of U_2 Lie algebra can be taken to be

$$K_1 = b_1^\dagger b_1, \ K_2 = b_2^\dagger b_2, \ K_+ = b_2^\dagger b_1, \ K_- = b_1^\dagger b_2. \tag{10}$$

U_2 linear Casimir operator is a number operator $N = b_2^+ b_2 + b_1^+ b_1$. All the higher U_2 Casimirs are functions of N due to the fact that in realization (10) we deal with fully symmetric U_2 representations that are labelled by the eigenvalues of N. As is known $U_2 = SU_2 \otimes U_1$ which means that U_2 algebra can be decomposed into the direct sum

$$\{K_+,\ K_-,\ K_0 = \frac{1}{2}(b_2^+ b_2 - b_1^+ b_1)\} \oplus \{N\}, \tag{11}$$

where generators K_+, K_-, K_0 span the SU_2 subalgebra:

$$[K_+, K_-] = 2K_0 \qquad [K_0, K\pm] = \pm K_\pm. \tag{12}$$

Algebra (10) is easily seen to have 3-grading decomposition with respect to the $U_1 \oplus U_1$ subalgebra generated by K_1, K_2:

$$L_+ = \{K_+\},\ L_- = \{K_-\},\ L_0 = \{K_1, K_2\},$$

grading being achieved with the generator K_2. The lowest weight state which is transformed irreducibly under $U_1 \otimes U_1$ group action and is annihilated by the K_- operator looks as follows

$$|lw\rangle = |n, 0\rangle, \tag{13}$$

where

$$b_1^+ b_1 \, |n,m\rangle = n\, |n,m\rangle\,,\ b_2^+ b_2 \, |n,m\rangle = m\, |n,m\rangle\,,\ n,m = 0,1,2,\ldots$$

Due to Eq.(8) the U_2 CS can be written in the form

$$|\alpha; n\rangle = (1+|\alpha|)^{-\frac{n}{2}} \exp(\alpha b_2^+ b_1)\, |n,0\rangle, \tag{14}$$

where the complex number α belongs to the coset space $U_2/U_1 \otimes U_1$ which is isomorphic to the complex projective space CP^1. Note that CS (14) depends upon the representation index $n \geq 0$ - the eigenvalue of linear Casimir operator. For every value of n the basis in the U_2 representation space can be chosen as

$$|e_p\rangle = |n-p, p\rangle\,,\quad p = 0, \ldots, n,$$

so that $dim\{|e_p\rangle\} = n+1$. The overlap of two states $|\alpha'; n\rangle$ and $|\alpha; n\rangle$ is given as

$$\langle \alpha'; n|\alpha; n\rangle = (1+|\alpha'|^2)^{-\frac{n}{2}}\, (1+|\alpha|^2)^{-\frac{n}{2}}\, (1+\bar{\alpha}'\alpha)^n. \tag{15}$$

An important property of these states is that they satisfy the completeness relation

$$\int |\alpha; n\rangle\langle\alpha; n|\, d\mu_n(\alpha) = I_n = \sum_{p=0}^{n} |e_p\rangle\langle e_p|, \tag{16}$$

where the U_2-invariant integration measure looks as follows:

$$d\mu_n(\alpha) = \frac{n+1}{\pi} \frac{d^2\alpha}{(1+|\alpha|^2)^2}\ . \tag{17}$$

It is easily seen that

$$\int \langle\alpha; n|e_p\rangle\langle e_q|\alpha; n\rangle\, d\mu_n(\alpha) = \delta_{pq}\ . \tag{18}$$

Due to Eq.(18) for any operator F acting in the $(n+1)D$ space spanned by $|e_p\rangle$ one has

$$Sp\, F = \sum_{p,q} \langle e_p|F|e_q\rangle\, \delta_{pq} = \int d\mu_n(\alpha)\, \langle\alpha; n|F|\alpha; n\rangle. \tag{19}$$

The averages over U_2 CS's look as follows

$$\begin{aligned}
\langle\alpha; n|K_1|\alpha; n\rangle &= \frac{n}{1+|\alpha|^2},\ \langle\alpha; n|K_2|\alpha; n\rangle = n\frac{|\alpha|^2}{1+|\alpha|^2} \\
\langle\alpha; n|(K_+)^p|\alpha; n\rangle &= \frac{n!}{(n-p)!}\frac{\bar{\alpha}^p}{(1+|\alpha|^2)^p}, \\
\langle\alpha; n|(K_-)^p|\alpha; n\rangle &= \frac{n!}{(n-p)!}\frac{\alpha^p}{(1+|\alpha|^2)^p}, \\
\langle\alpha; n|K_0|\alpha; n\rangle &= -\frac{n}{2}\frac{1-|\alpha|^2}{1+|\alpha|^2},\ p = 0, 1, \ldots
\end{aligned} \tag{20}$$

3 Path integral

Let us consider the path integral over U_2 CS for the partition function

$$Z = Sp\, e^{-\beta H},$$

where the Hamiltonian H belongs to the U_2 enveloping algebra. Due to Eq.(19) one has

$$Z = \sum_{n=0}^{\infty} \int d\mu_n(\alpha) \langle\alpha; n|e^{-\beta H}|\alpha; n\rangle.$$

Defining ϵ as β/N and using Eq.(16) we write in the usual manner

$$Z = \sum_{n=0}^{\infty} \int d\mu_n(\alpha)\langle\alpha;n|\alpha_N;n\rangle\langle\alpha_N;n|e^{-\epsilon H}|\alpha_{N-1};n\rangle\langle\alpha_{N-1};n|e^{-\epsilon H} \ldots e^{-\epsilon H}|\alpha_0;n\rangle\langle\alpha_0;n|\alpha;n\rangle\, d\mu_n(\alpha_N)\ldots d\mu_n(\alpha_0).$$

Up to the second order in ϵ one has

$$\langle\alpha_j;n|e^{-\epsilon H}|\alpha_i;n\rangle = \langle\alpha_j;n|\alpha_i;n\rangle \exp\left(-\epsilon\mathcal{H}_n(\bar{\alpha}_j,\alpha_i)\right),$$

where

$$\mathcal{H}_n(\bar{\alpha}_j,\alpha_i) = \frac{\langle\alpha_i;n|H|\alpha_j;n\rangle}{\langle\alpha_i;n|\alpha_j;n\rangle}.$$

The integration over $d\mu_n(\alpha)$ in accordance with Eq.(19) yields

$$\begin{aligned}\int d\mu_n(\alpha)\langle\alpha;n|\alpha_N;n\rangle\langle\alpha_0;n|\alpha;n\rangle &= Sp\,|\alpha_N;n\rangle\langle\alpha_0;n| \\ &= \langle\alpha_0;n|\alpha_N;n\rangle,\end{aligned}$$

so that

$$Z = \sum_{n=0}^{\infty}\int\prod_{i=0}^{N} d\mu_i \prod_{j=1}^{N}\langle\alpha_j|\alpha_{j-1}\rangle\langle\alpha_0|\alpha_N\rangle \exp\left(-\epsilon\sum_{j=1}^{N}\mathcal{H}_n(\bar{\alpha}_j,\alpha_{j-1})\right). \quad (21)$$

For any state vector $|\psi\rangle$, that belongs to the $(n+1)D$ Hilbert space with an element

$$P_m(\alpha)/(1+|\alpha|^2),$$

where $P_m(\alpha)$ is an arbitrary polynomial of degree $m \le n$ (Perelomov 1972), one has

$$\langle\psi| = \int d\mu_n(\alpha)\langle\psi|\alpha\rangle\langle\alpha|,$$

which in the components can be written as

$$\psi_n(\beta) = \int d\mu_n(\alpha)\langle\alpha;n|\beta;n\rangle\psi_n(\alpha), \quad \psi_n(\alpha) \equiv \langle\psi|\alpha;n\rangle. \quad (22)$$

Note that the reproducing kernel $\langle\alpha;n|\beta;n\rangle$ acts as a delta function with respect to the measure $d\mu_n(\alpha)$. Due to Eq. (22) the integration over $d\mu_0$ in (21) can be carried out explicitly to yield

$$Z = \sum_{n=0}^{\infty}\int d\mu_1\ldots d\mu_N \prod_{j=1}^{N}\langle\alpha_j|\alpha_{j-1}\rangle \exp\left(-\epsilon\sum_{j=1}^{N}\mathcal{H}_n(\bar{\alpha}_j,\alpha_{j-1})\right)_{\alpha_0=\alpha_N} \quad (23)$$

With $\alpha_{j-1} = \alpha_j - \delta_j$ it then follows

$$\ln\langle\alpha_j|\alpha_{j-1}\rangle = \frac{n}{2}\frac{\alpha_j\bar{\delta}_j - \bar{\alpha}_j\delta_j}{1+|\alpha_j|^2} + O(\delta_j^2). \quad (24)$$

In the continuous limit this yields

$$Z = \sum_{n=0}^{\infty}\int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha)\exp\left(-\frac{n}{2}\int_0^{\beta}\frac{\bar{\alpha}\dot{\alpha}-\dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}ds - \int_0^{\beta}\mathcal{H}_n(\bar{\alpha},\alpha)ds\right), \quad (25)$$

where the following normalization holds:

$$\int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha)\exp\left(-\frac{n}{2}\int_0^{\beta}\frac{\bar{\alpha}\dot{\alpha}-\dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}ds\right) = Sp\, I_n = n+1.$$

4 Evaluation of the U_2 path integral

In order to illustrate how the general formula (25) works, let us take H in the form

$$H = \omega_1 b_1^{\dagger}b_1 + \omega_2 b_2^{\dagger}b_2 + \bar{\lambda}b_2^{\dagger}b_1 + \lambda b_2 b_1^{\dagger}, \quad (26)$$

where $\omega_1\omega_2 \ge |\lambda|^2$ for the Hamiltonian (26) is to be bounded from below. For the partition function in accordance with Eqs.(25) and (20) one gets

$$Z = \sum_{n=0}^{\infty} e^{-\frac{n}{2}\beta(\omega_1+\omega_2)}\int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha)\exp\left(-\int_0^{\beta}\mathcal{L}_n(\bar{\alpha},\alpha)ds\right),$$

where the quantity

$$\mathcal{L}_n = \frac{n}{2}\frac{\bar{\alpha}\dot{\alpha}-\dot{\bar{\alpha}}\alpha}{1+|\alpha|^2} + \frac{n}{2}(\omega_1-\omega_2)\frac{1-|\alpha|^2}{1+|\alpha|^2} + n\lambda\frac{\bar{\alpha}}{1+|\alpha|^2} + n\lambda\frac{\alpha}{1+|\alpha|^2} \quad (27)$$

can be defined as the Lagrangian. The Euler-Lagrange equations lead to the equations of motion

$$\dot{\alpha} = \{\alpha, \mathcal{H}_n\},$$

where $\{,\}$ is a Poisson bracket defined by

$$\{A,B\} = \frac{(1+|\alpha|^2)^2}{n}\left[\frac{\partial A}{\partial\alpha}\frac{\partial B}{\partial\bar{\alpha}} - \frac{\partial A}{\partial\bar{\alpha}}\frac{\partial B}{\partial\alpha}\right].$$

This indicates that the classical phase space spanned by α and $\bar{\alpha}$ is curved - in fact the complex projective plane $CP^1 \simeq S^2$ (Berezin 1975).

Due to the relation $U_2 = SU_2 \otimes U_1$ the general U_2 transformation can be taken to be

$$U_2 = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} e^{i\phi},$$

where $|u|^2 + |v|^2 = 1$ and $0 \leq \phi < 2\pi$. The path integral in Eq.(25) can be evaluated with the help of transformations of the integration variables which are induced by the U_2 action in the coset space $U_2/U_1 \otimes U_1$. U_2 acts in the integration space $U_2/U_1 \otimes U_1$ through the following canonical transformations:

$$\alpha \to \alpha = \frac{u\alpha + v}{-\bar{v}\alpha + \bar{u}}. \qquad (28)$$

where the group parameters u and v are kept constant. The integration measure $d\mu_n(\alpha)$ is invariant under transformations (28). The same is true for the kinetic term, for example

$$\int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}ds \to \int_0^\beta \frac{\dot{\alpha}}{1+|\alpha|^2}\frac{\bar{u}\bar{\alpha}+\bar{v}}{-\bar{v}\alpha+\bar{u}}ds =$$

$$\int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}ds + \int_0^\beta \frac{ds}{1+|\alpha|^2}\left(\dot{\alpha}\frac{\bar{u}\bar{\alpha}+\bar{v}}{-\bar{v}\alpha+\bar{u}} - \bar{\alpha}\dot{\alpha}\right) =$$

$$\int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}ds - \int_0^\beta d\ln(\bar{v}\alpha - \bar{u}) = \int_0^\beta \frac{\bar{\alpha}\dot{\alpha}}{1+|\alpha|^2}ds,$$

where the total derivative can be dropped since $\alpha(0) = \alpha(\beta)$. Upon taking

$$u = \sqrt{\frac{\bar{\lambda}}{|\lambda|}}\sqrt{\frac{\omega+\Omega}{2\Omega}}, \quad v = \sqrt{\frac{\bar{\lambda}}{|\lambda|}}\frac{|\lambda|}{\sqrt{2\Omega(\omega+\Omega}},$$

$$\Omega = \sqrt{\omega^2+|\lambda|^2}, \quad \omega = \frac{\omega_1-\omega_2}{2},$$

one gets

$$Z = \sum_n^\infty e^{-n\beta\frac{\omega_1+\omega_2}{2}} Z_n,$$

where the path integral

$$Z_n = \int_{\alpha(0)=\alpha(\beta)} D\mu_n(\alpha)\exp\left(-\frac{n}{2}\int_0^\beta \frac{\bar{\alpha}\dot{\alpha}-\dot{\bar{\alpha}}\alpha}{1+|\alpha|^2} - n\Omega\int_0^\beta \frac{1-|\alpha|^2}{1+|\alpha|^2}ds\right) \qquad (29)$$

can be evaluated directly through the definition (23):

$$Z_n = \lim_{N\to\infty}\int_{\alpha_0=\alpha_N} d\mu_1\ldots d\mu_N \prod_{j=1}^N \langle\alpha_j|e^{2\epsilon\Omega K_0}|\alpha_{j-1}\rangle.$$

Namely,taking into account that

$$e^{2\epsilon\Omega K_0}|\alpha_{j-1}\rangle = e^{-\epsilon\Omega n}|\alpha_{j-1}e^{2\epsilon\Omega}\rangle$$

one gets

$$Z_n = e^{-\beta\Omega n}\lim_{N\to\infty}\int_{\alpha_0=\alpha_N} d\mu_1\ldots d\mu_N \prod_{j=1}^N \langle\alpha_j|\alpha_{j-1}e^{2\epsilon\Omega}\rangle. \qquad (30)$$

By using decomposition of unity (16) the integration over $d\mu_1\ldots d\mu_{N-1}$ can be carried out explicitly. This yields

$$Z_n e^{\beta\Omega n} = \int_{\alpha_N=\alpha_0} \langle\alpha_N|\alpha_0 e^{2N\epsilon\Omega}\rangle = \int d\mu_n(\alpha)\langle\alpha|\alpha e^{2\beta\Omega}\rangle$$

$$= (n+1)e^{2\beta\Omega(n+1)}\int_0^\infty dx\frac{(1+x)^n}{(e^{2\beta\Omega}+x)^{n+2}} = \frac{e^{2\beta\Omega(n+1)}-1}{e^{2\beta\Omega}-1},$$

which gives

$$Z = \sum_{n=0}^\infty \exp(-n\beta\frac{\omega_1+\omega_2}{2})\frac{\sinh\beta\Omega(n+1)}{\sinh\beta\Omega}.$$

It then follows that

$$Z = \sum_{n=0}^\infty \exp(-n\beta\frac{\omega_1+\omega_2}{2})\sum_{m=-n/2}^{m=n/2}\exp(-2\beta\Omega m),$$

which yields the correct spectrum

$$E_{n_1,n_2} = \frac{\omega_1+\omega_2}{2}(n_1+n_2) + \Omega(n_1-n_2), \quad n_1, n_2 \geq 0.$$

At the end of this section there should be pointed out that CS's (14) coincide in fact with those of the SU_2 group which can be parametrized by the points of the coset space SU_2/U_1 (Perelomov 1972). This is merely due to the fact that $U_2 = SU_2 \otimes U_1$. Thus, in order to construct path integral for a spin system with dimensionality $2j+1$ one can employ Eqs.(20) where one must put $j = \frac{n}{2}$. For example, for the linear spin Hamiltonian

$$H = \Omega K_0 + \bar{\lambda} K_+ + \lambda K_-,$$
$$K_0|m\rangle = m|m\rangle, \qquad m = -j, -j+1, \dots j,$$

one gets

$$Z_j = Sp\, e^{-\beta H_j} = \int D\mu_j(\alpha) \exp\left(-j\int_0^\beta \frac{\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha}{1+|\alpha|^2}ds \right.$$
$$\left. +\Omega j\int_0^\beta \frac{1-|\alpha|^2}{1+|\alpha|^2}ds - 2j\bar{\lambda}\int_0^\beta \frac{\bar{\alpha}}{1+|\alpha|^2}ds - 2j\lambda\int_0^\beta \frac{\alpha}{1+|\alpha|^2}ds\right), \quad (31)$$

which is readily evaluated with the help of substitution (28).

5 Path integral over $U_{2|1}$ CS

Thus far, we have discussed the path-integral representations for the U_2 Lie algebra. From the physical point of view ordinary Lie algebras are relevant for purely bosonic (or fermionic) systems. For example, the partition function for a superfluid helium model is expressed as a path integral over CS's associated with the noncompact $SU_{1,1}$ algebra (Gerry and Silverman 1982). In the models of quantum optics, however, there appear mostly Hamiltonians that include both bosonic and fermionic degrees of freedom. One needs then to consider path integral over super CS's associated with underling superalgebras. For example, the compact $U_{1|1}$ and noncompact $OSP_{2|2}$ superalgebras turn out to be SGA's for Jaynes-Cummings and Rabi Hamiltonians, respectively (Buzano et al 1989). Note also, that there has recently been considered the path integral for $OSP_{1|2}$ CS's (Schmitt and Mufti 1991). Here we consider the simplest $9D$ unitary $U_{2|1}$ supergroup that appears as the dynamical group for various quantum-optical Hamiltonians.

In the oscillator-like representation the $U_{2|1}$ generators can be taken to be (Bars, Günadyan 1983)

$$L_0 = \{b_1^\dagger b_1,\ f^\dagger f,\ b_1^\dagger f,\ b_1 f^\dagger\} \oplus \{b_2^\dagger b_2\}$$

$$L_+ = \{b_2^\dagger f,\ b_2^\dagger b_1\}, \quad L_- = \{b_2 f^\dagger,\ b_2 b_1^\dagger\}, \qquad (32)$$

where

$$\{f, f^\dagger\} = 1.$$

It then follows that Eq.(32) gives the 3-grading decomposition with respect to the maximal compact subsupergroup with superalgebra $U_{1|1} \oplus U_1$. Grading is achieved with the operator $b_2^\dagger b_2$. The operators in the first curly brackets in L_0 form the basis for the $U_{1|1}$ superalgebra. As is known, all the irreps. of $U_{1|1} \otimes U_1$ superalgebra are $1D$ or $2D$ (de Crombrugghe and Rittenberg 1983). The basis can be taken as

$$1D \text{ case}: \quad |e_0\rangle = |0, m\rangle_B \otimes |0\rangle_F$$

$$2D \text{ case}: \quad |e_1\rangle = |n-1, m\rangle_B \otimes |1\rangle_F, \quad |e_2\rangle = |n, m\rangle_B \otimes |0\rangle_F,$$

where the use is made of the standard notation so that

$$f|0\rangle_F = 0, \qquad f^\dagger|0\rangle_F = |1\rangle_F.$$

The $U_{2|1}$ supergroup acts in the superspace which is formed as Grassmann envelope of the $U_{2|1}$ superalgebra representation space (see, for example, Berezin and Tolstoy 1981). The basis of this superspace is given as $|e_0\rangle$ in the $1D$-case and $|e_2\rangle$, $\zeta|e_1\rangle$ in the $2D$- case. Here ζ is a Grassmann parameter and vector $|e_1\rangle$ is chosen to have an odd grading (consequently, $|e_2\rangle$ is even- graded).

The lowest weight vectors that are transformed irreducibly under the $U_{1|1} \otimes U_1$ supergroup action and are annihilated by all L_- operators are as follows

$$|0,0\rangle_B \otimes |0\rangle_F \quad \text{and} \quad |n,0\rangle_B \otimes |0\rangle_F + \zeta|n-1,0\rangle_B \otimes |1\rangle_F. \qquad (33)$$

Due to formula (8) $U_{2|1}$ CS can be represented in the form

$$|\alpha, \theta; n\rangle = (1+|\alpha|^2)^{-n/2} \exp\left(-\frac{n}{2}\frac{\bar{\theta}\theta}{1+|\alpha|^2}\right) \exp(\alpha b_2^\dagger b_1) \times$$
$$(\ |n,0\rangle_B|0\rangle_F + \sqrt{n}\theta|n-1,0\rangle_B|1\rangle_F\ ), \qquad (34)$$

Notice that CS (34) depends upon ordinary and Grassmann parameters simultaneously while the representation index n is now an eigenvalue of the $U_{2|1}$ linear Casimir operator

$$N = b_1^\dagger b_1 + b_2^\dagger b_2 + f^\dagger f.$$

The overlap of two states (34) is

$$\langle\alpha',\theta';n|\alpha,\theta;n\rangle = \langle\alpha';n|\alpha;n\rangle \times \exp\left(-\frac{n}{2}\frac{\bar\theta\theta}{1+|\alpha|^2} - \frac{n}{2}\frac{\bar\theta'\theta'}{1+|\alpha'|^2} + n\frac{\bar\theta'\theta}{1+\bar\alpha'\alpha}\right), \quad (35)$$

where $\langle\alpha';n|\alpha;n\rangle$ is given by Eq.(15). Unity in the representation n is resolved as

$$I_n = \int |\alpha,\theta;n\rangle\langle\alpha,\theta;n|\, d\mu_n(\alpha,\theta), \quad (36)$$

where the $U_{2|1}$ invariant measure reads as follows:

$$d\mu_n = \exp\left(\frac{\theta\bar\theta}{1+|\alpha|^2}\right)\frac{d^2\alpha}{1+|\alpha|^2}\frac{d\bar\theta d\theta}{\pi}. \quad (37)$$

The $U_{2|1}$ path integral can be obtained by the very same procedure as that in the U_2 case. The new point is that there appear a more complicated kinetic term and the antiperiodic boundary conditions for θ:

$$Z = Sp e^{-\beta H} = \sum_{\substack{n \\ \alpha(0)=\alpha(\beta),\ \theta(0)=-\theta(\beta)}} \int D\mu_n(\alpha,\theta)\exp\left(-\frac{n}{2}\int_0^\beta \frac{\dot{\bar\alpha}\alpha-\bar\alpha\dot\alpha}{1+|\alpha|^2}ds + \frac{n}{2}\int_0^\beta \frac{\bar\theta\dot\theta-\dot{\bar\theta}\theta}{1+|\alpha|^2}ds + \frac{n}{2}\int_0^\beta \frac{(\dot{\bar\alpha}\alpha-\bar\alpha\dot\alpha)\bar\theta\theta}{(1+|\alpha|^2)^2}ds - \int_0^\beta H(\alpha,\theta)ds\right), \quad (38)$$

where

$$H(\alpha,\theta) = \langle\alpha,\theta;n|H|\alpha,\theta;n\rangle.$$

In the case when $U_{2|1}$ is a spectrum generating algebra, i.e., H belongs to the $U_{2|1}$ even subalgebra, path integral (38) can be calculated by the change of integration variables in accordance with the $U_{2|1}$ group action in the classical phase space which is isomorphic to the coset $U_{2|1}/U_{1|1}\otimes U_1$.

Namely, $U_{2|1}$ supergroup element in the fundamental representation can be defined as

$$U = \begin{pmatrix} \omega_1 & \theta_1 & \lambda_1 \\ \theta_2 & \omega_2 & \theta_3 \\ \lambda_2 & \theta_4 & \omega_3 \end{pmatrix}, \quad U^\dagger U = 1,$$

where $\omega_{1,2,3}$ and $\lambda_{1,2}$ are even Grassmann parameters and $\theta_{1,2,3,4}$ are the odd ones. Then, under the $U_{2|1}$ action the supervariable $(\alpha,\theta)\in U_{2|1}/U_{1|1}\otimes U_1$

undergoes a linear fractional transformation (Bars and Günaydin 1983)

$$\alpha \to \frac{\omega_1\alpha+\theta_1\theta+\lambda_1}{\lambda_2\alpha+\theta_4\theta+\omega_3}, \qquad \theta \to \frac{\theta_2\alpha+\omega_2\theta+\theta_3}{\lambda_2\alpha+\theta_4\theta+\omega_3}. \quad (39)$$

The integration measure in Eq.(38) remains invariant under transformations (39). By specifying the $U_{2|1}$ parameters in Eq.(39) in the same manner as it was done in the previous section for the U_2 case, one can evaluate the path integral (38). This is equivalent to the direct diagonalization of the Hamiltonian by means of appropriate $U_{2|1}$ rotation in the super Fock space.

6 Conclusions

In conclusion, some remarks are to be made. First of all, there should be pointed out that we are dealing with CS's associated with finite-dimensional Lie algebras (superalgebras). As a consequence, quantum systems with finite degrees of freedom are only being considered.

The next point is that the path integral over U_2 and $U_{2|1}$ CS's turns out to be very convenient in the semiclassical treatment. In the general quantazation scheme for curved phase spaces developed by Berezin (1975) the representation index n labelling CS's associated with a group of motions of this phase space plays the role of $1/\hbar$. In the oscillator -like representation this means a large particle number limit. The stationary phase method for integrals (25,38) as $n\to\infty$ leads to the classical Euler-Lagrange equations. The important point is that by means of the substitution $\alpha\to\alpha/\sqrt{n}$ the integral (25) in the limit $n\to\infty$ goes over to the "flat" one with the standard measure $D\alpha D\bar\alpha$. Thus, one can employ the standard methods in dealing with the SU_2 path integral in the limit of a large total spin. This is in complete accordance with the fact that SU_2 CS's at large values of spin j go over into the ordinary (Glauber) CS's (Perelomov 1972). For example, the partition function (31) in the limit $j\to\infty$ reads as

$$Z_j = e^{\Omega\beta j}\int_{\alpha(0)=\alpha(\beta)} D\alpha D\bar\alpha \exp\left(\frac{1}{2}\int_0^\beta(\dot{\bar\alpha}\alpha-\bar\alpha\dot\alpha)ds - \Omega\int_0^\beta|\alpha|^2 ds - \sqrt{2j}\int_0^\beta \bar\lambda\bar\alpha ds - \sqrt{2j}\int_0^\beta \lambda\alpha ds\right), \quad (40)$$

where the coefficient λ is supposed to be time-dependent. The path integral (40) is seen to be easily calculated (see, for example, Dacol 1980).

As another example there could be considered a nuclear Hamiltonian proposed by Lipkin, Meshkov and Glik (LMG) which in the spin representation reads as (Lipkin et al 1965)

$$H = \epsilon \left[K_0 + \frac{r}{4j}(K^2{}_+ + K^2{}_-) \right],$$

where$K_i$are the SU_2 generators of dimensionality $2j+1$ with $j = \frac{n}{2}$. Here n is a total number of particles in the LMG model, ϵ and r are real parameters. Note that H belongs to the SU_2 enveloping algebra. With the help of Eqs.(20) at $p = 2$ one can readily obtain the path-integral representation for the LMG model in the form of Eq.(25).

There should be also pointed out that representations (25, 38) hold for Hamiltonians that belong to the U_2 and $U_{2|1}$ enveloping algebras, as it has been just mentioned for the SU_2 case.

The last remark concerns linear fractional transformations (28, 39). The U_2 and $U_{2|1}$ integration measures remain invariant under the corresponding local linear fractional transformations. This means that one could try to use them in calculating U_2 and $U_{2|1}$ path integrals with parameters depending on time. These and related problems will be discussed elsewhere.

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